MASTER'S DEGREE THESIS

MSc in Applied Mathematics

Title: Hamiltonian methods in stability and bifurcations problems for artificial satellite dynamics.

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Abstract

The dynamics of a rigid body in a central gravitational field can be modelled by a Hamiltonian system with continuous symmetries implemented by an action of the group $SO(3)$. Depending on the particular geometry of the body (as for instance if the body is axisymmetric) this symmetry group can even be enlarged.

There are many classical studies of steadily rotating solutions of this system based on various approximate models of the orbital-attitude coupling of artificial Earth satellites, but these models don't fully exploit the geometric structure of the problem.

[**WKM90**] provides a geometrical description of this problem and studies its relative equilibria (steady solutions) using Poisson reduction and the Energy-Cassimir method. Our approach consists in attacking this problem by means of the more recent Reduced-Energy-Momentum [**SLM91**], since it clarifies and improves the existing results and, furthermore, it allows for a systematic bifurcation analysis.

One novelty of this work with respect to previous approaches is that we also treat axisymmetric bodies using this geometric formalism. Employing these techniques we are able to explicitly find previously unknown relative equilibria and to study their stability and bifurcation patterns.

Contents

Introduction

This work studies the dynamics of a rigid body in a central gravitational field using geometric methods. If the bodies under consideration are not spherically symmetric non-trivial torques appear affecting the attitude (orientation) and the orbit. Physically this effect is known as "gravity-gradient torque".

In the literature related to this problem, there are studies of steady state solutions and quasi-periodic motions based on various approximate models of the coupling between orbital motion and attitude motion of earth satellites [**Bel65**]. In this classical treatment of this problem, the natural geometric and group-theoretic underpinnings of the problem are not exploited to the full possible extent.

Geometrically, the dynamics of a rigid body in the central field is the flow of a Hamiltonian system with symmetries. The classically studied steady state solutions correspond to what is known as relative equilibria of the Hamiltonian symmetric system. In the article [**WKM90**], the non-canonical Hamiltonian structure of the problem of motion of a rigid body in a central gravitational field is established. It is shown how the group SO(3) of three dimensional rotations emerges as a symmetry group. Poisson reduction by this action yields a nine-dimensional system that corresponds to observing the dynamics from a moving frame. In this body frame, the dynamics manifests the effect of a fictitious torque physically known as the gravity-gradient torque.

The dynamics of this flow may be very complicated. The point of view adopted by the qualitative study of symmetric Hamiltonian systems is to identify a special family of its solutions (integral curves) and then to carry out a local analysis of the properties of the flow near that solution. For this kind of symmetric Hamiltonian systems, the special family of solutions to consider are relative equilibria ([**Mar92**]). These are dynamical evolution orbits which are contained in group orbits of the symmetric action.

The reason for choosing relative equilibria as the starting point for a qualitative analysis is that, as is generally accepted, relative equilibria act as "organizing centres" for the dynamics of the system. That is, it suffices to study the flow at and near relative equilibria to understand to a big extent its general properties. This approach is especially relevant in cases where the exact integration of the flow is too difficult or even impossible.

This method is in contrast with other approaches classically employed towards the study of the rigid body in a gravitational field. One of the most popular consists in first treating the satellite as a point particle and finding its expected Keplerian orbit. Then the attitude dynamics is determined by solving the dynamics of the full body where the spatial part follows this prescribed Keplerian orbit. It is not clear if this procedure is a good method and if it could lead to wrong results. This approach, where the satellite is assumed to follow a Keplerian orbit, is named *restricted problem*.

On the other hand, other approaches, like [**WKM90**], rely on a potential series expansion and the coupling between orbital and attitude variables. This is the *unrestricted problem* and is the point of view adopted in this thesis.

As [**WKM90**] and [**Bec97**] point out, these geometric techniques can improve the classical calculations. As an example, classically the attitude of a body is described by a matrix parametrized by Euler angles. The use of Euler angles can result in considerable algebraic manipulations involving trigonometric functions and can lead to singularities at certain attitudes. This difficulties can be easily avoided using the geometric formalism.

Other classical treatments like [**Ste69**] or [**OT04**] use Routhian reduction and Euler angles as basic tools. In [**WKM90**] the major tool in studying this system was the Poisson reduction and energy-Cassimir method ([**Mar92**]). However they claim that the application of the reduced energy momentum method introduced in ([**SLM91**]) can be even more useful because it can simplify the computations and even provide tools to study the bifurcation behaviour of the system. The implementation of this program has been the main motivation for our study. In particular we find that the reduced energy momentum method has proved to be a powerful tool that produces in a simple way the already known results and actually allows to derive new, previously unknown ones like, most notably in some bifurcation studies.

Therefore, as a first step and following [**WKM90**], all results stated of that reference have been re-derived here using the energy momentum method as the main tool for the analysis of the problem.

In the second order approximation some previously unknown results have been found: all the relative equilibria for the order two approximation are explicitly computed. Numerically there were some studies of the oblique equilibria, like [**OT04**], also some properties of oblique equilibria were known [**Bec97**]. But the explicit algebraic solution remained unknown. In Section 8.1 we give all the explicit expressions.

[**Bec97**] suggests that a deeper study has to be done at very low orbits. In Section 8.7 we describe the transitions between equilibria families at low orbits. [**OT04**] makes a similar study numerically but here for the first time the reduced energy momentum method is applied to this problem leading to a detailed bifurcation description.

The geometric framework has also been applied to the axisymmetric case, buildibng on the work done by [**WKM90**], where only $SO(3)$ reduction was considered. [**Bec97**] also includes the axisymmetric reduction but it uses the energy-Cassimir method. Here we also treat the axisymmetric case also via the reduced energy momentum.

In the axisymmetric case new results have also been obtained: Oblique equilibria have been explicitly solved for arbitrary large orbits (Section 9.4). This is a new family of relative equilibria that was not previously known. Numerical investigations like [**OT04**] suggested that there were no oblique equilibria for large orbits (Routhian reduction and numerical continuation were used as the main tools). [**Bec97**] suggested the existence of a bound for oblique orbits as in the non-axisymmetric case. The new oblique equilibria have a very similar analogue in the restricted problem, called the conical family. The stability analysis done here shows that the results are almost identical to those corresponding to conical equilibria for the restricted problem. Also the detailed transition between the three families has been described (Section 9.5).

The first chapters of the work are organised as follows: In Chapter 1 the basic concepts of symplectic geometry and geometric mechanics are reviewed. In Chapter 2 the basic aspects of stability for dynamical systems are revisited, focusing in those that can be later applied to the study of Hamiltonian relative equilibria. Chapter 3 introduces symmetric Hamiltonian systems and the concept of relative equilibria. The special case of relative equilibria for simple mechanical systems is explained. In Chapter 4 the notion of stability of relative equilibria is discussed and the reduced energy momentum method is briefly explained. Its use as a tool for finding bifurcations is also presented in the last section.

Chapter 5 includes two introductory examples, one of them is the well known classical problem of one particle in a Keplerian field. This example is not complicated but is the first step towards more elaborated models. The second example is also a particle but now moving in a plane with a corrected Keplerian field. The low dimensionality of the problem allows for some explicit computations that can explain the stability loss mechanism that will later be observed in the full second order model.

The second part has less theoretical nature and everything is oriented towards the concrete problem of the rigid body in a gravitational field. In Chapter 6 the geometric model of the problem is given and an adimensionalization procedure that can get rid of most of the physical parameters in the problem is explained. Following the classical approach, the expansion of the potential integral as a power series is done, providing the usual potential approximations found in the literature. The various continuous actions that will be relevant to the problem are introduced. The $SO(3)$ action is due to the rotational symmetry of the system. This is the basic symmetry to consider (Section 6.4). In the case of a potential which ignores the orientation of the body (as for example the order zero approximation) the symmetry can be enlarged to $SO(3)\times SO(3)$ (Section 6.5). Finally if the body has an axis of symmetry a $SO(3) \times S^1$ action has to be considered in order to get optimal stability and bifurcation results.

Chapter 7 introduces the first realistic model of motion of a rigid body, the truncation of the potential at the dominant term. All the results given are classical. Stability of the motion is proved when the correct symmetry group is used (Proposition 7.2). If stability is understood with a group of symmetry different from $SO(3) \times SO(3)$, as is done in [**WKM90**], a instability appears. We will discuss this instability result in comparison with the previous stability result (Section 7.4).

Chapter 8 introduces the first model in which orbital and attitude motion present a non-trivial coupling. The orthogonal relative equilibria are shown to have the same stability properties as classically stated (Proposition 8.4).

A previously unknown algebraic characterization of oblique equilibria is obtained in Section 8.1. The bifurcation mechanism between classical and oblique equilibria for small orbits is explained in the following section. The results are in accordance with the numerical study done in [**OT04**] for the axisymmetric case. A complete classification of the relative equilibria is also obtained.

Chapter 9 includes the extra structure of the axisymmetric body and its relationship with the symmetry group. The geometric formalism developed is similar to the one exposed in [**Bec97**]. The conditions for relative equilibria are algebraically solved, therefore all the relative equilibria are explicitly found.

Examples of oblique equilibria are found contradicting the hypothesis stated in [**Bec97**] or in [**OT04**]. The stability analysis is done in all the families of equilibria for large enough orbits. That is, for all orbits physically meaningful. The main stability results for the conical equilibria are represented in Figure 7.

The transition mechanism between three kinds of equilibria is described in Section 9.5. It has been checked that the behaviour observed is exactly the same as the predicted one using the restricted problem (see [**Bec97**]). The only difference is that the restricted problem can not predict the obliqueness of the conical orbits. This obliqueness, although present, is very small. In Section 9.4 a rapidly decreasing behaviour is observed (r^{-2}).

Although well known, the stability results for the cylindrical equilibria (Proposition 9.2) lead to think that self spinning is a way of stabilizing a satellite. This is the idea of the dual-spin satellites where an internal rotor can stabilize unstable motions. This approach is actually used in many modern satellites but will not be exploited here.

Chapter 10 develops topological techniques devised mainly in [**WMK91**] that produce existence results for the exact potential. The example of a asymmetric molecule without orthogonal relative equilibria is briefly explained.

In order to make this thesis self-contained, Appendix A includes a brief review of relative equilibria for the classical free rigid body.

In Appendix B the restricted problem is written in Hamiltonian form, and the non-axisymmetric case is worked out. The agreement with the results obtained in Chapter 8 is almost complete for large orbits. The calculation effort in this case is much lower than in Chapter 8. A similar development can be done in the axisymmetric case leading to stability conditions very similar to those computed in Chapter 9. This case can be found, in the Poisson framework, in [**Bec97**].

Appendix C collects some long algebraic expressions computed in the previous chapters that are important but, most of the times, an approximation is enough.

Chapter 1
Geometric Mechanics

We will review some of the basic concepts of classical mechanics from a geometric point of view, following, mainly, the notation in [**Mar92**].

1.1. Symplectic Manifolds

Definition 1.1. A symplectic manifold is (M, ω) where M is a smooth manifold and $\omega \in \Omega^2(M)$ satisfying:

- ω is a non-degenerate bilinear form.
- $d\omega = 0$

A basic example of a symplectic manifold is constructed from a vector space V of dimension n. Set $M = V \times V^*$ and $\omega((a, \alpha), (b, \beta)) = \langle \beta, a \rangle - \langle \alpha, b \rangle$ it can be checked that in coordinates the expression for ω is:

$$\begin{bmatrix} 0 & Id_n \\ -Id_n & 0 \end{bmatrix}$$

In fact, every symplectic manifold is locally isomorphic to one of this models (Darboux's theorem [**AM78**]).

The symplectic structure can associate to each function a vector field:

Definition 1.2. If $f \in C^\infty(M)$ then there is a unique $X_f \in \mathfrak{X}(M)$ defined by

$$i_{X_f}\omega = df \tag{1}$$

this relation is the coordinate-free formulation of Hamilton's equations.

The symplectic form can be used to assign to each pair of functions another one:

Definition 1.3. Given $f, g \in C^\infty(M)$ the *Poisson bracket* is the function $\{f, g\}$ defined by $\{f, g\}(x) = \omega(X_f, X_g)(x)$

The flow generated by the fields of the form X_f, which are called *Hamiltonian*, preserve the symplectic structure, that is:

Proposition 1.4. *The flow $F^t : M \to M$ of X_f is made of symplectomorphisms (i.e. $(F^t)^*\omega = \omega \quad \forall t$)*

Usually in mechanics the symplectic manifold is a cotangent bundle. That is, we are given a *configuration space* Q and from which the cotangent bundle T^*Q is constructed, T^*Q will be the *phase space*. Local coordinates q^i in Q induce coordinates (q^i, p_j) in T^*Q called the *canonical cotangent coordinates* of T^*Q.

Remark 1.5. Usually in this mechanical setting points in the configuration space Q will be represented by $q \in Q$, points in phase space will be denoted $z \in T^*Q$ or $p_q \in T^*Q$ meaning that p_q lies in the fibre of q.

On cotangent manifold there is always a canonical symplectic structure:

Proposition 1.6. *There is a unique 1-form θ on T^*Q such that in any choice of canonical coordinates:*

$$\theta = p_i dq^i,$$

it's called the canonical 1-form, the canonical 2-form is: $\omega = -d\theta = dq^i \wedge dp_i$.

From this coordinate expression it's easy to see that it's non-degenerate so the pair (T^*Q, ω) form a symplectic manifold.

In this canonical coordinates and with the canonical 2-form as symplectic form, the Hamilton's equations (1) take the form:

$$\dot{q}^i = \frac{\partial H}{\partial p_i}$$
$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$

which are the usual equations used in physics.

1.2. Lagrangian Mechanics

Let Q be a manifold and TQ it's tangent bundle. Coordinates q^i on Q induce coordinates $(q^i, \dot{q}^i)$ on TQ, called *tangent coordinates.* A mapping $L : TQ \to \mathbb{R}$ is called a *Lagrangian.*

Typically in mechanics we choose L to be $L = K - V$ where K is the kinetic energy induced by some Riemannian structure on Q, that is $K(v) = \frac{1}{2}\langle\langle v, v\rangle\rangle$, and where $V : Q \to R$ is the *potential energy.*

The equations of motion of the system are determined from this data and:

Definition 1.7. The *principle of critical action* singles out particular curves $q(t)$ by the condition:

$$\delta \int_a^b L(q(t), \dot{q}(t))dt = 0 \tag{2}$$

where the variation is taken in the space of over smooth curves in Q with fixed endpoints.

Remark 1.8. Note that different Lagrangians can lead to the same equations, for example if we take $\tilde{L} = L + \lambda$ where $\lambda \in \mathbb{R}$. Obviously (2) is the same condition for $\tilde{L}$ and for L. More generally it can be shown that if L is replaced by $L + \frac{d}{dt}S(q,t)$ the equations doesn't change.

It's an elementary result of the calculus of variations that (2) is equivalent to the conditions:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = \frac{\partial L}{\partial q^i} \tag{3}$$

In the cases we will deal here both the Lagrangian and Hamiltonian approach are related through the *Legendre transformation.* The Legendre transformation here is the derivative of L in the fibre direction, leading to a map:

$$\mathbb{F}L : TQ \to T^*Q$$

in coordinates it's nothing more than the relation between velocities and momenta: $p_j = \frac{\partial L}{\partial \dot{q}^j}$. If $\mathbb{F}L$ defines a diffeomorphism between the tangent and cotangent bundle the Lagrangian is called *hyperregular*.

If L is hyperregular then the associated Hamiltonian is the function $H : T^*Q \to \mathbb{R}$, defined by:

$$H(q^i, p_j) = (\mathbb{F}L)_* \big(((\mathbb{F}L)^* p_i) q^i - L\big) \tag{4}$$

that is: $H = p_i \dot{q}^i - L$ expressed in position-momenta coordinates.

It can be check that in the case of hyperregular Lagrangian the equations (3) for L are equivalent to (1).

Chapter 2
Stability

One of the basic questions in the study of dynamics is the stability of an equilibrium point. Specifically, the question is whether a slight perturbation of a dynamical system from an equilibrium state will produce a motion confined to the neighbourhood of the equilibrium point or a motion tending to leave that neighbourhood. But this simple statement can lead to a large number of different concepts of stability that are going to be outlined here.

The main references for this section is [**Mei70**] and the review done in [**BH98**].

2.1. Stabilities of dynamical systems

Assume that we are given a vector field X on a manifold M with flow F^t and that $x_e \in M$ is an equilibrium point of the flow (i.e. $X(x_e) = 0$ or $F^t(x_e) = x_e \quad \forall t$)

Definition 2.1. The equilibrium point x_e is *stable in the sense of Lyapunov* if for any given neighbourhood U of x_e there is another neighbourhood $V \subset U$ of x_e such that $F^t(V) \subset U \quad \forall t > 0$.

If in addition M is a metric space with distance d, this condition can be restated in the following terms:

For any given $\varepsilon > 0$ there exists $\delta > 0$ such that if $d(x, x_e) < \delta$ then $d(F^t(x), x_e) < \varepsilon \quad \forall t > 0$.

An equilibrium point which is not stable is called unstable.

This one is the most general type of stability, we will often refer to it simply as *"stability"* or *"non-linear stability"*. A more restrictive stronger kind of stability is the asymptotic stability, assuming X complete:

Definition 2.2. The equilibrium point x_e is *asymptotically stable* if it is Lyapunov stable, and in addition, there exists a neighbourhood V of x_e such that if $x \in V$ then $x(t)$ converges to x_e as $t \to \infty$.

The asymptotic stability guarantees convergence to the equilibrium point but it does not control the speed of convergence, another related notion (for dynamics in a metric space) is:

Definition 2.3. The equilibrium point x_e is *exponentially stable* if there exists a neighbourhood V of x_e, and constants $m, \alpha > 0$ such that if $x \in V$:

$$d(F^t(x), x_e) < me^{-\alpha t} d(x, x_e)$$

that is, all the solutions nearby go to the equilibrium point exponentially fast.

2.2. Linearisation of a field near a equilibrium

If we have a dynamical system defined by a vector field X defined in some manifold M which has a equilibrium point at $x_e \in M$, i.e. $X(x_e) = 0$, then we can define the linearisation $dX(x_e)$ (following the notation of [**Pat95**]) as the linear map from $T_{x_e}M$ to itself satisfying:

$$dX(x_e)V_{x_e} = [V, X](x_e)$$

where V is a vector field in M such that $V(x_e) = V_{x_e}$, expanding the Lie bracket it's easy to see that it is independent of the extension chosen for V_{x_e} because $X(x_e) = 0$ ($[V, X]f(x_e) = V(X(f))(x_e) - X(V(f))(x_e) = V(X(f))(x_e) \quad \forall f$).

Thus from a dynamical system given by the data (M, X) and the equilibrium point x_e we have constructed another dynamical system given by $(T_{x_e}M, dX(x_e))$. This new system is, after taking coordinates, a linear system with constant coefficients in a vector space with 0 as an equilibrium point (and maybe more). This simpler system gives, in some sense, a local model of the behaviour of the original system near the equilibrium point.

Remark 2.4. A more geometric interpretation of the linearisation process is through the complete lift that can be applied not only to fixed point equilibria and can linearise the system about a trajectory [**BL05**].

Recall that the system of ordinary differential equations in $\mathbb{R}^n$:

$$\dot{x} = Ax, \quad x(t) = x_0$$

with constant coefficient matrix $A \in \mathbb{R}^{n^2}$ has as solution $x(t) = \exp(At)x_0$.

It is well known (for example [**Arn78**]) that if the matrix A has different eigenvalues and all of them have non-positive real part then 0 is a stable point for the system $\dot{x} = Ax$. If it has one eigenvalue with positive real part then 0 is a unstable equilibrium.

This notions lead to the following concepts:

Definition 2.5. The equilibrium point x_e of a dynamical system (M, X) is *spectrally stable* if the linearisation $dX(x_e)$ has all the eigenvalues all with non-positive real part.

It's *spectrally unstable* if the linearisation has, at least, one eigenvalue with positive real part.

It's *strongly spectrally unstable* if the linearisation has all the eigenvalues different and with negative real part.

Definition 2.6. The equilibrium point x_e of a dynamical system (M, X) is *linearly stable* if the linearisation $dX(x_e)$ has the origin as a stable point. It is called *linearly unstable* if the linearisation has the origin as a unstable point. It is called *strongly linearly stable* if the linearisation has the origin as asymptotically stable point.

The spectral stability it's simpler to check, one only has to take care of the eigenvalues. But in the case there are repeated eigenvalues one has to check the Jordan decomposition of the matrix, this leads to the linear stability concept.

There is a relationship between the different notions of equilibria:

$$\text{Stability} \implies \text{Linear Stability} \implies \text{Spectral Stability}$$

There is a similar chain of implications:

$$\text{Stability} \Longleftarrow \text{Asymptotic Stability} \Longleftarrow \text{Strong Linear Stability} \Longleftarrow \text{Strong Spectral Stability}$$

In the Hamiltonian setting Liouville's theorem gives conservation of the volume, in particular, fixed points will never be strongly spectrally stable.

2.3. Linearisation of Hamiltonian fields

In the Hamiltonian case, a equilibrium point is $x_e \in M$ such that $dH(x_e) = 0$, in this case the Hessian at x_e, $d^2H(x_e)$ is a well defined bilinear form defined by

$$d^2H(x_e)(V_1(x_e), V_2(x_e)) = V_1(V_2(H))(x_e)$$

in this case it can be shown [**Pat95**] that:

$$\omega(x_e)(dX_H(x_e)v_1, v_2) = d^2H(x_e)(v_1, v_2) \tag{5}$$

for all $v_1, v_2 \in T_{x_e}M$. So $dX_H(x_e)$ is the linear Hamiltonian for the quadratic Hamiltonian $v \in T_{x_e}M \mapsto \frac{1}{2}d^2H(x_e)(v,v)$ on the linear symplectic space $(T_{x_e}, \omega(x_e))$.

Remark 2.7. In the Hamiltonian setting, the linearised field near an equilibrium is an infinitesimally symplectic map, and the infinitesimally symplectic eigenvalue theorem [**AM78**] states that the eigenvalues come in pairs $\{\lambda_i, -\lambda_i\}$. So, a linearised Hamiltonian system will never be strongly linearly stable. The linearisation can *only be used to prove instability* in the Hamiltonian case.

Proposition 2.8. *Let (V, ω) is a linear symplectic space equipped with a quadratic Hamiltonian $v \mapsto A(v,v)$ defined by a symmetric bilinear linear operator A. If the number of negative eigenvalues is odd and A is invertible then the equilibrium 0 is unstable.*

PROOF. Using Darboux's theorem we can take basis in the vector space $V \cong W \oplus W^*$ that take the symplectic form to the usual matrix expression. In this basis the linear endomorphism dX_A is given by (see (5)):

$$dX_A = \mathbb{J}^{-1}A = \begin{bmatrix} 0 & Id_n \\ -Id_n & 0 \end{bmatrix}^{-1} A$$

Note that $\det \mathbb{J} = 1$. In the hypothesis of the proposition $\det A < 0$, and applying this last expression also $\det dX_A < 0$ therefore the linearisation has and odd number of eigenvalues with negative real part, and being a infinitesimally symplectic map it also has an odd number of eigenvalues with positive real part. That is, there is at least one eigenvalue that implies spectral instability of the system. $\square$

Remark 2.9. This result implies that we can conclude, in some cases, linear instability computing only the eigenvalues of the Hessian of the Hamiltonian at a relative equilibrium.

2.4. Lyapunov functions

As we will see, in many cases the study of the linearised system is not enough to conclude stability, for this cases there is another method, due to Lyapunov, which is based on the following theorem:

Theorem 2.10. *If we have a dynamical system (M, X) and an equilibrium point $x_e \in M$ and there exists a function V (called a* Lyapunov function*) satisfying:*

- $V(x) \geq 0$ *with equality only if* $x = x_e$
- $\frac{d}{dt}V(\gamma(t)) \leq 0$ *for all the integral curves γ of the field X. (i.e. $X(V) \leq 0$)*

then the equilibrium point x_e is stable.

If we require negative-definiteness of the derivative of V then the equilibrium point is not only stable but asymptotically stable.

The proof of the theorem can be found in may texts, for example [**Mei70**]. The geometric idea behind is very simple: a positive definite function must be locally like a "cup", the non-positive time derivative forces the point to decrease, or at least not increase the value of the function avoiding the escape from a neighbourhood of x_e (see Figure 1).

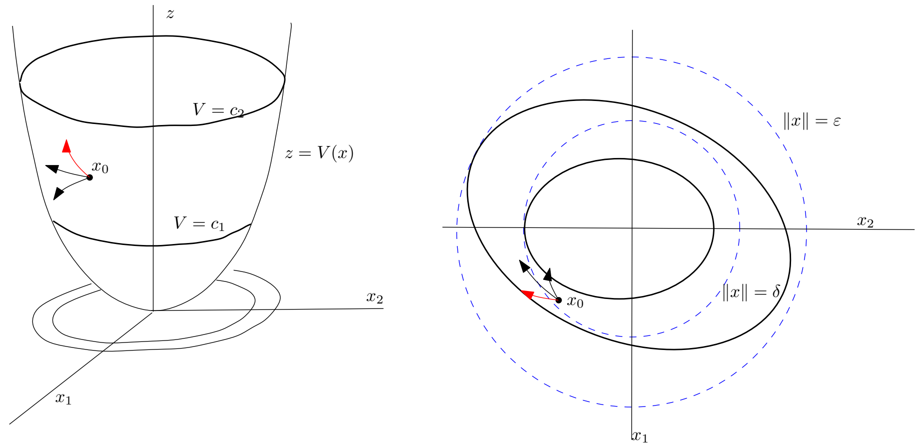


FIG. 1. Geometric interpretation of Lyapunov function [**Mei70**]

The Lyapunov functions can be used to determine stability of Hamiltonian systems, one simple candidate for a Lyapunov function is the Hamiltonian H itself if it is positive definite near the equilibrium z_e. This gives the classical result:

Theorem 2.11 (Dirichlet)**.** *Let (M, ω) be a symplectic manifold, and $H \in \mathcal{C}^\infty(M)$ a Hamiltonian. If $z_e \in M$ is a equilibrium point and the Hessian $d^2H(z_e)$ is positive definite then the equilibrium is stable.*

2.5. Simple Examples

Consider the non-linear differential equation in $\mathbb{R}^2$ defined as:

$$\begin{aligned} \dot{x} &= y + x^3 - y^2 + xy \\ \dot{y} &= \cos(x) - 1 \end{aligned}$$

we want to study the behaviour of the fixed point $(x, y) = (0, 0)$. As a first approximation we can linearise it and it gives:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= 0 \end{aligned}$$

The associated endomorphism of $T_{x_e}\mathbb{R}^2$ is $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. It's already in Jordan form so:

- It is spectrally stable, both eigenvalues have real part 0.
- It is linearly unstable, because the non-trivial Jordan block of eigenvalue 0 gives rise to the growth of solutions therefore instability.

- Because of the linear instability we can conclude non-linear instability.

Consider now the system, with equilibrium point $x_e = (0, 0)$:

$$\dot{q} = p$$
$$\dot{p} = -q$$

it's already in linear form, the associated operator is $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ which has i and $-i$ as eigenvalues. We have spectral and linear stability but we can not conclude anything about non-linear stability.

But if we set $V = p^2 + q^2$ we can check that this function is a Lyapunov function:

- It is positive definite and $V(x) = 0 \implies x = x_e$
- The time derivative is non-positive: $\dot{V} = X(V) = 2pq - 2qp = 0 \leq 0$ because $X = p\frac{\partial}{\partial q} - q\frac{\partial}{\partial p}$

so the system is non-linearly stable.

Remark 2.12. This system was, in fact, Hamiltonian with $H = \frac{1}{2}(p^2 + q^2)$. Physically it corresponds to a non-dimensionalization of the harmonic oscillator system. It models the movement of a mass attached with a spring. The equilibrium can also be shown to be stable using the Dirichlet criterion 2.11

Chapter 3
Symmetric systems

We will try to exploit the symmetry that some mechanical systems have in order to make them more tractable. So we will need to study the structure associated with this symmetry. This section follows the treatment of [**Mar92**].

3.1. Lie groups and actions

Definition 3.1. A left-group action of G over a set X is a mapping

$$\varphi : G \times X \to X$$

such that:

- $\varphi(g, \varphi(h, x)) = \varphi(gh, x) \quad \forall x \in X$
- $\varphi(e, x) = x \quad \forall x \in X$ if e denotes the identity element of G.

Fixing a group element leads to a bijective map from X to itself denoted by $\varphi_g : X \to X$. Sometimes the action will be denoted simply by left multiplication by the group: $\varphi(g, x) = g \cdot x$.

If G is a Lie group and M is a smooth manifold a smooth left-action is a group action $\varphi : G \times M \to M$ which is a left-G action and in addition it is a smooth map. We will always work in this situation so we will always assume smoothness of the actions.

Given an element $p \in M$ the stabilizer of p is

$$G_p = \{g \in G | g \cdot p = p\},$$

the action on M will be free if G_p is trivial for all $p \in M$.

We will identify the Lie algebra $\mathfrak{g}$ of the group G with the tangent space at the identity, recall from Lie group theory (for example [**AM78**]) that there exists a exponential mapping

$$\exp : \mathfrak{g} \to G$$

which is a local diffeomorphism near 0.

Definition 3.2. Given and action G on M for each $\xi \in M$ there is a vector field, called *the fundamental vector field of* ξ, $\xi_M \in \mathfrak{X}(M)$ defined as:

$$\xi_M(p) = \frac{d}{d\varepsilon} \exp(\varepsilon \xi) \cdot p \Big|_{\varepsilon=0}$$

3.2. Examples

A Lie group can act on itself in several ways:

- Left action:
$$\begin{aligned} L : G \times G &\to G \\ (g, p) &\mapsto gp \end{aligned}$$
- Right action:
$$\begin{aligned} \tilde{R} : G \times G &\to G \\ (g, p) &\mapsto pg^{-1} \end{aligned}$$
(this is indeed a left action $\tilde{R}_g \circ \tilde{R}_h = \tilde{R}_{gh}$)
- Inner automorphisms (or conjugation)
$$\begin{aligned} \mathrm{Conj} : G \times G &\to G \\ (g, p) &\mapsto gpg^{-1} \end{aligned}$$

The tangent lifts of the left and right trivializations lead to the left and right trivializations of the tangent bundle of a Lie group:

$$\begin{aligned} TL :&TG \xrightarrow{\sim} G \times \mathfrak{g} \\ T\tilde{R} :&TG \xrightarrow{\sim} G \times \mathfrak{g} \end{aligned}$$

The conjugation action as it leaves the identity invariant gives the adjoint action restricting it to T_eG:

$$\begin{aligned} \mathrm{Ad} :&G \times \mathfrak{g} \to \mathfrak{g} \\ &(g, \xi) \mapsto \frac{d}{d\varepsilon} g \exp(\varepsilon\xi) g^{-1}\Big|_{\varepsilon=0} = T_e\mathrm{Conj}_g \xi \end{aligned} \tag{6}$$

by duality the coadjoint action is:

$$\begin{aligned} \mathrm{Ad}^* :&G \times \mathfrak{g}^* \to \mathfrak{g}^* \\ &(g, \mu) \mapsto T_e^*\mathrm{Conj}_g \mu \end{aligned} \tag{7}$$

So in $\mathfrak{g}$ we have a natural left-G action given by the adjoint action and in $\mathfrak{g}^*$ we have a right-G-action. Given $\xi \in \mathfrak{g}$ and $\mu \in \mathfrak{g}^*$, G_ξ and G_μ will refer to the stabilizers of the elements under this actions.

3.3. Reduced dynamics

Suppose we are given a manifold M with an action of a Lie group on it and a vector field $X \in \mathfrak{X}(M)$, G-invariant. It seems reasonable that under some hypothesis the dynamics can be "reduced" using the symmetry given.

Associated with the field X we have its flow F^t, G-invariance of X (i.e. $\varphi_g^* X = X, \quad \forall g \in G$) is the same as requiring equivariance of the flow (i.e. $F^t(g \cdot x) = g \cdot F^t(x), \quad \forall g \in G$)

Definition 3.3. An integral curve $F^t(x_0)$ of X is called a relative equilibria if

$$F^t(x_0) = \exp(t\xi) \cdot x_0$$

for some $\xi \in \mathfrak{g}$.

This means that the integral curve is also the orbit of the one-parameter subgroup generated by $\xi \in \mathfrak{g}$.

If the action is free and G is compact, then the orbit space M/G can be given a smooth manifold structure, and the projection $\pi : M \to M/G$ is indeed a principal fibration with fibre G. Elements of M/G are denoted by $[x]$ where $x \in M$, that is $\pi(x) = [x]$.

By invariance of X and equivariance of F^t:

- X is π-projectable, that is there exists $\overline{X} \in \mathfrak{X}(M/G)$ such that
$$\overline{X} = \pi_* X$$
- The flow F^t is π-projectable and its projection is the flow $\overline{F}^t$ of $\overline{X}$:
$$\pi(F^t(x)) = \overline{F}^t([x])$$

The dynamics on M/G defined by $\overline{X}$ is called the *reduced dynamics* of X.

Remark 3.4. If $x \in M$ is a relative equilibria of X then $[x] \in M/G$ is a fixed point of the field $\overline{X}$.

3.4. Hamiltonian G-spaces

Definition 3.5. An action $\varphi : G \times M \to M$, where (M, ω) is a symplectic manifold, is called a *Hamiltonian action* if

(1) G acts by symplectomorphisms, i.e. $\varphi_g^* \omega = \omega \quad \forall g \in G$
(2) $\exists \mathbf{J} : M \to \mathfrak{g}^*$ called a *momentum map* such that:
$$i_{\xi_M}\omega = d\langle \mathbf{J}(\cdot), \xi \rangle \quad \forall \xi \in \mathfrak{g}$$
(3) $\mathbf{J}$ is Ad^*-equivariant:
$$\mathbf{J}(g \cdot x) = Ad^*_{g^{-1}}(\mathbf{J}(x))$$

The tuple $(M, \omega, \mathbf{J}, G)$ will be called a *Hamiltonian G-space.*

The action of G in M gives rise to the fundamentals fields of the action, if this fields are Hamiltonian, there should be a function $J : M \times \mathfrak{g} \to \mathbb{R}$ such that $\xi_M = X_{J(\cdot,\xi)}$ for all $\xi \in \mathfrak{g}$, as this function has to be linear in ξ, this induces the map $\mathbf{J} : M \to \mathfrak{g}^*$. The existence of the momentum map (not necessarily equivariant) is the requirement that the group action is not preserves the symplectic structure but also given by a Hamiltonian vector field.

The momentum map is a key element of the theory, it provides conserved quantities for all the G-invariant Hamiltonians, in the following sense:

Theorem 3.6 (Noether's Theorem)**.** *If H is a G-invariant Hamiltonian on the G-space $(M, \omega, \mathbf{J}, G)$, then $\mathbf{J}$ is conserved on the trajectories of the Hamiltonian vector field X_H associated with H.*

The existence and equivariance of momentum maps for a given action is not a trivial problem, but for cotangent bundles and lifted actions it's easy.

If G acts on Q through $\varphi : G \times Q \to Q$ on the left, then taking, for each fixed g the transpose inverse of the tangent lift we get: $T^*\varphi_{g^{-1}} : T^*Q \to T^*Q$ which fit together to give a action of G on T^*Q on the left, this is called *cotangent lifted action.* It can be check that this action preserves the symplectic structure. Moreover there exists a momentum map given by:
$$\langle \mathbf{J}, \xi \rangle (p_q) = \langle p_q, \xi_Q(q) \rangle$$

thus if we are given and action of G on Q we have and associated Hamiltonian G-space given by $(T^*Q, \omega, \mathbf{J}, G)$.

More details and proofs can be found in [**Mar92**]

Remark 3.7. Although for some results is not really necessary we will always assume from here on that the *action of the group G is free and G is a compact group.* This conditions are essential for many stability results and for the energy-momentum method, at least in the version stated here. There is a version of the reduced energy-momentum method that doesn't require the action to be free [**RO06**] but it is not going to be needed here.

A *simple mechanical G-system* will be a Hamiltonian G-system with phase space T^*Q where the base manifold Q is assumed to have a Riemannian metric $\langle\langle , \rangle\rangle$ such that the action of G is by isometries, the Hamiltonian is of the form

$$H(q,p) = \frac{1}{2}\|p\|_q^2 + V(q) \tag{8}$$

where $\|\cdot\|$ is the norm induced on T_q^*Q, and where V is a G-invariant function.

Remark 3.8. The concept of simple mechanical G-system corresponds to the Hamiltonian formulation of a physical problem in which both the kinetic energy and the potential energy are invariant by the group action. The Lagrangian is as always $L = T - V$, the non-degeneracy of the kinetic energy ensures that the Lagrangian is hyperregular and the Legendre transform (4) gives (8)

Remark 3.9. The map $\mathbb{F}L : TQ \to T^*Q$ in this case is simply the metric tensor regarded as a map from vectors to covectors, in coordinates if g_{ij} are the coordinates of the metric tensor $p_i = g_{ij}v^j$.

Remark 3.10. Given Q with a Riemannian metric $\langle\langle , \rangle\rangle$, the induced norm on T_qQ will be represented as $|v| = \sqrt{\langle\langle v, v\rangle\rangle}$ whereas the induced norm on T_q^*Q will be denoted as $\|p\|$.

3.5. Example

Consider N particles interacting among themselves in $\mathbb{R}^3$ through conservative forces depending on the distance. For this problem, the configuration space will be $Q = (\mathbb{R}^3)^N$ with euclidean coordinates $(q^j)_i$, $1 \le i \le N, 1 \le j \le 3$. The kinetic energy will be (in vector notation):

$$K = \frac{1}{2}m_1\dot{\mathbf{q}}^1 \cdot \dot{\mathbf{q}}^1 + \cdots + \frac{1}{2}m_N\dot{\mathbf{q}}^N \cdot \dot{\mathbf{q}}^N$$

and the potential energy will depend only on variables of the form $|\mathbf{q}_i - \mathbf{q}_j|$.

Consider the action of the group $G = \mathbb{R}^3$ on Q via translations (i.e. $\mathbf{q}_i \mapsto \mathbf{q}_i + \mathbf{a}$), clearly this action leaves both the kinetic energy and the potential invariant, this is an example of a mechanical G-system. The fundamental field of the action has coordinates $\dot{\mathbf{q}}_i = \boldsymbol{\xi}$, so the momentum map is:

$$\langle \mathbf{J}, \xi\rangle(p_q) = \langle p, \xi_Q(q)\rangle = \xi \cdot (\mathbf{p}_1 + \cdots + \mathbf{p}_N)$$

thus the momentum map is nothing more than the linear momentum of the system, Theorem 3.6 states that the linear momentum is conserved for this system.

3.6. Locked inertia tensor and mechanical connection

For each $q \in Q$ the action of G can map $\mathfrak{g}$ in some subspace of T_qQ, through the map $\xi \mapsto \xi_Q(q)$. So the metric on q induces a bilinear form on $\mathfrak{g}$:

Definition 3.11. For each $q \in Q$, let the *locked inertia tensor* be the map $\mathbb{I}(q) : \mathfrak{g} \to \mathfrak{g}^*$ defined by:

$$\langle \mathbb{I}(q)\eta, \zeta \rangle = \langle\!\langle \eta_Q(q), \zeta_Q(q) \rangle\!\rangle \tag{9}$$

If the action is free this map is invertible an it defines a metric on $\mathfrak{g}$ for each $q \in Q$. In a similar way we can define a map $\alpha : TQ \to \mathfrak{g}$ which assigns to each point of TQ the corresponding angular velocity of the locked system:

$$\alpha(q, v) = \mathbb{I}^{-1}(q)(\mathbf{J}(\mathbb{F}L(q, v)))$$

This map α is named *mechanical connection* because it can be thought as a connection on the principal bundle $Q \to Q/G$ ([**Mar92**]).

Contracting this $\mathfrak{g}$-valued form with an element of $\mathfrak{g}^*$ gives a usual 1-form on Q that will be denoted by $\alpha_\mu : TQ \to \mathbb{R}$:

$$\langle \alpha_\mu(q), v \rangle = \langle \mu, \alpha(q, v) \rangle$$

3.7. Relative equilibria

Let $(M, \omega, G, \mathbf{J})$ be a Hamiltonian G-space with Hamiltonian H.

Definition 3.12. A point $z_e \in M$ is called a *relative equilibrium* if

$$X_H(z_e) \in T_{z_e}(G \cdot z_e)$$

Remark 3.13. This definition may seem more general that the Definition 3.3 given for general vector fields, but the next result will prove its equivalence.

The basic tool for studying relative equilibria is:

Definition 3.14. Given a velocity $\xi \in \mathfrak{g}$ the *augmented Hamiltonian* is the function:

$$H_\xi(z) = H(z) - \langle \mathbf{J}(z) - \mu, \xi \rangle$$

Theorem 3.15. *(*[**Mar92**] *Theorem 4.1) Let $z_e \in M$ and let $z_e(t)$ be the dynamic orbit of X_H with $z_e(0) = z_e$ and let $\mu = \mathbf{J}(z_e)$, a regular point of $\mathbf{J}$. The following assertions are equivalent:*

(1) *z_e is a relative equilibrium*
(2) *$\exists \xi \in \mathfrak{g}$ such that z_e is a critical point of the augmented Hamiltonian:*
(3) *$\exists \xi \in \mathfrak{g}$ such that $z_e(t) = \exp(t\xi) \cdot z_e$*
(4) *$z_e(t) \in G_\mu \cdot z_e \subset G \cdot z_e$*
(5) *z_e is a critical point of the energy-momentum map: $H \times \mathbf{J} : M \to \mathbb{R} \times \mathfrak{g}^*$*
(6) *z_e is a critical point of H restricted to $\mathbf{J}^{-1}(\mu)$*
(7) *z_e is a critical point of H restricted to $\mathbf{J}^{-1}(\mathcal{O})$ where $\mathcal{O} = G \cdot \mu \subset \mathfrak{g}^*$*
(8) *$[z_e] \in M_\mu$ is a critical point of the reduced Hamiltonian H_μ.*

PROOF. We will start showing that $1 \implies 2 \implies 3 \implies 4 \implies 1$.

If 1 is assumed, $X_H(z_e) = \xi_M(z_e)$ for some $\xi \in \mathfrak{g}$, but, by definition of the momentum map, this gives $X_H(z_e) = H_{\langle J,\xi \rangle}(z_e)$ and thus $X_{H - \langle J,\xi \rangle}(z_e) = 0$ this implies that $H - \langle J, \xi \rangle$ has a critical point at z_e which is the assertion of 2.

Assume 2 now, if φ_t is the flow of X_H and ψ_t^ξ the flow of $X_{\langle J,\xi \rangle}$ so $\psi_t^\xi(z) = \exp(t\xi) \cdot z$. Since H is G-invariant this means that the flows φ_t and ψ_t^ξ commute, so that the flow of the augmented Hamiltonian is $\varphi_t \circ \psi_{-t}^\xi$ as the augmented Hamiltonian has a critical point: $\varphi_t(\exp(-t\xi) \cdot z_e) = z_e$ for all $t \in \mathbb{R}$. Thus, $\varphi_t(z) = \exp(t\xi) \cdot z_e$, which is 3.

Condition 3 shows that $z_e(t) \in G \cdot z_e$; but $z_e(t) \in \mathbf{J}^{-1}(\mu)$ and $G \cdot z_e \cap \mathbf{J}^{-1}(\mu) = G_\mu \cdot z_e$ by equivariance so 3 implies 4. 1 is only the tangent version of 4, so 4 implies 1.

5 and 6 are equivalent because of the Lagrange multipliers theorem. The condition for critical points of the restricted problem 6 is that $dH = \lambda \cdot d\mathbf{J}$ which is the same as requiring that $H \times \mathbf{J}$ has a critical point if μ is a regular value of $\mathbf{J}$.

Now we will show that $2 \implies 5, 6 \implies 7 \implies 8 \implies 4$.

If 2 holds then 6 also holds because $H = H_\xi$ if both are restricted to $\mathbf{J}^{-1}(\mu)$ (the correction term vanishes there).

By G-equivariance of $\mathbf{J}$ and as H is G-invariant 6 implies 7.

6 implies 8 by G-invariance and passing to the quotient.

If 8 is assumed the dynamic orbit $z_e(t)$ projects to a fixed point, that is $z_e(t) \in G_\mu \cdot z_e$ which is 4. $\square$

ξ will be the group-velocity of the steady-motion, or relative equilibria starting at z_e. $\xi(z)$ will be the angular velocity of the relative equilibria starting at z, indicating the dependence on z.

Proposition 3.16. *(*[**Mar92**] *Proposition 4.1) Let z_e be a relative equilibrium. Then*

- *so is $g \cdot z_e$ for any $g \in G$ with $\xi(g \cdot z_e) = Ad_g(\xi(z_e))$*
- *$Ad^*_{\exp t\xi}\mu = \mu$, i.e. $ad^*_\xi \mu = 0$*

Proof. As we have $H_{Ad_g\xi}(gz) = H_\xi(z)$ by equivariance of the momentum map if (z_e, ξ) is a relative equilibria, so is $(g \cdot z_e, Ad_g(z_e))$.

For the second assertion:

$$z_e(t) = \exp(t\xi) \cdot z_e \in \mathbf{J}^{-1}(\mu) \cap G \cdot z_e = G_\mu \cdot z_e$$

therefore $\exp(t\xi) \in G_\mu$ and $\xi \in \mathfrak{g}_\mu$. $\square$

3.8. Relative equilibria for simple mechanical systems

In the case of a simple mechanical system, the expression of the augmented potential can be simplified.

Recall that the phase space is the cotangent bundle T^*Q of the configuration space Q, the Hamiltonian is the sum of a potential energy and a kinetic term $H = K + V$. In this setting:

$$H_\xi(z) = K_\xi(z) + V_\xi(q) + \langle \mu, \xi \rangle \tag{10}$$

where $z = (q, p) \in T^*Q$,

$$K_\xi = \frac{1}{2}\|p - \mathbb{F}L\xi_Q(q)\|^2$$

and

$$V_\xi(q) = V(q) - \frac{1}{2}\langle \xi, \mathbb{I}(q)\xi \rangle$$

Because:

$$\begin{aligned}
&\frac{1}{2}\|p-\mathbb{F}L\xi_Q(q)\|^2+V(q)-\frac{1}{2}\langle\xi,\mathbb{I}(q)\xi\rangle+\langle\mu,\xi\rangle=\\
&=\frac{1}{2}\|p\|^2-\langle\langle p,\mathbb{F}L\xi_Q(q)\rangle\rangle+\frac{1}{2}\|\mathbb{F}L\xi_Q(q)\|^2+V(q)-\frac{1}{2}\langle\langle\xi_Q(q),\xi_Q(q)\rangle\rangle+\langle\mu,\xi\rangle\\
&=\frac{1}{2}\|p\|^2-\langle p,\xi_Q(q)\rangle+\frac{1}{2}\langle\langle\xi_Q(q),\xi_Q(q)\rangle\rangle+V(q)-\frac{1}{2}\langle\langle\xi_Q(q),\xi_Q(q)\rangle\rangle+\langle\mu,\xi\rangle\\
&=H(q,p)-\langle\mathbf{J}(q,p)-\mu,\xi\rangle=H_\xi(z)
\end{aligned}$$

The expression (10) for the augmented potential as a sum of one part only depending on q and the other one of quadratic character provides:

Proposition 3.17. *A point $z_e=(q,p)$ is a relative equilibrium if and only if there is a $\xi\in\mathfrak{g}$ such that:*

- *q_e is a critical point of V_ξ*
- *$p_e=\mathbb{F}L(\xi_Q(q_e))$*

This proposition makes the job of finding relative equilibria easier because now we have to look for critical points of V_ξ (which is a function in Q) instead of working with H_ξ in $M=T^*Q$.

Remark 3.18. This proposition is also true if the action of G in Q is not free. We will use it to locate the relative equilibria even if the action is not free. After finding a relative equilibrium one can check if that point has isotropy or not. If the action is trivial near that point all the following results could be applied near that point.

The augmented Hamiltonian also can be seen as the Hamiltonian from a "moving-frame" with velocity $\xi\in\mathfrak{g}$. A point $q\in Q$ for each t will be the point $\overline{q}=\exp(-t\xi)q$ in the moving frame, the coordinates of a fixed point in Q as seen from a moving-frame are time-dependent.

The Lagrangian in fixed coordinates is:

$$L=\frac{1}{2}|\dot{q}|^2-V(q)$$

can be written in time-dependent coordinates as:

$$L=\frac{1}{2}|\exp(t\xi)\dot{\overline{q}}-\xi_Q(\exp(t\xi)\overline{q})|^2-V(\exp(t\xi)\overline{q})=\frac{1}{2}|\exp(t\xi)(\dot{\overline{q}}-\xi_Q(\overline{q}))|^2-V(\exp(t\xi)\overline{q})$$

where $\exp(t\xi)$ also means the tangent lifted action. Now if the metric and the potential are G invariant:

$$L=\frac{1}{2}|\dot{\overline{q}}-\xi_Q(\overline{q})|^2-V(\overline{q})$$

the Lagrangian for the time dependent variables becomes time independent.

The associated Hamiltonian will be, after using (4):

$$H=\frac{1}{2}\|\overline{p}-\mathbb{F}L(\xi_Q(\overline{q}))\|^2+V(\overline{q})-\frac{1}{2}\|\xi_Q(\overline{q})\|^2$$

where $\overline{p}=\exp(-t\xi)p$ through the cotangent lifted action.

This expression is the same as (10) thus the augmented Hamiltonian gives the evolution of the system as seen from a moving-frame.

3.9. Poisson and symplectic reduction

We have seen that if we have a G-invariant vector field it projects to a vector field on the reduced space M/G and relative equilibria correspond to fixed points.

If the fields are Hamiltonian, this extra structure also is present in the reduced dynamics but in a more subtle way:

Theorem 3.19. *Let $(M, \omega, G, \mathbf{J})$ be a Hamiltonian G-space. And let $\{,\}$ be the associated Poisson bracket. Then we have:*

- Poisson reduction *The orbit space is a Poisson manifold endowed with the reduced Poisson bracket $\{,\}_{red}$ defined by:*
$$\{f, g\}_{red}([x]) = \{\pi^* f, \pi^* g\}(x)$$
where $f, g \in C^\infty(M/G)$ and $\pi : M \to M/G$.
- Symplectic reduction *Let $\mu \in \mathfrak{g}^*$ then $\mathbf{J}^{-1}(\mu)$ is G_μ-invariant submanifold of M, and the orbit space $\mathbf{J}^{-1}(\mu)/G_\mu$ is a symplectic manifold with a reduced symplectic form defined by:*
$$\pi_\mu^* \omega_\mu = \iota_\mu^* \omega$$
where $\iota_\mu : \mathbf{J}^{-1}(\mu) \to M$ and $\pi_\mu : \mathbf{J}^{-1}(\mu) \to \mathbf{J}^{-1}(\mu)/G_\mu$.

There is a natural inclusion of $\mathbf{J}^{-1}(\mu)/G_\mu$ in M/G, with respect to this inclusion the first space is a symplectic leaf of the Poisson manifold $(M/G, \{,\}_{red})$.

If H is a G-invariant function on M it gives a well defined function in the orbit space M/G called h. In this way the Hamiltonian vector field X_H on M descends to a vector field on M/G given by h and the Poisson tensor, in the sense:
$$\overline{X}_H = \{\cdot, h\}_{red}$$

From the symplectic point of view, the restriction of H to $\mathbf{J}^{-1}(\mu)$ is G_μ invariant so it gives a well defined function h^μ on $\mathbf{J}^{-1}(\mu)/G_\mu$. Also by Theorem 3.6 X_H is tangent to $\mathbf{J}^{-1}(\mu)$ and by G_μ-invariance gives a vector field on the quotient $\overline{X}_H^\mu$, this field is Hamiltonian on the reduced space according to:

$$\omega_\mu(\overline{X}_H^\mu, \cdot) = dh^\mu$$

Chapter 4
Stability of relative equilibria

The different types of stability for relative equilibria are going to be introduced. General definitions are introduced in [**AM78**] and the reduced energy momentum will follow [**SLM91**].

4.1. Orbital stability

In the case of symmetric systems we are interested not only on the existence of relative equilibria but also on it's stability. The stabilities introduced before were all of them for fixed point, relative equilibria analogues are needed.

Definition 4.1. Let $x(t)$ be a integral curve of X with initial condition $x_0 \in M$, then the orbit x is said to be *orbitally stable* if for every neighbourhood U of the whole orbit exists V neighbourhood of x such that $V \subset U$ and

$$F^t(V) \subset U$$

for all t.

This definition of stability is, in principle, too restrictive for being used as the right stability for Hamiltonian relative equilibria. The nearby solutions can "drift" in directions different to that of the orbit.

The other important classical definition is:

Definition 4.2. Let x_0 be a relative equilibrium, then it's called *relatively stable* if in the reduced dynamics $[x_0]$ is a stable equilibrium.

In principle the relationship between relative stability and orbital stability is by no means trivial. We need to introduce a third stability concept, the stability modulo a subgroup of G:

Definition 4.3. Let x_0 be a relative equilibria with velocity ξ and $A \subset G$ a subgroup. $\exp(\xi\mathbb{R}) \cdot x_0$ is *stable modulo A* if for every A-invariant neighbourhood U of $\exp(\xi\mathbb{R}) \cdot x_0$ there is a neighbourhood V of $\exp(\xi\mathbb{R}) \cdot x_0$ such that $F^t(V) \subset U$ for all t.

Is like orbital stability but requiring that the initial neighbourhood is A-invariant.

The right relationship between relative stability and dynamics in the unreduced phase space was given in [**Pat92**]:

Theorem 4.4. *Let $(M, \omega, G, \mathbf{J})$ be a Hamiltonian G-space with Hamiltonian H. Let $x \in M$ be a relative equilibrium with momentum $\mathbf{J}(x) = \mu$. Then x is G_μ-stable if the reduced equilibrium $[x] \in M/G$ is stable*

Remark 4.5. Compactness of G is important for this result, otherwise technical assumptions on the action and the group should be made.

Thus this theorem states that the only directions in which nearby integral curves can drift are the directions in G_μ. A priori stability of the reduced dynamics would imply G stability in the unreduced phase space but this result is much stronger, in general $G_\mu \subsetneq G$

4.2. Energy-Momentum Method

The problem of applying Theorem 4.4 is that we still need to work in the reduced space in order to conclude formal stability.

But this is not necessary, we can realize the tangent space at $\mathbf{J}^{-1}(\mu)/G_\mu$ as a subspace of the tangent space at M and then we can check the formal stability there.

Definition 4.6. Let $(M, \omega, G, \mathbf{J})$ be a Hamiltonian G-space and $x \in M$, let $\mu = \mathbf{J}(x)$, if $\mathbf{g}_\mu$ is the Lie algebra of G_μ then any subspace $S \subset T_xM$ such that:

$$\ker T_x\mathbf{J} = \mathfrak{g}_\mu \cdot x \oplus S$$

is called a *symplectic slice.*

This symplectic slice encodes the symplectic geometry a neighbourhood of a given orbit via the Marle-Guillemin-Stenberg (MGS) form [**RO07**].

Note that N is isomorphic to $T_{[x]}(\mathbf{J}^{-1}(\mu)/G_\mu)$. It can be proved (for example [**Pat95**]) that:

$$T_x\pi_\mu^* d^2_{[x]} h^\mu = d^2_x H_\xi|_S$$

so definiteness of h^μ at $[x]$ can be checked by computing the Hessian of the augmented Hamiltonian and checking its definiteness restricted to a symplectic slice. Definiteness of this bilinear form is formal stability:

Definition 4.7. Let $x_e \in M$ be a relative equilibrium, if the restriction of the augmented Hamiltonian to a symplectic slice at x_e is positive or negative definite, then x_e is called *formally stable.*

This gives conditions for G_μ stability which can be tested on the original phase space, this is the *Energy-Momentum method*:

Theorem 4.8. *Let $(M, \omega, G, \mathbf{J})$ be a Hamiltonian G-space with Hamiltonian H. Let $x \in M$ be a relative equilibrium with associated momentum μ. Then the augmented Hamiltonian $H_\xi = H - \langle \mathbf{J}(\cdot), \xi\rangle$ has a critical point at ξ.*

If the bilinear form

$$d^2_x H_\xi|_S$$

is definite for some (and hence any) symplectic slice S at x, then the relative equilibrium is G_μ-stable.

4.3. Reduced Energy-Momentum

The energy-momentum method gives a way of computing relative equilibria and to check stability in the unreduced phase space. Most of the systems of interest are simple mechanical systems, that is, with a cotangent bundle structure and a Riemannian metric. It seems reasonable that all this extra structure can be used to simplify the computation of stability. This extra structure leads to the Reduced Energy-Momentum method [**SLM91**] which we will outline.

For simple mechanical systems it was already observed that the augmented Hamiltonian can be written as (10):

$$H_\xi(z) = \frac{1}{2}\|p - \mathbb{F}L\xi_Q(q)\|^2 + V_\xi(q) + \langle \mu, \xi \rangle$$

once a relative equilibrium is found using Proposition 3.17, the definiteness of this function has to be tested on a symplectic slice S, if it's definite we can conclude stability. Therefore we have the following almost trivial result:

Proposition 4.9. *If q_e is a relative equilibrium for a mechanical system with associated velocity $\xi \in \mathfrak{g}$, then the relative equilibrium is G_μ-stable if $V_\xi(q_e)$ is positive definite.*

This rough condition for stability is equivalent to the inequality

$$d^2 H_\xi \geq d^2 V_\xi$$

In order to test the definiteness of H_ξ both V_ξ and the kinetic term in (10) must be controlled because we need the second variation restricted to the symplectic leaf and position change implies momenta change to satisfy the linearised momentum conservation.

One important observation [**SLM91**] if $z_e \in T^*Q$ is a relative equilibrium the associated velocity is given by:

$$\xi_e = \mathbb{I}^{-1}(q_e)\mathbf{J}(z_e)$$

this follows from a simple calculation:

$$\langle \mathbf{J}(z_e), \zeta \rangle = \langle p_e, \zeta_Q(q_e) \rangle = \langle \mathbb{F}L((\xi_e)_Q(q_e)), \zeta_Q(q_e) \rangle = \langle \zeta, \mathbb{I}(q_e)\xi_e \rangle$$

This observation leads to a "locked velocity" field in T^*Q:

$$z \mapsto \mathbb{I}^{-1}(q)\mathbf{J}(z) \in \mathfrak{g}$$

taking the fundamental field and the Legendre transform:

$$z \mapsto (q, p_{\mathbf{J}}(z)) = (q, \mathbb{F}L((\mathbb{I}^{-1}(q)\mathbf{J}(z))_Q)$$

with this mapping:

Proposition 4.10. *The Hamiltonian $H : T^*Q \to \mathbb{R}$ can be expressed as:*

$$H(z) = V(q) + \frac{1}{2}\langle \mathbf{J}(z), \mathbb{I}^{-1}(q)\mathbf{J}(z) \rangle + \frac{1}{2}\|p - p_{\mathbf{J}}(z)\|^2$$

The constraint condition $z \in \mathbf{J}^{-1}(\mu_e)$ will be easier with this last representation; The computation of the augmented Hamiltonian with this new expression gives:

$$H_\xi = H(z) - \langle \mathbf{J}(z) - \mu_e, \xi(z) \rangle = V(q) + \langle \mu_e - \frac{1}{2}\mathbf{J}(z), \xi(z) \rangle + \frac{1}{2}\|p - p_{\mathbf{J}}(z)\|^2$$

If $z \in \mathbf{J}^{-1}(\mu_e)$ then it reduces to

$$H|_{\mathbf{J}^{-1}(\mu_e)} = V(q) + \frac{1}{2}\langle \mu_e, \mathbb{I}^{-1}(q)\mu_e \rangle + \frac{1}{2}\|\tilde{p}\|^2 \tag{11}$$

where $\tilde{p} = p - p_{\mathbf{J}}(z)$. The first two terms form *Smale's amended potential*:

$$V_\mu(q) = V(q) + \frac{1}{2}\langle \mu_e, \mathbb{I}^{-1}\mu \rangle$$

the idea will be to use $(q, \tilde{p})$ as coordinates, then the stability analysis can be done considering only the configuration space Q instead of the whole T^*Q, that is, the momenta $\tilde{p}$ will play no role in the stability analysis.

The shifting map $\Sigma : P \to P$ $(P = T^*Q)$ defined as

$$\Sigma(q, p) = (q, p - p_{\mathbf{J}}(z))$$

maps P onto the level set of zero total angular momentum, $\Sigma(P) = \mathbf{J}^{-1}(0)$. This shifting map restricted to $\mathbf{J}^{-1}(\mu)$:

$$\begin{aligned} \Sigma : \mathbf{J}^{-1}(\mu) &\to \mathbf{J}^{-1}(0) \\ (q, p) &\mapsto (q, p - p_{\mathbf{J}}(z)) \end{aligned}$$

is bijective, an if $z \in \mathbf{J}^{-1}(\mu)$ the shifted momenta is exactly the mechanical form introduced before, that is $(q, p_{\mathbf{J}}(z)) = \alpha_\mu(q)$.

We will try to solve the problem of stability for the Hamiltonian defined on $(q, \tilde{p}) \in \mathbf{J}^{-1}(0)$ for the Hamiltonian:

$$h = V_\mu(q) + \frac{1}{2}\|\tilde{p}\|^2$$

if (q, p) is a critical point for the augmented Hamiltonian the shifted point is a critical point for h, because, using the chain rule[1]:

$$\begin{aligned} \delta V_\mu(q_e) &= \delta V(q_e) - \frac{1}{2}(\mathbb{I}(q_e)^{-1}\mu_e) \cdot (\delta\mathbb{I}(q_e))(\mathbb{I}(q_e)^{-1}\mu_e) \\ &= \delta V(q_e) - \frac{1}{2}\xi_e \cdot (\delta\mathbb{I}(q_e))\xi_e = \delta V_\xi(q_e) \end{aligned}$$

Proposition 4.11. *Critical points of H_ξ and critical points of h are in correspondence through the shifting map Σ, that is $\Sigma((q_e, p_e)) = (q_e, 0)$.*

As H is G-invariant all infinitesimal group motions correspond to neutral variations of H. We are concerned to the restriction of H to $\mathbf{J}^{-1}(\mu)$, we only need to consider the group motions that preserve $\mathbf{J}^{-1}(\mu)$. Equivariance of the momentum map implies that:

$$\mathfrak{g} \cdot z \in \mathbf{J}^{-1}(\mu) \Leftrightarrow g \in G_\mu$$

in consequence, we define the space of admissible configuration variations $\mathcal{V} \in T_{q_e}Q$ as the orthogonal complement to $\mathfrak{g}_\mu \cdot q_e$.

Definition 4.12. *The space of admissible configuration variations* around a relative equilibria q_e is:

$$\mathcal{V} = \{\delta q \in T_{q_e}Q | \langle\!\langle \delta q, \zeta_Q(q)_e \rangle\!\rangle = 0 \quad \forall \zeta \in \mathfrak{g}_\mu\}$$

Now we can characterize the variations of the shifted momenta that preserve the zero momentum condition. Let $\delta\tilde{z} = (\delta q, \delta\tilde{p}) \in T_{(q_e,0)}T^*Q$, it must satisfy $\delta\tilde{z} \in T_{\tilde{z}_e}\mathbf{J}^{-1}(0)$, and because of linearity of the momentum map with respect to momenta:

$$T_{\tilde{z}_e}(\delta q, \delta\tilde{p}) \cdot \eta = \mathbf{J}(q_e, \delta\tilde{p}) \cdot \eta = \langle \delta\tilde{p}, \eta_Q(q_e) \rangle = 0$$

Therefore one possible realization for the symplectic slice at $\tilde{z}_e$ for h is:

[1] given a real valued function $f : M \to \mathbb{R}$ and a path $\gamma : (-\varepsilon, \varepsilon) \to M$ such that $\gamma(0) = x_0$ the first variation of f at x_0 along γ is $\delta f = \frac{d}{dt}f(\gamma(t)) = \langle df, \gamma'(0) \rangle$.

$$\mathcal{S}_0 = \{(\delta q, \delta\tilde{p}) \in \mathcal{V} \times T^*_{q_e} Q | \langle \delta\tilde{p}, \eta_Q(q_e)\rangle = 0 \quad \forall \eta \in \mathfrak{g}\}$$

this symplectic slice can be written as the direct sum $\mathcal{S}_0 = \mathcal{V} \oplus (\mathfrak{g} \cdot q_e)^0$, where the second term is the annihilator of the group orbit.

The second variation of h restricted to $\mathcal{S}_0$ is easily computed:

$$\delta^2 h = \delta^2 V_\mu + \|\delta\tilde{p}\|^2 \text{ in } \mathcal{S}_0$$

This Second variation of Smale's amended potential depends only on the position $q \in Q$ but it's definition requires the inversion of the locked inertia tensor which is computationally difficult in many cases. But using the chain rule:

$$\delta^2 V_\mu = \delta^2 V_\xi + (\delta(\mathbb{I}\xi)) \cdot \mathbb{I}(q_e)^{-1}(\delta(\mathbb{I}\xi))$$

therefore we only need to compute the Hessian of the augmented potential and add a correction term which only needs the inversion of the locked inertia tensor at the equilibrium.

At this point we have:

Proposition 4.13. *Given a relative equilibrium (q_e, p_e), if $\dim G < \dim Q$ then positive definiteness of*

$$\delta^2 V_\xi + (\delta(\mathbb{I}\xi)) \cdot \mathbb{I}(q_e)^{-1}(\delta(\mathbb{I}\xi)) = \delta^2 V_\mu$$

restricted to $\mathcal{V}$ implies formal-stability of the relative equilibrium.

If $\dim G = \dim Q$ then definiteness (positive or negative) of the same bilinear form implies stability of the relative equilibrium.

The conditions of stability given by this results are sharp in contrast to Proposition 4.9 which only give a rough condition. For example the lasts computations can be used to give an instability criteria. Based on Proposition 2.8:

Proposition 4.14. *Given a relative equilibria (q_e, p_e), if $\dim G < \dim Q$ and*

$$\delta^2 V_\xi + (\delta(\mathbb{I}\xi)) \cdot \mathbb{I}(q_e)^{-1}(\delta(\mathbb{I}\xi)) = \delta^2 V_\mu$$

restricted to $\mathcal{V}$ has an odd number of negative eigenvalues then the relative equilibria is unstable

4.4. Block diagonalization of $\delta^2 V_\mu$

If $\dim Q = n$, the energy-momentum method (Theorem 4.8) required to test the stability of a relative equilibria computing definiteness on the symplectic slice, a vector space of dimension:

$$\dim \mathcal{S} = 2n - \dim G - \dim G_\mu$$

The reduced energy-momentum method (Proposition 4.13) only requires the to test the definiteness in a vector space of dimension

$$\dim \mathcal{V} = n - \dim G_\mu$$

but this computation can be improved decomposing the space $\mathcal{V}$ in such a way that the Hessian of the amended potential block diagonalizes [**SLM91**].

The idea is to exploit the symmetry properties of V_μ and decompose the admissible variation space $\mathcal{V}$ into rigid variations and internal variations $\mathcal{V} = \mathcal{V}_{RIG} \oplus \mathcal{V}_{INT}$.

$\mathcal{V}_{RIG}$ would correspond to motions in rotational modes, that is, induced by the group action, whereas $\mathcal{V}_{INT}$ can be interpreted as "shape" variations variations in the "shape" space or internal configurations [**Mar92**].

Let $\mathfrak{g}_\mu^\perp \subset \mathfrak{g}$ be the orthogonal complement of $\mathfrak{g}_\mu$ with respect to the locked inertia metric at the equilibrium configuration:

$$\mathfrak{g}_\mu^\perp = \{\eta \in \mathfrak{g} | \eta \cdot \mathbb{I}(q_e)\zeta = 0 \quad \forall \zeta \in \mathfrak{g}_\mu\}$$

so $\mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{g}_\mu^\perp$.

Infinitesimal variations in the group G correspond to variations in the configuration space through the group action in Q and the fundamental fields, so the space of rigid variations is:

$$\mathcal{V}_{RIG} = \{\eta_Q(q_e) \in T_{q_e}Q | \eta \in \mathfrak{g}_\mu^\perp\} \subset \mathcal{V}$$

The space $\mathcal{V}_{INT}$ of internal vibration modes is chosen as a "energy-orthogonal" complement to $\mathcal{V}_{RIG}$ such that variations in one space decouple from the other one. Specifically:

$$\mathcal{V}_{INT} = \{\delta q \in \mathcal{V} | \eta \cdot \delta(\mathbb{I}\xi) = 0 \quad \forall \eta \in \mathfrak{g}_\mu^\perp\} = \{\delta q \in \mathcal{V} | \mathbb{I}(q_e)^{-1}\delta(\mathbb{I}\xi) \in \mathfrak{g}_\mu\}$$

After some computations [**SLM91**], with the above definitions:

Proposition 4.15.

$$d^2V_\mu(q_e)(\eta_Q(q_e), \delta q) = 0$$

if $\eta_Q(q_e) \in \mathcal{V}_{RIG}$ and $\delta q \in \mathcal{V}_{INT}$. Thus the bilinear form $\delta^2 V_\mu$ block diagonalizes.

With some manipulations, the restriction of d^2V_μ to $\mathcal{V}_{RIG}$ is independent of the potential function and can be expressed as:

Proposition 4.16. *The restriction of d^2V_μ to $\mathcal{V}_{RIG}$ is:*

$$d^2V_\mu(q_e)(\eta_Q(q_e), \nu_Q(q_e)) = ad_\eta^*\mu \cdot \mathbb{I}(q_e)^{-1} ad_\nu^*\mu + ad_\eta^*\mu \cdot [\nu, \xi] \tag{12}$$

for any $\eta, \nu \in \mathfrak{g}_\mu^\perp$. This is expression will be referred as Arnold form.

If the Arnold form is non-degenerate it can be shown that $\mathcal{V} = \mathcal{V}_{RIG} \oplus \mathcal{V}_{INT}$. Thus:

Proposition 4.17. *Given a relative equilibria (q_e, p_e), if $\dim G < \dim Q$ then positive definiteness of*

$$\delta^2 V_\xi + (\delta(\mathbb{I}\xi)) \cdot \mathbb{I}(q_e)^{-1}(\delta(\mathbb{I}\xi)) = \delta^2 V_\mu$$

restricted to $\mathcal{V}_{INT}$ and positive definiteness of the Arnold form (12) *implies formal-stability of the relative equilibrium.*

If $\dim G = \dim Q$ then definiteness (positive or negative) of the Arnold form implies stability of the relative equilibrium.

4.5. Splitting of $\mathcal{S}$

The decomposition described in the preceding sections is closely related to a decomposition of a symplectic slice $\mathcal{S} \subset T_{z_e}P$, in the present context, the symplectic slice can be realized as:

$$\mathcal{S} = \{\delta z = (\delta q, \delta p) \in \ker T_{z_e}\mathbf{J} | \delta q \in \mathcal{V}\}$$

We will see that the decomposition $\mathcal{V} = \mathcal{V}_{RIG} \oplus \mathcal{V}_{INT}$ will induce a decomposition $\mathcal{S} = \mathcal{S}_{RIG} \oplus \mathcal{S}_{INT}$ and $\mathcal{S}_{INT} = \mathcal{W}_{INT} \oplus \mathcal{W}_{INT}^*$ such that the Hessian of the augmented potential block-diagonalizes.

The subspaces can be interpreted as:

- $\mathcal{S}_{RIG}$ is the space of superposed rigid body variations which satisfy the linearised constant angular momentum condition.
- $\mathcal{W}_{INT}$ is the space of internal configuration ("shape") variations.
- $\mathcal{W}^*_{INT}$ is the space of internal momentum variations.

First of all, we will decompose the space of shifted variations $\mathcal{S}_0$. Lifting the decomposition of $\mathcal{V}$ and vertically lifting the annihilator of $\mathfrak{g} \cdot q_e$:

$$\begin{aligned}\mathcal{S}_{0,RIG} &= \{(\delta q, 0) | \delta q \in \mathcal{V}_{RIG}\}\\ \mathcal{W}_{0,INT} &= \{(\delta q, 0) | \delta q \in \mathcal{V}_{INT}\}\\ \mathcal{W}^*_{0,INT} &= \{(0, \delta\tilde{p}) | \delta\tilde{p} \in (\mathfrak{g} \cdot q_e)^0\}\end{aligned}$$

The decomposition on $\mathcal{S}$ can be defined by using the shift map $T_{\tilde{z}_e}\Sigma_\mu^{-1}$. Using the definition of the mechanical connection we can write:

$$\begin{aligned}\mathcal{S}_{RIG} &= T\alpha_\mu \mathcal{V}_{RIG}\\ \mathcal{W}_{INT} &= T\alpha_\mu \mathcal{V}_{INT}\\ \mathcal{W}^*_{INT} &= \mathrm{vlift}(\mathfrak{g} \cdot q_e)^0\end{aligned}$$

That is:

$$\mathcal{S} = S_{RIG} \oplus \mathcal{S}_{INT} = S_{RIG} \oplus (\mathcal{W}_{INT} \oplus \mathcal{W}^*_{INT})$$

if the Arnold form is non-degenerate at the relative equilibrium.

Theorem 4.18. *Let $z_e \in P$ be a relative equilibria and let $\mathcal{S} = S_{RIG} \oplus \mathcal{S}_{INT} = S_{RIG} \oplus (\mathcal{W}_{INT} \oplus \mathcal{W}^*_{INT})$ as above. Then*

$$d^2 H_\xi(z_e)((\Delta z, 0), (\delta z, 0)) = 0$$

if $(\Delta z, \delta z) \in \mathcal{S}_{RIG} \times \mathcal{S}_{INT}$ or $(\Delta z, \delta z) \in \mathcal{W}_{INT} \times \mathcal{W}^*_{INT}$. Therefore:

$$d^2 H_\xi(z_e)|_{\mathcal{S}} = \begin{bmatrix} \begin{bmatrix} \textit{Arnold} \\ \textit{form} \end{bmatrix} & 0 & 0 \\ 0 & d^2 V_\mu|_{\mathcal{V}_{INT}} & 0 \\ 0 & 0 & \langle , \rangle \end{bmatrix} \tag{13}$$

with respect to the splitting $\mathcal{S} = S_{RIG} \oplus \mathcal{W}_{INT} \oplus \mathcal{W}^*_{INT}$.

Remark 4.19. The three subspaces can be shown to be isomorphic to:

$$\mathcal{S}_{RIG} \cong \mathcal{O}_\mu, \quad \mathcal{W}_{INT} \cong T_{[q_e]}(Q/G), \quad \mathcal{W}^*_{INT} \cong T^*_{[q_e]}(Q/G)$$

Although the splitting of $\mathcal{S}$ is not a dynamical splitting, in the sense that it doesn't decouple the variables in different spaces the symplectic form has a block structure in the same splitting, allowing the linearisation of the system easily.

Proposition 4.20. *Let $z_e \in P$ be a relative equilibria and let $\mathcal{S} = S_{RIG} \oplus (\mathcal{W}_{INT} \oplus \mathcal{W}^*_{INT})$ be the previous splitting, the symplectic form becomes:*

$$\omega(z_e)|_{\mathcal{S}} = \begin{bmatrix} \begin{bmatrix} \textit{coadjoint} \\ \textit{orbit form} \end{bmatrix} & \begin{bmatrix} \textit{internal-rigid} \\ \textit{coupling} \end{bmatrix} & 0 \\ -\begin{bmatrix} \textit{internal-rigid} \\ \textit{coupling} \end{bmatrix}^T & S & Id \\ 0 & -Id & 0 \end{bmatrix} \tag{14}$$

where:

- Coadjoint orbit form*: If $\Delta_1 z, \Delta_2 z$ are rigid body motions with generators η_1, η_2 then:*
$$\omega(z_e)(\Delta_1 z, \Delta_2 z) = -\langle \mu, [\eta_1, \eta_2] \rangle$$
- Internal-rigid coupling*: If $\Delta z \in \mathcal{S}_{RIG}$ with generator ζ and $\delta z \in \mathcal{S}_{INT}$ then:*
$$\omega(z_e)(\Delta z, \delta z) = -\langle\!\langle \zeta_Q(q_e), \delta q \rangle\!\rangle$$
- Internal self-coupling *(Coriollis term): If $\delta_1 z = T\alpha_\mu \delta_1 q$ and $\delta_2 z = T\alpha_\mu \delta_2 q \in \mathcal{W}_{INT}$ where $\delta_1 q, \delta_2 q \in \mathcal{V}_{INT}$ then*
$$\omega(z_e)(\delta_1 z, \delta_2 z) = -d\alpha_\mu(\delta_1 q, \delta_2 q)$$

4.6. Persistence and Bifurcations

The stability analysis conditions given by Theorem 4.8 can also be useful to look for bifurcations and study the qualitative behaviour of the system.

The key result is [**Pat95**] the smoothness of the set of relative equilibria near a relative equilibria with non-degenerate $d^2 H_\xi$

Theorem 4.21 (Persistence theorem)**.** *Let $(M, \omega, \mathbf{J}, G)$ by a Hamiltonian G-space equipped with a Hamiltonian H. Let $M_e \subset M$ be the set of relative equilibria and let $p_e \in M$ be a relative equilibrium with generator ξ_e, and momentum $\mu_e = \mathbf{J}(p_e)$ such that*

$$\ker d\mathbf{J}(p_e) \cap \ker d^2 H_{\xi_e}(p_e) \subset \mathfrak{g}_\mu \cdot p_e$$

Then there is an open neighbourhood U of p_e such that:

- *$U \cap M_e$ is a $\dim G + \operatorname{rank} G$ dimensional manifold of M*
- *$T_{p_e}(U \cap M_e) = dX_{H_{\xi_e}}(p_e)^{-1}(\mathfrak{g} p_e)$*

This result can also be proved using the MGS-form [**RO07**].

In view of this persistence result, a *regular branch of relative equilibria* can be defined as a subset $\Sigma \subset M_e$ such that, for all $p \in \Sigma$, there is an injective parametrization $\sigma : \mathbb{R}^n \to \Sigma$ such that $\sigma(0) = p$ where $n = \dim G + \operatorname{rank} G$.

The points belonging to two or more different regular branches will be called *bifurcation points.*

Remark 4.22. In view of the persistence theorem a bifurcation in the set of relative equilibria can only occur in the relative equilibria such that $d^2 H_\xi|\mathcal{S}$ is not invertible. This theorem provides only a necessary condition for bifurcation.

To study the bifurcations we will search for points where the restricted Hessian is degenerate. This are only candidate points, a detailed analysis near that point is required to decide if it is o it is not a bifurcation point.

Seen in the reduced space M/G, a regular branch of relative equilibria will be a branch of fixed points of dimension rank G. Each of fixed point in the reduced space represents a whole G-orbit of relative equilibria.

Chapter 5
Introductory Examples

5.1. Keplerian particle

Before treating the problem of the motion of a rigid body in the gravitational field, the motion of a particle in a Keplerian field is going to be studied.

In this case the configuration space can be taken as $Q = \mathbb{R}^3$, coordinates will be denoted as r. We can endow the space with the usual euclidean metric. Consider the classical Kepler's potential, a central potential proportional to the inverse of the radius. We can eliminate all the physical constants through an adimesionalization process that will be explained in Section 6.2, the keplerian potential in those units is:

$$V = -\frac{1}{|r|}$$

There is a linear action of the group of rotations $G = SO(3)$ on Q. We will use the hat notation introduced in Section A.1 and the left trivialization of G.

If $r, \delta r$ are coordinates in TQ then the fundamental field associated with $\eta \in \mathfrak{g}$ is $\delta r = \hat{\eta} r$ (the complete computations are similar to those done in Section 6.5). From this expression is easy to see that the action is not free, the isotropy group of each point is isomorphic to S^1.

But the cotangent lifted action is free for almost all the points, as it will be computed in Section 6.5, the fundamental field for the action in T^*Q is $\delta r = \hat{\eta} r, \delta p = \hat{\eta} p$. For all the points such that r, p are not aligned the action is free. In this points all the techniques developed in the previous chapters can be applied. We will see that this assumption is correct. As the action in the configuration space is always non-free we will work directly with the Energy-Momentum method instead of the reduced Energy-Momentum method.

Take the augmented Hamiltonian $H_\eta = \frac{-1}{|r|} + \frac{p \cdot p}{2} - \hat{r} p \cdot \eta$. A relative equilibrium is a critical point of this functional. Computing the first variation:

$$\delta H_\eta = \left(\frac{r}{|r|^3} + \hat{\eta} p \right) \cdot \delta r + (p + \hat{r} \eta) \cdot \delta p$$

vanishing of this variation implies $p = \hat{\eta} r$ and $\frac{r}{|r|^3} + \hat{\eta}\hat{\eta} r = 0 = \frac{r}{|r|^3} + \eta(\eta \cdot r) - r|\eta|^2$ that is $\eta \cdot r = 0$ and $\frac{1}{|r|^3} = |\eta|^2$ which is Kepler's formula in this units.

Note that as $\eta \cdot r = 0$, $p = \hat{\eta} r$ and assuming $r \neq 0$ $\eta \neq 0$, r, p are linearly independent. So the action is free around that relative equilibrium and all the methods for free actions can be applied.

The second variation has as matrix (in the $\delta r, \delta p$ basis):

$$d^2 H_\eta = \begin{bmatrix} -\frac{3rr^T}{|r|^5} + \frac{1}{|r|^3} & \hat{\eta} \\ -\hat{\eta} & Id \end{bmatrix}$$

A orthogonal basis can be chosen such that $\eta = (\eta, 0, 0)$ and $r = (0, r, 0)$. Therefore $p = (0, 0, \eta r)$.

The momentum map is $\mathbf{J}(z) = \hat{r} p$ so the tangent application has as matrix $TJ = (-\hat{p}, \hat{r})$. Evaluating at the relative equilibrium point:

$$TJ = \begin{bmatrix} 0 & \eta r & 0 & 0 & 0 & r \\ -\eta r & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -r & 0 & 0 \end{bmatrix}$$

A basis of $T\mathbf{J}^{-1}(\mu)$ at this point can be:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \\ -\eta & 0 & 0 \\ 0 & 0 & -\eta \end{bmatrix}$$

the first element generates $\mathfrak{g}_\mu$.

The Hessian $d^2 H_\eta$ restricted to the subspace generated by the other two vectors is:

$$\begin{bmatrix} R_2^{-3} & 0 \\ 0 & R_2^{-3} \end{bmatrix}$$

formal stability of all the relative equilibria can be concluded.

Proposition 5.1. *For a Keplerian particle r is a relative equilibrium with velocity $\eta \in \mathbb{R}^3$ if $\eta \cdot r = 0$ and Kepler's formula is satisfied: $\frac{1}{|r|^3} = |\eta|^2$.*

All the relative equilibria are formally stable.

5.2. Modified planar potential

Consider the movement of a particle in the plane under the influence of a Keplerian potential and a cubic correction. The potential will be:

$$V = -\frac{1}{|r|} - \frac{1}{2|r|^3}$$

this potential has no physical foundations, but it has a similar expression to the potential that is going to appear in more realistic models.

The cubic term can appear if the potential comes from Taylor's expansion in powers of r^{-1}. For r large the cubic term is a correction term, improving the approximation but for small r the cubic term can be dominant. This term can affect the qualitative behaviour in a substantial way as this model will show.

5.2.1. Relative equilibria analysis. Let $Q = \mathbb{R}^2$ be the configuration space. On this plane polar coordinates (r, θ) can be taken.

There is a natural action of $G = S^1$ in Q by rotations. If $(\delta r, \delta\theta)$ are coordinates in TQ induced by (r, θ) then the fundamental field of the action is given by: $\delta r = 0, \delta\theta = \xi$ for each $\xi \in \mathfrak{g} \cong \mathbb{R}$.

The metric considered in Q will be the usual one, in polar coordinates has the expression:

$$\langle\langle(\delta r_1, \delta\theta_1), (\delta r_2, \delta\theta_2)\rangle\rangle = \delta r_1 \delta r_2 + r^2 \delta\theta_1 \delta\theta_2$$

from this expression the locked inertia tensor is given by: $\eta \cdot \mathbb{I}\xi = \langle\langle \eta_Q, \xi_Q \rangle\rangle = r^2 \eta \xi$, in other words given an angular velocity ξ the associated angular momentum is $\mu = r^2 \xi$.

The Reduced Energy Momentum Method will be used to study the relative equilibria.

The group is abelian so $\mathfrak{g}_\mu = \mathfrak{g}$, the fundamental fields of the group span the space $\mathbb{R} \cdot \delta\theta$ at each tangent space. Therefore space of admissible variations will be $\mathcal{V} = \mathbb{R} \cdot \delta r$.

The augmented potential is

$$V_\xi = \frac{-1}{r} + \frac{-1}{2r^3} - \frac{1}{2}\xi^2 r^2$$

as this is only a function of r the vanishing of the variation δV_ξ is the same as:

$$\frac{\partial}{\partial r} V_\xi = \frac{1}{2} \frac{2r^2 + 3 - 2\xi^2 r^5}{r^4} = 0$$

so, the critical points of the augmented potential are: (r, θ) such that $\xi^2 = \frac{2r^2+3}{2r^5}$. In terms of the angular momentum $\mu^2 = \frac{2r^2+3}{2r}$.

In this example the metric is easily inverted so a closed form for the amended potential can be given:

$$V_\mu = \frac{-1}{r} + \frac{-1}{2r^3} + \frac{1}{2}\mu^2 \frac{1}{r^2}$$

the second variation of this expression restricted to $\mathcal{V}$ is given by the second derivative in the r direction, which at the critical point is:

$$\frac{\partial^2}{\partial r^2} V_\mu = \frac{3\mu^2 r - 2r^2 - 6}{r^5} = \frac{2r^2 - 3}{2r^5}$$

positiveness of this expression implies formal stability and if it is negative instability of the relative equilibria will follow. In other words, the relative equilibrium (r, θ) is stable if $r^2 > \frac{3}{2}$ and unstable otherwise.

5.2.2. Behaviour justification. In this idealized example the dimension is low, most of the functionals can be represented to figure out this sudden stability loss.

The first important function is the augmented potential V_ξ, in the next figure several plots of V_ξ for different values of ξ.

In this figure the existence of relative equilibria can be observed (the critical points of the function). For each value of ξ there is one and only one relative equilibria. All the relative equilibria are maximums of this functions. The second variation of V_ξ in each of those points is negative definite, but, of course, this doesn't imply instability. Positive definiteness of V_ξ implies formal stability (Proposition (4.9)) but this is only a rough condition.

The other relevant functional is the amended potential V_μ, represented here for different values of μ:

For each value of the momentum μ, V_μ can have two critical points, one or zero critical points. For μ small enough V_μ has no critical points whereas for big μ V_μ has a maximum and a minimum.

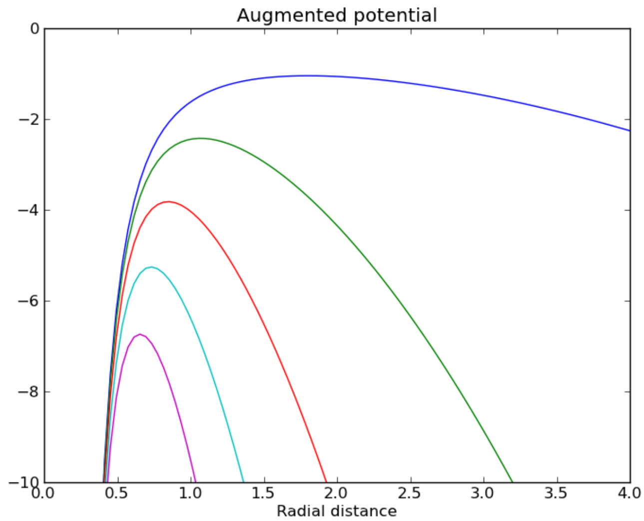


FIG. 1. Augmented potential

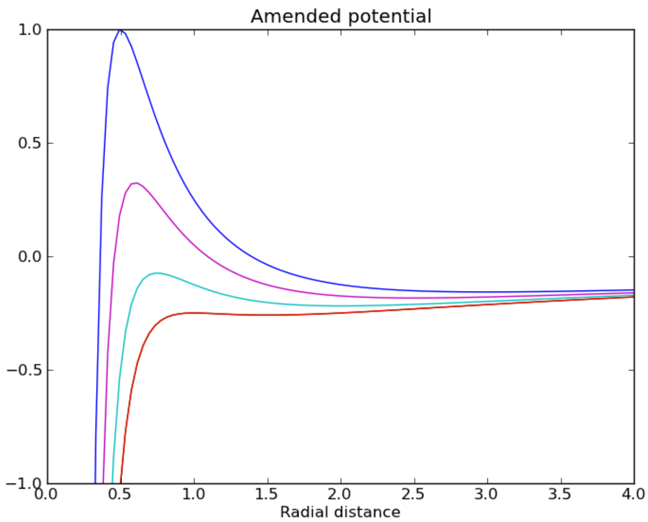


FIG. 2. Amended potential

Recall that Theorem (4.13) implies that the maximum will be a unstable relative equilibrium and the minimum will be a stable relative equilibrium.

The behaviour of relative equilibria for different values of μ is shown in the following diagram:

for each value of angular momentum μ there is one stable and one unstable relative equilibrium. When μ gets closer to a critical value, this two branches collapse and the relative equilibria disappear for $\mu < 6^{1/4}$. In green the behaviour for the classical Keplerian potential is shown, recall that in the classical Kepler potential all the equilibria were stable.

The cubic term causes the stability loss at low orbits.

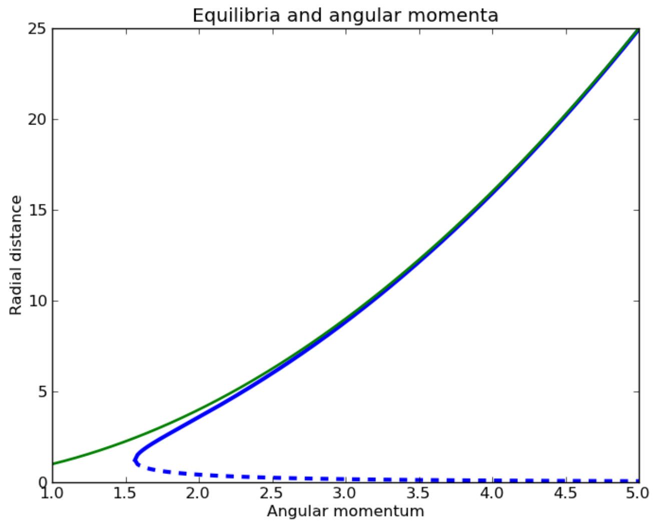


FIG. 3. Relative equilibria

Following the notation of [**AM78**], the set of relative equilibria, in this case, form a *orbit cylinder*. It can be shown that the energy of the relative equilibria also attains a minimum at the critical value of the angular momentum. The orbit cylinder becomes tangent to the energy surface, one of the sheets is stable and the other one is unstable. For lower values of the energy there are not relative equilibria. This seems to correspond with the bifurcation type called Creation [**AM78**].

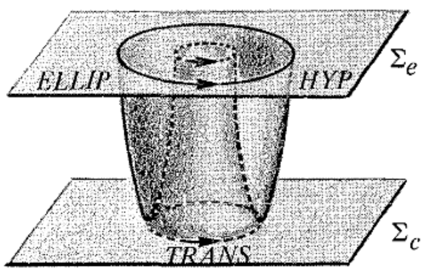


FIG. 4. Creation, reproduced from [**AM78**]

Chapter 6
The rigid body in a gravitational field.

The problem we will try to model and study qualitatively is the dynamics of two bodies of finite extent evolving according to their mutual gravitational attraction.

We will make some assumptions about one of the bodies:

- one of the bodies (*the primary*) is assumed to have a total mass which is much greater that of the other body (*the satellite*).
- the mass distribution of the primary is assumed to be *spherically symmetric*.

The first assumption implies that the movement of the primary is unaffected by the dynamics of the satellite, whereas the second one implies that the dynamics of the satellite is unaffected by the rotational motion of the primary.

The second assumption is the more physically unrealistic if we think about an artificial satellite orbiting around the Earth. However this symmetry assumption allows considerable simplification and provides a valuable first point approximation from which additional effects can be added with perturbation theory. The problem is equivalent to the dynamics of one single body in a central gravitational field. The dynamics of this problem has the mathematical structure of *a Hamiltonian system with symmetries.*

Traditionally ([**Bel65**],[**Mei70**],[**Hal02**]) another assumption was made:

- the satellite is "inertially-small" so that the motion of its centre of mass is unaffected by the finite extent of the body. So the orbit described by the centre of mass of the satellite is the same as if the satellite were replaced by a point mass.

We will not suppose this "smallness" assumption. The correctness of this approximation will be one of our main concerns.

Of course, the dynamics of this system may be very complicated. The point of view adopted by the qualitative study of symmetric Hamiltonian systems is to identify a special family of solutions of the flow, and to carry out a local analysis of the properties of the flow near the solution. For this system the special family of solution under consideration is that of relative equilibria: that is, dynamical evolution orbits which are also group orbits.

There are several studies of the coupling between orbital and attitude motion (e.g. [**SH83**]) but in most of them the natural geometric and group-theoretic underpinnings of the problem are not exploited to the full possible extent.

In [**WKM90**] the major tool in studying this system was the Poisson reduction and energy-Cassimir method [**Mar92**]. We will reproduce their results using the energy-momentum method that even provide tools to study the bifurcation behaviour of the system.

6.1. Potential models

Usually the gravitational of a body in a central gravitational field is given by the expression:

$$V = -GM\frac{m}{r}$$

but this expression is only an approximation if the bodies are not dimensionless.

If a body in a central force gravitational field does not possess spherical symmetry, the centre of gravity does not coincide with the centre of mass and, in addition to a resultant force, differential-gravity torques are also present. These gravitational torques become significant in the dynamics.

We will consider a system consisting of one body m_1 possessing spherical symmetry and another body of total mass m_2 having an arbitrary mass distribution. Because m_1 is symmetric, it can be assumed that its geometric centre coincides with the centre of the force. The interest lies in specific expressions for the gravitational potential as well as differential-gravity torques acting upon m_2. See Figure 1.

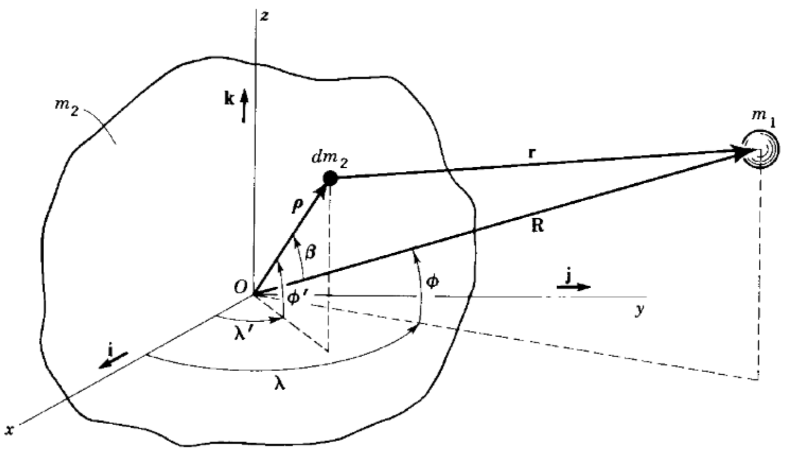


FIG. 1. Notation for potential expansions [**Mei70**]

The potential energy corresponding to an inverse square force field can be written in the integral form:

$$V = -Gm_1 \int_{m_2} \frac{dm_2}{r} \tag{15}$$

where $r = |\mathbf{R} - \boldsymbol{\rho}|$ is the distance between the centre of the field and the differential element of mass dm_2. As usually the radial distance r is going to be much larger than m_2 dimensions we can expand r in power series of $R = |\mathbf{R}|$:

$$
\begin{aligned}
r^{-1} &= R^{-1}\left[1-\frac{2\mathbf{R}\cdot\boldsymbol{\rho}}{R^2}+\left(\frac{\rho}{R}\right)^2\right]^{-\frac{1}{2}} \\
&= R^{-1}+R^{-3}\mathbf{R}\cdot\boldsymbol{\rho}+\frac{1}{2}R^{-3}\left(3R^{-2}(\mathbf{R}\cdot\boldsymbol{\rho})^2-\rho^2\right)+\ldots
\end{aligned}
$$

Letting $\boldsymbol{R}\cdot\boldsymbol{\rho}=R\rho\cos\beta$ the previous expansion can be written in the compact formula:

$$
\frac{1}{r}=\frac{1}{R}\sum_{n=0}^{\infty}P_n(\cos\beta) \tag{16}
$$

where P_n is the Legendre polynomial of degree n.

The first few Legendre polynomials have the form:

$$
\begin{aligned}
P_0(\mu) &= 1 \\
P_1(\mu) &= \mu \\
P_2(\mu) &= \frac{1}{2}\left(3\mu^2-1\right) \\
P_3(\mu) &= \frac{1}{2}\left(5\mu^3-3\mu\right) \\
P_4(\mu) &= \frac{1}{8}\left(35\mu^4-30\mu^2+3\right)
\end{aligned}
$$

they obey the recurrence formula:

$$
nP_n(\mu)=(2n-1)\mu P_{n-1}(\mu)-(n-1)P_{n-2}(\mu)
$$

We have the expression:

$$
V=-\frac{Gm_1}{R}\sum_{n=0}^{\infty}\int_{m_2}\left(\frac{\rho}{R}\right)^n P_n(\cos\beta)dm_2
$$

introducing spherical coordinates R,λ,ϕ and ρ,λ',ϕ' and expressing $\cos\beta$ in this coordinates we can define the Laplacian coefficient $L_n(\phi,\lambda,\phi',\lambda')$ as the expression $L_n(\lambda,\phi,\lambda',\phi')=P_n(\cos\beta)$, doing the computation explicitly we can separate the ρ dependence from the R dependence in L_n:

$$
\begin{aligned}
L_n(\lambda,\phi,\lambda',\phi') &= P_n(\cos\beta)=P_n(\sin\phi\sin\phi'+\cos\phi\cos\phi'\cos(\lambda-\lambda')) \\
&= P_n(\sin\phi)P_n(\sin\phi')+2\sum_{m=1}^{k}\left[\frac{(n-m)!}{(n+m)!}P_{nm}(\sin\phi)P_{nm}(\sin\phi')\cos m(\lambda-\lambda')\right]
\end{aligned}
$$

where P_{nm} are the called Legendre functions of nth degree and mth order. Introducing this expansion in the potential expression we get:

$$
\begin{aligned}
V = &-\frac{Gm_1}{R}\sum_{n=0}^{\infty}\int_{m_2}\left(\frac{\rho}{R}\right)^n L_n(\lambda,\phi,\lambda',\phi')dm_2 = \\
= &-\frac{Gm_1m_2}{R} - \frac{Gm_1}{R}\sum_{n=1}^{\infty}\Big[\frac{P_n(\sin\phi)}{R^n}\int_{m_2}\rho^n P_n(\sin\phi')dm_2 \\
&+ 2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}P_{nm}(\sin\phi)\cos m\lambda\int_{m_2}\rho^n P_{nm}(\sin\phi')\cos m(\lambda')dm_2 \\
&+ 2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}P_{nm}(\sin\phi)\cos m\lambda\int_{m_2}\rho^n P_{nm}(\sin\phi')\sin m(\lambda')dm_2\Big]
\end{aligned}
$$

which is:

$$
\begin{aligned}
V = &-\frac{Gm_1m_2}{R}\Big[1+\sum_{n=1}^{\infty}\Big[\left(\frac{a}{R}\right)^n C_{n0}P_n(\sin\phi)+ \\
&+2\sum_{m=1}^{n}\left(\frac{a}{R}\right)^n P_{nm}(\sin\phi)(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda)\Big]\Big]
\end{aligned}
$$

where the coefficients C_{n0}, C_{nm}, S_{nm} reflect the mass distribution as well as the shape of the body. This equation represents the solution of the harmonic equation $\nabla^2 V = 0$ in terms of spherical coordinates. The terms $P_{nm}(\sin\phi)\cos m\lambda$ and $P_{nm}(\sin\phi)\sin m\lambda$ are called tesseral harmonics of nth degree and mth order. The harmonics of order 0, $P_{n0} = P_n$ are called zonal harmonics, they depend on the latitude but not on the longitude λ. The combination of tesseral harmonics of degree n represents a surface spherical harmonic of degree n.

If we neglect terms of order four or more the coefficients C_{nm}, S_{nm} can be expressed in terms of the inertia tensor, leading after a lengthy computation in Cartesian coordinates [**Mei70**] to the classical second order approximation of the potential:

$$
V_2 = -\frac{Gm_1m_2}{R} - \frac{Gm_1\mathrm{tr}(\mathbf{I})}{2R^3} - \frac{3Gm_1}{2R^5}\mathbf{R}\cdot\mathbf{IR} \tag{17}
$$

In the case that all the moments of inertia are equal this expression collapses to the zeroth order approximation:

$$
V_0 = -\frac{Gm_1m_2}{R}
$$

thus if all the moments of inertia are equal it's expected that the second order approximation won't be correct enough, specially when we'll deal with torques.

Defining the higher order inertia integrals [**Mei70**] as:

$$
J_{x^py^qz^r} = \int_{m_2} x^py^qz^r dm_2
$$

where p, q, r are integers and x, y, z are Cartesian coordinates terms of higher order in R^{-n} can be expressed in terms of these numbers leading to complicated correction expressions. Related higher order inertias are used, for example, in [**SH83**].

6.2. Adimensionalization

In order to simplify the algebraic manipulation of expressions that will appear we will choose, following [**Bec97**] [**WMK91**], the following physical units:

$$m = \overline{m}_2, \quad l = \sqrt{\frac{tr(\overline{\mathbf{I}})}{\overline{m}_2}}, \quad t = \sqrt{\frac{l^3}{\overline{Gm_1}}}$$

The dimensional physical values (noted here with an overbar) are obtained following the transformations:

$$\overline{\mathbf{I}} = m^1 l^2 \mathbf{I}, \overline{R} = l^1 R, ...$$

according to the dimensions of the magnitude to be converted.

Recall from the last section that m_1 represents the mass of the primary, m_2 the mass of the secondary body, ...

In this units the trace of the dimensionless inertia tensor has always an unity value:

$$\mathrm{tr}\overline{\mathbf{I}} = l^2 m \; \mathrm{tr}\mathbf{I} = \mathrm{tr}\overline{\mathbf{I}} \; \mathrm{tr}\mathbf{I} \implies \mathrm{tr}\mathbf{I} = 1 \tag{18}$$

The second order approximation of the potential (17) can be transformed into:

$$\begin{aligned}
\overline{V}_2 &= -\frac{\overline{Gm_1 m_2}}{\overline{R}} - \frac{\overline{Gm_1}\mathrm{tr}(\overline{\mathbf{I}})}{2\overline{R}^3} - \frac{3\overline{Gm_1}}{2\overline{R}^5}\overline{\mathbf{R}} \cdot \overline{\mathbf{IR}} \\
&= -\frac{\overline{Gm_1}}{R} m l^{-1} - \frac{\overline{Gm_1}}{2R^3} m l^{-1} - \frac{3\overline{Gm_1}}{2R^5}\mathbf{R} \cdot \mathbf{IR} m l^{-1} \\
&= \left(-\frac{1}{R} - \frac{1}{2R^3} - \frac{3}{2R^5}\mathbf{R} \cdot \mathbf{IR}\right) \overline{Gm_1} m l^{-1} \\
&= \left(-\frac{1}{R} - \frac{1}{2R^3} - \frac{3}{2R^5}\mathbf{R} \cdot \mathbf{IR}\right) m l^2 t^{-2}
\end{aligned}$$

Thus, the non-dimensional version of the second-order potential we will use in next section is:

$$V_2(R) = -\frac{1}{R} - \frac{1}{2R^3} + \frac{3\mathbf{R} \cdot \mathbf{IR}}{2R^5}$$

once principal axes have been chosen this formula only depends on two parameters (for example I_1, I_2 because $I_3 = 1 - I_1 - I_2$), whereas (17) depends on many more constants in a non-trivial way.

Following the same procedure the kinetic energy term gets transformed to a simpler expression:

$$\overline{K} = \frac{1}{2}\overline{\mathbf{\Omega}} \cdot \overline{\mathbf{I\Omega}} + \frac{1}{2}\overline{m_2}\;\overline{\mathbf{v}} \cdot \overline{\mathbf{v}} = \frac{1}{2}\mathbf{\Omega} \cdot \mathbf{I\Omega} m l^2 t^{-2} + \frac{1}{2}\mathbf{v} \cdot \mathbf{v} m l^2 t^{-2} = \left(\frac{1}{2}\mathbf{\Omega} \cdot \mathbf{I\Omega} + \frac{1}{2}\mathbf{v} \cdot \mathbf{v}\right) m l^2 t^{-2}$$

The adimensionalization process is bijective, as an example take the adimensional expression we will derive in the next chapter:

$$\xi^2 = \frac{1}{2}\frac{2R^2 + 3 - 9I_2}{R^5}$$

gets transformed into:

$$\begin{aligned}
\overline{\xi}^2 t^2 &= \frac{1}{2}\frac{2\overline{R}^2 l^{-2} + 3 - 9\overline{I}_2 l^{-2} m^{-1}}{\overline{R}^5 l^{-5}} \\
\overline{\xi}^2 \frac{l^3}{\overline{Gm_1}} &= \frac{1}{2}\frac{2\overline{R}^2 + 3l^2 - 9\overline{I}_2 m^{-1}}{\overline{R}^5 l^{-3}} \\
\overline{\xi}^2 \frac{1}{\overline{Gm_1}} &= \frac{1}{2}\frac{2\overline{R}^2 + 3(\overline{I}_1 + \overline{I}_2 + \overline{I}_3)m^{-1} - 9\overline{I}_2 m^{-1}}{\overline{R}^5} \\
\overline{\xi}^2 &= \frac{\overline{Gm_1}}{\overline{R}^3}\left(1 + \frac{3(\overline{I}_1 - 2\overline{I}_2 + \overline{I}_3)}{2\overline{m}_2\overline{R}^2}\right)
\end{aligned}$$

which is the dimensional version of Kepler's frequency formula with a correction term [**WKM90**].

As an example, the next table contains the distance to the centre of attraction ($|R|$) for several satellites in the previously introduced units. Note that for all artificial satellites the value is large.

Satellite	Orbital radius ($\lvert R\rvert$)
GPS system	$1.7 \cdot 10^7$
International Space Station	$2.1 \cdot 10^5$
Moon	64.2

6.3. Configuration Space

Let $Q = SO(3) \times \mathbb{R}^3$ be the configuration manifold. The product coordinates in this manifold will be denoted (B, r), $B \in SO(3)$, $r \in \mathbb{R}^3$. B is the orthogonal transformation that maps the rigid body from a reference configuration to the orientation it has in the space. The vector between mass centres will be denoted by r, whereas in the previous section it was called $\mathbf{R}$.

The kinetic energy after the adimensionalization process described in the previous section is given by the following metric on Q:

$$\langle\langle (B\widehat{\Omega_1}, \dot{r}_1), (B\widehat{\Omega_2}, \dot{r}_2) \rangle\rangle = \Omega_1 \cdot \mathbf{I}\Omega_2 + \dot{r}_1 \cdot \dot{r}_2$$

where $\cdot$ represents the usual scalar product in $\mathbb{R}^3$. The hat notation is explained in Section A.1.

The vector r can be expressed in body coordinates, that is $R = B^T r$. We will use capital letters to represent the magnitudes in body coordinates. In (B, R) coordinates using the left trivialization of $SO(3)$ the metric has matrix:

$$\langle\langle , \rangle\rangle = \begin{bmatrix} \mathbf{I} - \hat{R}\hat{R} & \hat{R} \\ -\hat{R} & Id \end{bmatrix} \tag{19}$$

Using the block inversion matrix formula[1] the metric induced on T^*Q has the matrix expression:

[1]Given a matrix of the form

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where A, D are square invertible matrices then the inverse can be written as:

$$M^{-1} = \begin{bmatrix} A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{bmatrix} \tag{20}$$

if $D - CA^{-1}B$ is invertible.

$$\langle\!\langle , \rangle\!\rangle^{-1} = \begin{bmatrix} \mathbf{I}^{-1} & -\mathbf{I}^{-1}\hat{R} \\ \hat{R}\mathbf{I}^{-1} & Id - \hat{R}\mathbf{I}^{-1}\hat{R} \end{bmatrix} \tag{21}$$

Given a tangent vector $v = (\delta\theta, \delta R) \in T_{(B,R)}Q$ the Legendre transformation is given by:

$$\mathbb{F}L(v) = \langle\!\langle \cdot, v \rangle\!\rangle = \begin{bmatrix} \mathbf{I} - \hat{R}\hat{R} & \hat{R} \\ -\hat{R} & Id \end{bmatrix} \begin{pmatrix} \delta\theta \\ \delta R \end{pmatrix} = \begin{pmatrix} (\mathbf{I} - \hat{R}\hat{R})\delta\theta + \hat{R}\delta R \\ -\hat{R}\delta\theta + \delta R \end{pmatrix} = \begin{pmatrix} \Pi \\ P \end{pmatrix} \tag{22}$$

And the inverse transformation is:

$$\mathbb{F}L^{-1}(\Pi, P) = \begin{bmatrix} \mathbf{I}^{-1} & -\mathbf{I}^{-1}\hat{R} \\ \hat{R}\mathbf{I}^{-1} & Id - \hat{R}\mathbf{I}^{-1}\hat{R} \end{bmatrix} \begin{pmatrix} \Pi \\ P \end{pmatrix} = \begin{pmatrix} \mathbf{I}^{-1}(\Pi - \hat{R}P) \\ \hat{R}\mathbf{I}^{-1}\Pi + P - \hat{R}\mathbf{I}^{-1}\hat{R}P \end{pmatrix} = \begin{pmatrix} \delta\theta \\ \delta R \end{pmatrix} \tag{23}$$

Remark 6.1. We will always trivialize TQ using the left trivialization of $SO(3)$ and the linear structure of $\mathbb{R}^3$. So $(\delta\theta, \delta R)$ will be the equivalence class of the path:

$$B_\varepsilon = B\exp(t\widehat{\delta\theta}), \quad R_\varepsilon = R + \varepsilon\delta R$$

Tangent vectors at $(B, R) \in Q$ will be represented as a column vector of 6 components with $\delta\theta$ as the first three components and δR as the other three.

Remark 6.2. The gravitational potential can be expressed also as:

$$V(B, r) = \int_{\mathcal{B}} \frac{\rho(Q)dV(Q)}{|r + BQ|} = \int_{\mathcal{B}} \frac{\rho(Q)dV(Q)}{|R + Q|} = V(R)$$

even for the exact potential, the potential depends only on the variable R, not on B, this is one for choosing this body-fixed coordinates (B, R)

6.4. $SO(3)$ action.

In the configuration space Q described in the previous section there is one natural action of $SO(3)$, given by the left action on the $SO(3)$ factor and the linear representation of $SO(3)$ in $\mathbb{R}^3$, that is:

$$M \cdot (B, r) = (MB, Mr)$$

or in body coordinates:

$$M \cdot (B, R) = (MB, R)$$

The action is free because:

$$M \cdot (B, R) = M' \cdot (B, R) \implies MB = M'B \implies M = M'$$

The fundamental fields of this action are:

$$\xi_Q(B, r) = \left.\frac{d}{dt}\right|_{t=0} (\exp(t\hat{\xi})B, \exp(t\hat{\xi})r) = (\hat{\xi}B, \hat{\xi}r) = (B\widehat{B^{-1}\xi}, \hat{\xi}r) \tag{24}$$

and in the body coordinates and using left trivialization of $TSO(3)$:

$$\delta\theta = B^T\xi, \quad \delta R = 0 \tag{25}$$

.

The locked inertia tensor is:

$$\langle\!\langle \xi_Q, \eta_Q \rangle\!\rangle(B,R) = \xi \cdot B(\mathbf{I} - \hat{R}\hat{R})B^T\eta \implies \mathbb{I} = B(\mathbf{I} - \hat{R}\hat{R})B^T \tag{26}$$

The momentum map is:

$$\langle \mathbf{J}(z), \xi \rangle = \langle\!\langle \xi_Q, \mathbb{F}L^{-1}(z) \rangle\!\rangle = \Pi \cdot B^T\xi \implies \mathbf{J}(z) = B\Pi$$

Physically the conservation of $\mathbf{J}$ is the conservation of the total angular momentum, that is the sum of the angular momentum due to orbital motion and the spinning motion.

Finally a expression for the mechanical form and its tangent can be computed:

$$\langle \alpha_\mu(q), v \rangle = \langle \mu, \alpha(q,v) \rangle = \langle \mu, \mathbb{I}^{-1}\mathbf{J}\mathbb{F}L(q,v) \rangle = \langle\!\langle (\mathbb{I}^{-1}\mu)_Q, v \rangle\!\rangle$$

that is,

$$\begin{pmatrix} \Pi \\ P \end{pmatrix} = \begin{bmatrix} \mathbf{I} - \hat{R}\hat{R} & \hat{R} \\ -\hat{R} & Id \end{bmatrix} \begin{pmatrix} B^T\mathbb{I}^{-1}\mu \\ 0 \end{pmatrix} = \begin{pmatrix} B^T\mu \\ -\hat{R}B^T\mathbb{I}^{-1}\mu \end{pmatrix}$$

Using the chain rule, and some vector product identities, $T\alpha_\mu : T_qQ \to T_zT_q^*Q$ is given by:

$$\begin{pmatrix} \delta\Pi \\ \delta P \end{pmatrix} = \begin{bmatrix} \widehat{B^T\mu} & 0 \\ -\hat{R}B^T\mathbb{I}^{-1}B\widehat{B^T\mu} & \hat{R}B^T\mathbb{I}^{-1}(-BR^TB^T\xi - BR\xi^TB + 2\xi R^T) + \widehat{B^T\xi} \end{bmatrix} \begin{pmatrix} \delta\theta \\ \delta R \end{pmatrix} \tag{27}$$

6.4.1. Relative equilibria. It's interesting to describe what is the trajectory in space of a rigid body subject to a relative equilibrium.

In spatial coordinates (B, r) the evolution with angular velocity ξ is given by: $B(t) = \exp(t\hat{\xi}) \cdot B_0$ and $r(t) = \exp(t\hat{\xi}) \cdot r_0$.

Remark 6.3. Relative equilibria motion forces that the rigid body rotates as quickly as it describes an orbit, in more astronomical terms, a satellite in relative equilibrium has the "day" and the "year" of the same period, as the moon does.

Obviously $|r(t)|^2 = |\exp(t\hat{\xi})r_0|^2 = |r_0|^2 \quad \forall t$, but also:

$$\left| r(t) - \frac{(r(t)\cdot\xi)\xi}{|\xi|^2} \right|^2 = |r(t)|^2 - \frac{r(t)\cdot\xi}{|\xi|^2} = |r_0|^2 - \frac{r_0 \cdot \exp(-t\hat{\xi})\xi}{|\xi|^2} = |r_0|^2 - \frac{r_0\cdot\xi}{|\xi|^2}$$

that is, the vector r describes a right circular cone with vertex at the centre of attraction O and centre at $C = \frac{r_0\cdot\xi}{|\xi|^2}\xi$.

If $\xi \cdot r_0 \neq 0$ the orbit centre C does not coincide with the centre of attraction. This case is going to be called oblique equilibria.

Definition 6.4. In the case of $SO(3)$ symmetry, a relative equilibrium such that $r \cdot \xi = 0$ is called *orthogonal equilibrium*, otherwise it will be called *oblique equilibrium*.

For orthogonal equilibria the centre of mass the rigid body describes a circle whose centre coincides with the centre of attraction, (i.e. $C = O$) but in the oblique equilibria $C \neq O$. The orbit is a circle but the centre of attraction O lies in another plane.

Remark 6.5. In the literature several different names are given for the orthogonal/oblique dichotomy, in [**OT04**] they are called coplanar/non-coplanar motions, in [**WKM90**] great-circle and non-great circle motions.

[**Ste69**] defines the offset angle $\varkappa$ as the angle of the orbit with respect to the classical plane, that is $\sin\varkappa = \frac{r_0\cdot\xi}{|\xi||r_0|}$. Although one may think that the only physically meaningful value for $\varkappa$ is $\varkappa = 0$ this is not always the case. The coupling between spinning and orbital motion can make $\varkappa \neq 0$.

6.5. $SO(3)\times SO(3)$ action

In some potential approximations (zeroth order) the system has more symmetries that the ones considered before. The system can be also symmetric to body rotations leaving the space ($\mathbb{R}^3$ component of Q) fixed.

We will now consider the action of the direct product $G = SO(3)\times SO(3)$ in Q acting on the left on the first factor and with the standard linear representation on the second, that is:

$$(M,N)\cdot(B,r) = (MB, Nr)$$

The $SO(3)$ action in the previous section is the action of the diagonal subgroup of $SO(3)\times SO(3)$.

The Lie algebra will be identified with the left trivialization, giving $\mathfrak{g} = \mathbb{R}^3\oplus\mathbb{R}^3$. The adjoint action of $(\xi,\eta)\in\mathfrak{g}$ is given by the linear endomorphism:

$$\begin{bmatrix}\widehat{\xi} & 0\\ 0 & \widehat{\eta}\end{bmatrix} : \mathfrak{g}\to\mathfrak{g}$$

In spatial coordinates (B,r), the fundamental fields of the action are (a computation similar to (24)):

$$(\xi,\eta)_Q(B,r) = (B^T\xi, \widehat{\eta}r)$$

Recall that the metric matrix in this coordinates is:

$$\langle\!\langle , \rangle\!\rangle = \begin{bmatrix}\mathbf{I} & 0\\ 0 & Id\end{bmatrix}$$

and the Legendre transformation is $\mathbb{F}L(\delta\theta,\delta r) = (\mathbf{I}\delta\theta,\delta r) = (\Pi, p)$ The momentum map is:

$$\langle\mathbf{J}(z),(\xi,\eta)\rangle = \langle\!\langle \mathbb{F}L^{-1}(\Pi,p),(\xi,\eta)_Q\rangle\!\rangle = B^T\xi\cdot\Pi + \hat{\eta}r\cdot p \implies \mathbf{J}(z) = (B\Pi, \hat{r}p)$$

The first component of the momentum map is the spinning angular momentum and the second one is the orbital angular momentum.

In this case the action is not free

$$(M,N)\cdot(B,r) = (B,r) \implies M = Id, \quad N = e^{t\hat{r}}$$

the isotropy subgroup of any point is isomorphic to S^1.

But the cotangent lifted action to T^*Q:

$$(M,N)(B,r;\Pi,p) = (MB, Nr;\Pi, Np)$$

is free for almost all the points. The fundamental field is: $(B^T\xi, \hat{\eta}r, 0, \hat{\eta}p)$, if r and p are not aligned the isotropy of (B,r,Π,p) is zero.

6.6. $SO(3) \times S^1$ action

If the body possesses an axis of symmetry the symmetry group can be enlarged. Take a body fixed orthogonal frame such that the first vector corresponds to the symmetry axes, in this basis the rigid body inertia tensor will be:

$$\mathbf{I} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_2 \end{bmatrix}$$

and by (18) $I_1 + 2I_2 = 1$

Define an action of the direct product $SO(3) \times S^1 = G$ by:

$$(M, R_\theta)(B, r) = (MBR_\theta, Mr)$$

where

$$R_\theta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

rotates the rigid body along it's axis an angle θ. It is a left action because of commutativity of S^1:

$$(M_1M_2, R_{\theta_1}R_{\theta_2})(B, r) = (M_1M_2BR_{\theta_1}R_{\theta_2}, M_1M_2r) = (M_1M_2BR_{\theta_2}R_{\theta_1}, M_1M_2r) =$$
$$= (M_1, R_{\theta_1})(M_2BR_{\theta_2}, M_2r) = (M_1, R_{\theta_1})((M_2, R_{\theta_2})(B, r))$$

The Lie algebra of G can be identified with the direct sum: $\mathbb{R}^3 \oplus \mathbb{R}$, such that:

$$\exp(t(\xi, \eta))(B, r) = (\exp(t\hat{\xi})B\exp(t\eta\hat{e_1}), \exp(t\hat{\xi})r)$$

the adjoint action of $(\xi, \eta) \in \mathfrak{g}$ is given by:

$$\begin{bmatrix} \widehat{\xi} & 0 \\ 0 & 0 \end{bmatrix} : \mathfrak{g} \to \mathfrak{g}$$

In body coordinates the action is:

$$(M, R_\theta)(B, R) = (MBR_\theta, R_{-\theta}R)$$

differentiating this expression the fundamental field is given by:

$$(\xi, \eta)_Q(B, R) = (B^T\xi + \eta e_1, -\eta\widehat{e_1}R) = (\delta\theta, \delta R)$$

The fundamental field is in coordinates associated to (B, r):

$$(\xi, \eta)_Q = (B(\widehat{B^T\xi_1 + \eta e_1}), \widehat{\xi}r)$$

In (B, R) coordinates it has the expression:

$$(\xi, \eta)_Q = (B^T\xi + \eta e_1, -\eta\widehat{e_1}R)$$

applying this vectors to the metric (19), the locked inertia tensor is:

$$\mathbb{I} = \begin{bmatrix} B(\mathbf{I} - \hat{R}\hat{R})B^T & Be_1I_1 \\ (Be_1I_1)^T & I_1 \end{bmatrix}$$

Chapter 7
Order zero approximation

The first approach to the problem is truncating the potential function to the dominant term and solve the equations of motion with this approximate Lagrangian.

The problem of this approach is that the attitude and orbital motion are uncoupled, more terms are needed to get realistic models. Nevertheless this is a easy model for extremely large orbits, where the “gravity-gradient torque” is very small.

With this approximation the problem has $SO(3) \times SO(3)$ symmetry, but first of all we will reduce only by $SO(3)$, as is done in [**WKM90**]. With this reduction the relative equilibria can be easily compared with those in the order two model. In following sections the full reduction and stability analysis is done.

7.1. $SO(3)$ augmented potential

According to Proposition 3.17 the relative equilibria are the critical points of the augmented potential functional:

$$V_\xi(q) = V(q) - \frac{1}{2}\langle\!\langle \xi_Q(q), \xi_Q(q) \rangle\!\rangle$$

in this case, using (19) and (25):

$$V_\xi(B, R) = V(R) - \frac{1}{2}\langle \xi, \mathbb{I}(B, R)\xi \rangle = V(R) - \frac{1}{2}\xi \cdot B(\mathbf{I} - \hat{R}\hat{R})B^{-1}\xi$$

As it was said above, taking variations such that: $B_t = B\exp(t\widehat{\delta\theta})$ y $R_t = R + t\delta R$ and using the chain rule:

$$\begin{aligned}
\delta V_\xi(B, R) &= \nabla_R V(R) \cdot \delta R - \frac{1}{2}\xi \cdot \Big(B\widehat{\delta\theta} \left(\mathbf{I} - \widehat{R}\widehat{R} \right) B^T - B \left(\mathbf{I} - \widehat{R}\widehat{R} \right) \widehat{\delta\theta} B^T \\
&\quad - B\widehat{\delta R}\widehat{R}B^T - B\widehat{R}\widehat{\delta R}B^T \Big)\xi \\
&= \frac{1}{2}\xi \cdot B(\left(\mathbf{I} - \widehat{R}\widehat{R} \right) B^T\xi)\hat{}\,\delta\theta - \frac{1}{2}\xi \cdot B \left(\mathbf{I} - \widehat{R}\widehat{R} \right) \widehat{B^T\xi}\delta\theta \\
&\quad + \nabla_R V(R) \cdot \delta R + \frac{1}{2}\xi \cdot B(\widehat{\delta R}\widehat{R} + \widehat{R}\widehat{\delta R})B^T\xi
\end{aligned}$$

where $\nabla_R V(R)$ is the vector valued function such that $\delta V = \nabla_R V(R) \cdot \delta R$ for all paths with tangent vector δR.

Using $a \times (b \times c) = b(a \cdot c) - c(a \cdot b) = (ba^T - b^T a Id)c$, that is, $\widehat{a}\widehat{b} = ba^T - b^T a Id$:

$$
\begin{aligned}
\delta V_\xi(B,R) &= \widehat{B}^T\xi \left(\mathbf{I} - \widehat{R}\widehat{R}\right) B^T\xi \cdot \delta\theta + \nabla_R V(R) \cdot \delta R \\
&+ \frac{1}{2}\xi \cdot B(R\xi^T B - 2(B^T\xi)R^T + R^T B^T \xi)\delta R \\
&= \widehat{B^T\xi}(\mathbf{I} - \widehat{R}\widehat{R})B^T\xi \cdot \delta\theta \\
&+ \left(\nabla_R V(R) + B^T\xi(R^T B^T \xi) - R\xi^T\xi\right) \cdot \delta R
\end{aligned} \tag{28}
$$

The relative equilibria conditions are:

$$\widehat{B^T\xi}(\mathbf{I} - \widehat{R}\widehat{R})B^T\xi = 0 \tag{29a}$$

$$\nabla_R V(R) + B^T\xi(R^T B^T\xi) - R\xi^T\xi = 0 \tag{29b}$$

Using Proposition 3.16 we can assume that $B = Id$, so the relative equilibria conditions are simplified to:

$$\widehat{\xi}(\mathbf{I} - RR^T + |R|^2)\xi = 0 \tag{30a}$$

$$\nabla_R V(R) + \xi(R \cdot \xi) - R|\xi|^2 = 0 \tag{30b}$$

Remark 7.1. In the relative equilibria with velocity ξ, the momentum map satisfies $\mu = \mathbf{J}(z_{eq}) = \mathbb{F}L(\xi_Q(q_{eq})) = \mathbb{I}(q_{eq})\xi$, in our case

$$\mu = (\mathbf{I} - \hat{R}\hat{R})\xi$$

thus the condition (30a) is the condition is $-\hat{\xi}\mu = \mathrm{ad}^*_\xi \mu = 0$ which is also stated on the second part of Proposition 3.16

The point in Q is determined by the conditions (30), the associated momenta are given by Proposition 3.17 ($p = \mathbb{F}L(\xi_Q)$), that is, in the relative equilibria:

$$\Pi = (\mathbf{I} - \hat{R}\hat{R})\xi, \quad P = -\hat{R}\xi$$

7.2. Order zero approximation, $SO(3)$ symmetry

If we take as the potential function the first order approximation:

$$V(R) = \frac{-1}{|R|}$$

the gradient of this potential gives: $\nabla_R V(R) = \frac{R}{|R|^3}$, so the relative equilibria conditions (30) are:

$$\widehat{\xi}(\mathbf{I} - RR^T + |R|^2)\xi = 0 \tag{31a}$$

$$R\left(\frac{1}{|R|^3} - |\xi|^2\right) + \xi(\xi \cdot R) = 0 \tag{31b}$$

If R and ξ are independent vectors the last equation decouples into two independent scalar equations. The ξ-component imposes orthogonality between R and ξ, the R-component establishes a relationship between the lengths of the two vectors: $|\xi| = |R|^{-3/2}$, dimensionalizing this expression:

$$|\bar{\xi}|^2 = |\xi|^2 t^{-2} = \frac{\overline{GM}}{l^3}\frac{1}{|R|^3} = \frac{\overline{GM}}{\overline{R}^3} \tag{32}$$

which is the Kepler's frequency formula.

The condition (31a) is, in this case, $\widehat{\xi}(\mathbf{I} + |R|^2)\xi = 0$, this forces ξ to be an eigenvector of the inertia tensor.

If we now suppose that ξ and R are linearly dependent, for example $\xi = \rho R$ then (31b) is $\frac{R}{|R|^3} = 0$; only possible if $R = 0$, which is not a valid solution because the potential becomes singular there.

Thus for the zeroth order approximation for the potential the relative equilibria are the ones in which the angular velocity ξ is an eigenvector of the inertia matrix $\mathbf{I}$ and the module of ξ and R are related through the Kepler frequency formula.

The condition is equivalent to say that the orbital plane is the same as a principal plane of the body. The situation is, in this case, like in the following figure.

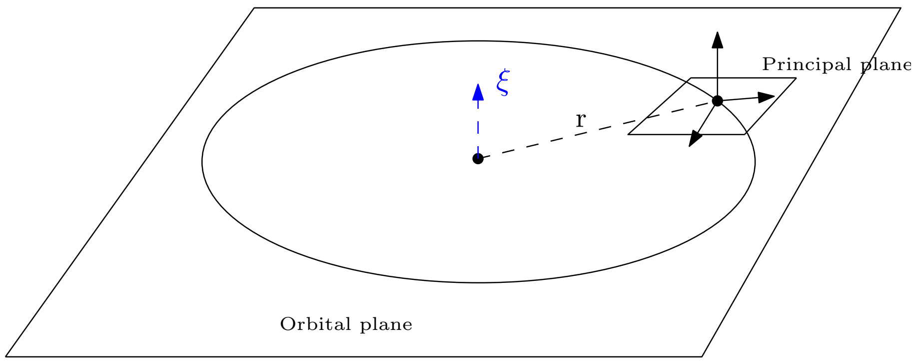


FIG. 1. Relative equilibria for the zeroth order approximation

7.3. $SO(3) \times SO(3)$ symmetry

In the order zero potential we have the expression $V = \frac{1}{|r|}$, so any rotation of the rigid body leaves the potential invariant. The system not only possesses a $SO(3)$ it has a $SO(3) \times SO(3)$ symmetry. The Hamiltonian is invariant by rotations of the space, and by rotations of the rigid body, this corresponds to the $SO(3) \times SO(3)$ symmetry introduced in Section 6.5.

The action of the group is not free, the isotropy of each point is isomorphic to S^1, we can't apply the reduced energy momentum, but we can work with the classical energy momentum method 4.8.

The augmented Hamiltonian will be:

$$H_{\xi,\eta} = \frac{1}{2}\Pi \cdot \mathbf{I}\Pi - B\Pi \cdot \xi + \frac{-1}{|r|} + \frac{p \cdot p}{2} - \hat{r}p \cdot \eta = H^{\mathrm{rig}}_{\xi}(B, \Pi) + H^{\mathrm{kepl}}_{\eta}(r, p)$$

it is the sum of two terms, one of which only depends on the rigid-body variables (B, Π) whereas the second one only depends on orbital variables (r, p). Really we are dealing with two uncoupled systems, a free rigid-body (see Appendix A) and a particle following the Kepler potential (see Section 5.1).

Proposition 7.2. *For the order zero approximation the relative equilibria are:*

- *The centre of mass of the rigid-body describes a circular orbit of radius r with angular velocity satisfying* $|\eta| = |r|^{-\frac{3}{2}}, \quad \eta \cdot r = 0.$

- *The rigid body rotates with angular velocity ξ along one of the principal axes and the direction of rotation doesn't change along the orbit. The angular velocity always "points" to the same direction in inertial space*

Proposition 7.3. *Of the relative equilibria for the order zero approximation, if the angular velocity of the rigid body ξ is along the major of minor principal axis the equilibrium is formally stable. Otherwise the rotation is around the middle axis and the equilibrium is unstable.*

In this case $\mathfrak{g}_\mu = \mathbb{R}\xi \oplus \mathbb{R}\eta$, there are two drift directions following the interpretation of [**Pat92**]. One of them drifts the equilibrium along the orbit, whereas the other rotates the rigid body.

7.4. Instability of the order zero model

In [**WKM90**] it is proved that all the relative equilibria for the potential of order zero are unstable. This contradiction with the conclusions of the previous section is explained by the use of different groups of symmetry. In [**WKM90**] only the $SO(3)$ reduction is used, so the relative equilibria in Section 7.2 were found.

The Energy-Cassimir method is used to test the stability but it gives inconclusive results, so the linearisation is computed; a nilpotent block of order two is found leading to linear instability.

The same calculations can be reproduced with the Reduced-Energy Momentum, as we will compute in the next section, adequate basis can be found: (42) (43) and with an analogous calculation as the one for the second order potential, Smale's form is going to be, in the relative equilibrium $B = Id, R = (0, R, 0)$:

$$\begin{bmatrix} \frac{R_2^2-3I_1}{I_1+R_2^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (R_2^2+I_1-I_2)(I_1-I_2) \end{bmatrix}$$

the zero implies nothing and the method is inconclusive.

If the linearisation is considered a nilpotent block of order two appears, leading to linear instability.

The zero in Smale's form is due to the other "drift direction". Recall that with this reduction $\mathfrak{g}_\mu$ is one-dimensional. The nilpotent block introduces a instability that can be explained as a "drift" in another direction.

Chapter 8
Order two approximation

If two terms of the power expansion of the potential are considered a non-trivial coupling between attitude and orbital motion appears.

For large orbits, 24 relative equilibria appear. This equilibria were known to Lagrange. For very small orbits the situation is more complicated, non-orthogonal equilibria can appear.

All possible relative equilibria will be found, the non-orthogonal equilibria are explicitly solved. The reduced energy-momentum method is applied to the orthogonal equilibria. Before interpreting the results the linearisation around this equilibria will be done. Then the study of the stability will lead to the analysis of the parallel family. The two main bifurcation phenomena that appear as the orbit gets smaller are studied.

8.1. Existence of relative equilibria

The relative equilibria are characterized as critical points of the augmented potential (Proposition 3.17) in (29) the first variation of V_ξ with the $SO(3)$ action was computed (28). Using Proposition 3.16 and assuming $B = Id$, the relative equilibria conditions are:

$$\widehat{\xi}(\mathbf{I} - RR^T + |R|^2)\xi = 0 \tag{33a}$$

$$\nabla_R V(R) + \xi(R \cdot \xi) - R|\xi|^2 = 0 \tag{33b}$$

With the order two approximation:

$$V(R) = -\frac{1}{|R|} - \frac{1}{2|R|^3} + \frac{3R \cdot \mathbf{I}R}{2|R|^5}$$

the gradient is:

$$\nabla_R V(R) = \frac{R}{|R|^3} + \frac{3R}{2|R|^5} + \frac{3\mathbf{I}R}{|R|^5} - \frac{15R(R \cdot \mathbf{I}R)}{2|R|^7} \tag{34}$$

the conditions (33) are:

$$(\mathbf{I} - RR^T)\xi = \alpha\xi \implies R(R \cdot \xi) = \mathbf{I}\xi - \alpha\xi \tag{35a}$$

$$\frac{R}{|R|^3} + \frac{3R}{2|R|^5} + \frac{3\mathbf{I}R}{|R|^5} - \frac{15R(R \cdot \mathbf{I}R)}{2|R|^7} = -\xi(R \cdot \xi) + R|\xi|^2 \tag{35b}$$

for some real α.

First of all we are going to suppose that $\xi \cdot R = 0$, this condition imposing the orthogonality of the angular velocity and the radial vector will lead to the orthogonal relative equilibria.

Orthogonal order two approximation. If $\xi \cdot R = 0$ (35b) forces R and $\mathbf{I}R$ to be collinear, thus R has to be an eigenvector of $\mathbf{I}$. On the other hand (30a) forces ξ to be an eigenvector of $\mathbf{I}$ so both ξ and R are different eigenvectors of $\mathbf{I}$.

The lengths of the vectors are related through a "corrected Kepler's formula" similar to (32). If we take basis such that $\mathbf{I}$ is in diagonal form $\mathbf{I} = diag(I_1, I_2, I_3)$ and ξ, R are aligned with the first two eigenvectors of the matrix then (35b) gives:

$$0 = \frac{1}{2}R\frac{2R^4 + 3R^2 - 9R^2 I_2 - 2\xi^2|R|^7}{|R|^7} \implies \xi^2 = \frac{1}{2}\frac{2R^2 + 3 - 9I_2}{|R|^5} = \frac{1}{R^3} + \frac{1}{2}\frac{3 - 9I_2}{|R|^5}$$

The situation here is like in the following picture, where R and ξ are locked to be eigenvectors of the inertia tensor:

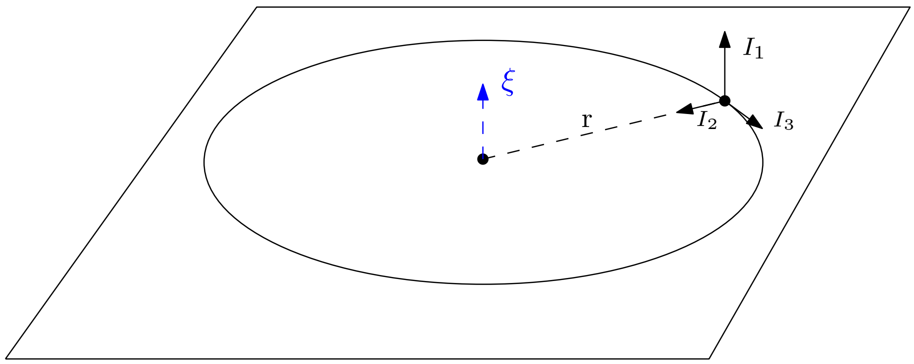


FIG. 1. Relative equilibria for the second order approximation

If we choose a basis such that $\mathbf{I}$ is in diagonal form $\mathbf{I} = diag(I_1, I_2, I_3)$, then we have 3 different axes to choose for ξ, each with 2 orientations. Once ξ is fixed, R has to be in one of the two orthogonal principal axes and with two orientations each, so we have in total $3 \cdot 2 \cdot 2 \cdot 2 = 24$ orthogonal relative equilibria. This equilibria were known to Lagrange.

Remark 8.1. The angular velocity expression:

$$\xi^2 = \frac{1}{|R|^3} + \frac{1}{2}\frac{3 - 9I_2}{|R|^5}$$

implies that if $I_2 > \frac{1}{3}$ this equilibria can't exist for R small enough.

Large radii. We are going to see that the orthogonal assumption for this model is reasonable and, in fact correct, for large radii.

The condition (30a) can be written as $(\mathbf{I} - \widehat{R}\widehat{R})\xi = \alpha\xi$ for some suitable scalar $\alpha > 0$ which is positive because of the positive definiteness of the locked inertia tensor. Taking the dot product of this expression with R we get the relation: $R \cdot \mathbf{I}\xi = \alpha R \cdot \xi$.

On the other hand if we take the dot product of (35b) with ξ:

$$(\xi \cdot R)\left(\frac{1}{|R|^3} + \frac{3}{2|R|^5} - \frac{15(R \cdot \mathbf{I}R)}{2|R|^7}\right) + \frac{3(\xi \cdot \mathbf{I}R)}{|R|^5} = 0$$

using α we can factor it:

$$(\xi \cdot R)\left(\frac{1}{|R|^3} + \frac{3}{2|R|^5} - \frac{15(R \cdot \mathbf{I}R)}{2|R|^7} + \frac{3\alpha}{|R|^5}\right) = 0$$

either we have $\xi \cdot R = 0$ or:

$$\left(1 + \frac{3}{2|R|^2} - \frac{15(R \cdot \mathbf{I}R)}{2|R|^4} + \frac{3\alpha}{|R|^2}\right) = 0$$

but

$$\left(1 + \frac{3}{2|R|^2} - \frac{15(R \cdot \mathbf{I}R)}{2|R|^4} + \frac{3\alpha}{|R|^2}\right) > \left(1 + \frac{3}{2|R|^2} - \frac{15(R \cdot \mathbf{I}R)}{2|R|^4}\right) > \left(1 + \frac{3}{2|R|^2} - \frac{15 I_{max}}{2|R|^2}\right) >$$
$$> \left(1 + \frac{3}{2|R|^2} - \frac{15}{4|R|^2}\right) = \left(1 - \frac{9}{4|R|^2}\right) \geq 0 \tag{36}$$

Therefore if $|R| \geq 3/2$ in a relative equilibria then we have $\xi \cdot R = 0$, thus for large radii there are only orthogonal relative equilibria.

Proposition 8.2. *In the second order model for a fixed orbital radius $|R| > \frac{3}{2}$ there are only 24 different relative equilibria: the Lagrangian equilibria. R and ξ are perpendicular and aligned with two principal axis of the body. The orbital frequency is fixed by the corrected Kepler's formula:*

$$|\xi|^2 = \frac{1}{|R|^3} + \frac{1}{2}\frac{3 - 9I_R}{|R|^5}$$

where $I_R = (R \cdot \mathbf{I}R)/|R|^2$.

The non existence of non-orthogonal relative equilibria for large radii was first shown in [**WKM90**] and improved to this inequality in [**Bec97**]

For the second order approximation, the relative equilibria conditions (30) can be written as:

$$(\mathbf{I} - RR^T)\xi = \alpha\xi \implies R(R \cdot \xi) = \mathbf{I}\xi - \alpha\xi \tag{37a}$$
$$\frac{R}{|R|^3} + \frac{3R}{2|R|^5} + \frac{3\mathbf{I}R}{|R|^5} - \frac{15R(R \cdot \mathbf{I}R)}{2|R|^7} = -\xi(R \cdot \xi) + R|\xi|^2 \tag{37b}$$

for some real α.

Parallel equilibria. If $\xi \cdot R \neq 0$ and R is aligned with a principal axes of $\mathbf{I}$ then $\nabla_R V(R)$ is aligned with R so ξ has also to be aligned with R and the second equation in this case is $\nabla_R V = 0$. The two conditions are in this case ($R = \beta\xi$):

$$R(R \cdot \xi) = \mathbf{I}\xi - \alpha\xi \implies \mathbf{I}\xi = \tilde{\alpha}\xi \tag{38a}$$
$$\frac{R}{|R|^3} + \frac{3R}{2|R|^5} + \frac{3\mathbf{I}R}{|R|^5} - \frac{15R(R \cdot \mathbf{I}R)}{2|R|^7} = 0 \tag{38b}$$

Without loss of generality we can take basis such that R is aligned with the second axis, i.e. $R = (0, R_2, 0)$ then the second equation determines R_2 completely and $\xi = (0, \xi_2, 0)$ can assume any value.

The physical reason for the existence of this equilibria is due only to the approximation used. Near 0 the second order potential can be repulsive.

In the set of points defined by

$$1 + \frac{3}{2|R|^2} - \frac{9R \cdot \mathbf{I}R}{2|R|^4} = 0$$

is where $\nabla_R V = 0$, a rigid body in that point will be in static equilibrium. And also it can spin along one of it's axis without moving away from that equilibrium point.

Oblique equilibria. From the first equation of (35), if the equilibria is not orthogonal, R is in the plane spanned by ξ and $\mathbf{I}\xi$. Substituting this in the second condition it gives a linear relationship between $\xi, \mathbf{I}\xi, \mathbf{I}^2\xi$. If we take a basis aligned with the principal axes, the following matrix must be singular:

$$\begin{bmatrix} \xi_1 & I_1\xi_1 & I_1^2\xi_1 \\ \xi_2 & I_2\xi_2 & I_2^2\xi_2 \\ \xi_3 & I_3\xi_3 & I_3^2\xi_3 \end{bmatrix} \tag{39}$$

computing the determinant it vanishes only if two I_i are equal (axisymmetric case) or if one component of ξ is zero. The axisymmetric case will be treated in the next chapter.

Without loss of generality we will suppose that $\xi_3 = 0$, of course this implies $R_3 = 0$ from the first equation. We have reduced the dimensionality of the system.

Suppose that $R = (r\cos\psi, r\sin\psi, 0)$. The gradient of the potential at the equilibrium point satisfies:

$$\xi \cdot \nabla_R V(R) = 0$$

(because of the second condition), it also satisfies that the third component of $\nabla_R V(R)$ is zero. As in a plane there is only one orthogonal vector up to scaling: $\nabla_R V(R) = k(-\xi_2, \xi_1)$. Denote $\nabla_R V(R) = (g_1, g_2, 0)$. Note that g_1, g_2 are functions of r, ψ only.

On the other hand the first condition is equivalent to:

$$\det\left[\left(\begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix} - RR^T\right)\begin{pmatrix}\xi_1 \\ \xi_2\end{pmatrix}; \begin{pmatrix}\xi_1 \\ \xi_2\end{pmatrix}\right] = 0$$

using linearity this is the same as:

$$\det\left[\left(\begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix} - RR^T\right)\begin{pmatrix}g_2 \\ -g_1\end{pmatrix}; \begin{pmatrix}g_2 \\ -g_1\end{pmatrix}\right] = 0$$

this equation only involves ψ, r, in expanded form it is:

$$(A_4\cos^4\psi + A_2\cos^2\psi + A_0)(\cos\psi)(I_1 - I_2)(\sin\psi) = 0$$

where:

$$\begin{aligned} A_4 &= 225(I_2 - I_1)^2 \\ A_2 &= 6(I_2 - I_1)(19r^2 + 15I_1 - 60I_2 + 15) \\ A_0 &= (2r^2 - 9I_2 + 3)(8r^2 + 6I_1 - 15I_2 + 3) \end{aligned}$$

only the first factor can be zero, otherwise we will be in the axisymmetric or orthogonal one. This factor can be rewritten as

$$A_4S^2 + A_2S + A_0 = 0$$

where $S = \cos^2\psi$.

The second equation becomes:

$$\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \frac{1}{k^2}\begin{bmatrix} (g_1^2 + g_2^2)r\cos\psi - (g_2r\cos\psi - g_1r\sin\psi)g_2 \\ (g_1^2 + g_2^2)r\sin\psi + (g_2r\cos\psi - g_1r\sin\psi)g_1 \end{bmatrix} = \frac{1}{k^2}\begin{bmatrix} g_1^2r\cos\psi + g_1g_2r\sin\psi \\ g_2^2r\sin\psi + g_1g_2r\cos\psi \end{bmatrix}$$

That is, the following equation has to be solved for k:

$$k^2 = g_1 r \cos\psi + g_2 r \sin\psi$$

but this is the same as requiring[1] $\nabla_R V(R) \cdot R > 0$, which is:

$$\frac{1}{|R|} + \frac{3}{2|R|^3} + \frac{3R \cdot \mathbf{I}R}{|R|^5} - \frac{15(R \cdot \mathbf{I}R)}{2|R|^5} > 0$$

or

$$1 + \frac{3}{2|R|^2} - \frac{9R \cdot \mathbf{I}R}{2|R|^4} > 0$$

Therefore algebraic conditions for oblique equilibria have been found: Given a value of $r = |R|$:

- Check if $A_4 S^2 + A_2 S + A_0 = 0$ has a solution $0 < S < 1$. Let $S = \cos\psi$
- From this S construct (g_1, g_2) using (r, ψ) and check if $g_1 r \cos\psi + g_2 r \sin\psi > 0$

Given one value of r we can compute in this way all the oblique solutions. In Figure 4 for each fixed radius r the points in the Smelt plane (see Section A.4) where a oblique equilibria exist are marked in green.

According to the inequalities in the previous section relative equilibria can only exist in the region of the configuration space where:

$$1 + \frac{3}{2|R|^2} - \frac{9R \cdot \mathbf{I}R}{2|R|^4} > 0$$
$$1 + \frac{3}{2|R|^2} - \frac{15R \cdot \mathbf{I}R}{2|R|^4} < 0$$

the first condition requires that the potential is attractive, the second one is a necessary condition for the existence of oblique equilibria (see (36)).

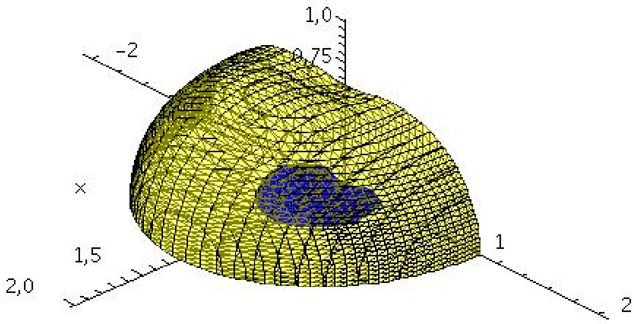


FIG. 2. Repulsive barrier (blue) and orthogonal barrier (yellow) for the rigid body with: $I_1 = 0.4, I_2 = 0.35, I_3 = 0.25$. In body coordinates R_1, R_2, R_3. The oblique equilibria can exist only in the region bounded by this two barriers.

All the relative have been found. The equilibria can be classified in the following families:

[1]from a physics point of view this is requiring that the potential is attractive

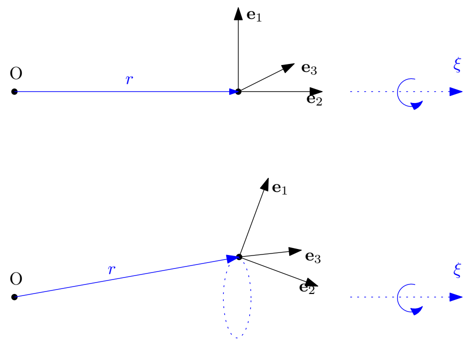


FIG. 3. Parallel and oblique equilibria. The three principal axes represent the satellite. O represents the centre of attraction.

- Orth_i^j orthogonal equilibria with spinning aligned with the j axis and R aligned with i axis. There are 6 different families. If signs of angular velocity and R are taken into account this will lead to the 24 Lagrangian equilibria.
- Par_i parallel equilibria rotating along the i axis. There are 3 possible choices for i.
- $\mathrm{Obl}_{i,j}$ oblique family with angular velocity in the plane spanned by i, j axis. There are 3 families, one for each principal plane.

This notation will be used in the figures to denote what kind of equilibria is represented. In Figure 1 the orthogonal equilibria is represented. In Figure 3 the paralel equilibria and the oblique equilibria are represented.

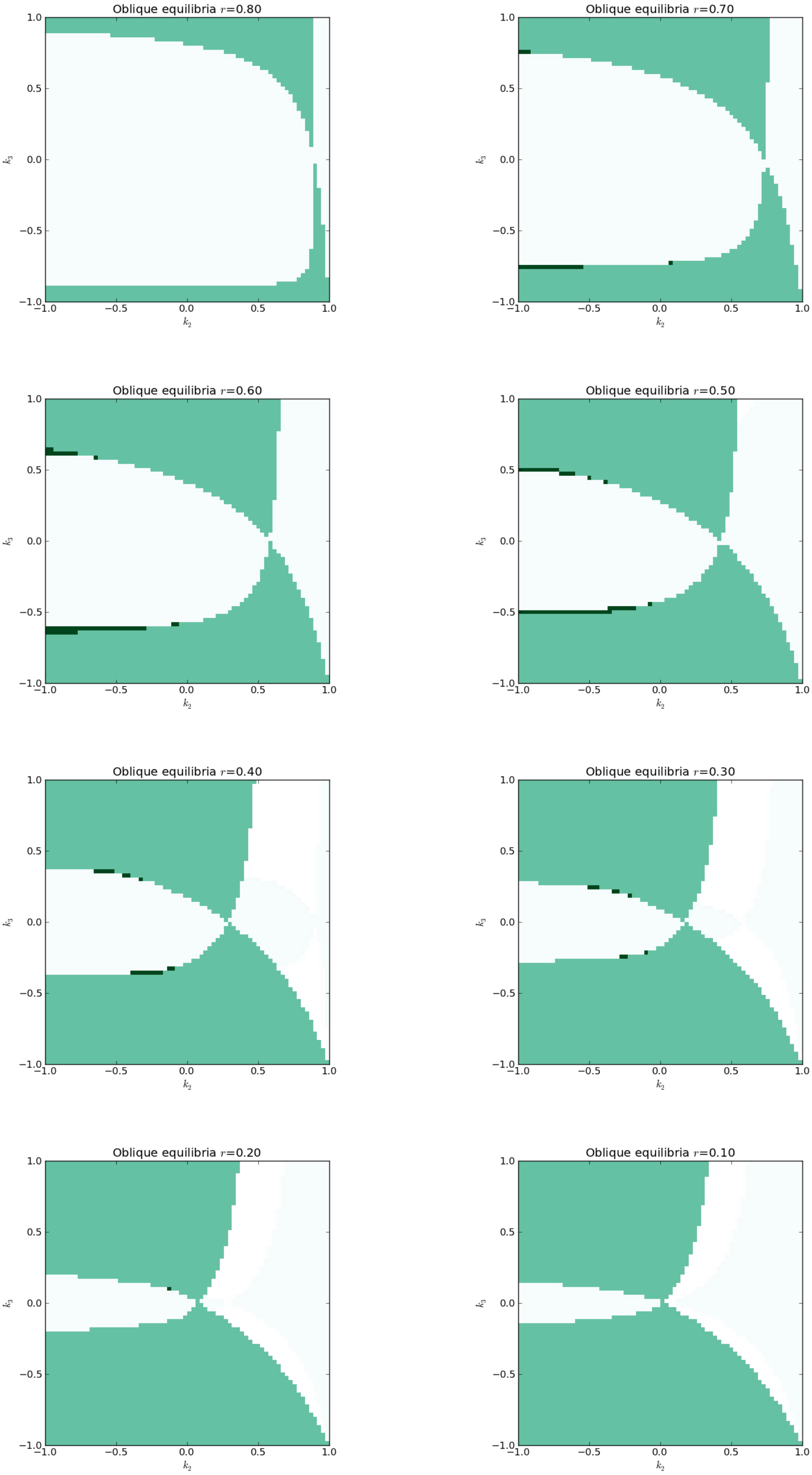


FIG. 4. Existence of oblique equilibria

8.2. General stability considerations

To compute the stabilities of the different kinds of equilibria the second variation of the augmented potential is needed. Using the expression (28), part of the second variation of V_ξ is:

$$\delta(\widehat{B^T\xi}(\mathbf{I}-\widehat{R}\widehat{R})B^T\xi) = HAT(-\widehat{\delta\theta}B^T\xi)(\mathbf{I}-\widehat{R}\widehat{R})B^T\xi - \widehat{B^T\xi}(\mathbf{I}-\widehat{R}\widehat{R})\widehat{\delta\theta}B^T\xi + \\ +\widehat{B^T\xi}(-R\xi^T B + 2B^T\xi R^T - R^T B^T\xi)\delta R$$

The first two terms can be rewritten as ($\mu = \mathbb{I}\xi$)

$$(-\widehat{B^T\mu}\widehat{B^T\xi} + B^T\widehat{\xi}\mathbb{I}\widehat{\xi}B)\delta\theta$$

The variation of $\nabla_R V(R) + B^T\xi(R^T B^T\xi) - R\xi^T\xi$ is:

$$\nabla^2_R V(R) - \xi^T\xi\delta R + B^T\xi\xi^T B\delta R + (R^T B^T\xi)\widehat{B^T\xi}\delta\theta + B^T\xi R^T B^T\hat{\xi}\delta\theta$$

So the second variation of V_ξ is:

$$\begin{bmatrix} -\widehat{B^T\mu}\widehat{B^T\xi} + B^T\widehat{\xi}\mathbb{I}\widehat{\xi}B & -\hat{\xi}R\xi^T - \hat{\xi}R^T\xi \\ R^T\xi\hat{\xi} + \xi R^T\hat{\xi} & \nabla^2 V - \xi^T\xi + \xi\xi^T \end{bmatrix}$$

To compute the space of internal variations the variation of the locked inertia tensor is needed, according to (26):

$$\delta\mathbb{I} = B\widehat{\delta\theta}(\mathbf{I}-\hat{R}\hat{R})B^T - B(\mathbf{I}-\hat{R}\hat{R})\widehat{\delta\theta}B^T + B(-\widehat{\delta R}\widehat{R} - \widehat{R}\widehat{\delta R})B^T$$

applying to the velocity $\xi \in \mathfrak{g}$ and rearranging terms:

$$\delta(\mathbb{I}\xi) = (-R^T B^T\xi - RB\xi^T + 2B^T\xi R^T)\delta R + (-B\widehat{B^T\mathbb{I}\xi} + \mathbb{I}B\widehat{B^T\xi})\delta\theta \tag{40}$$

8.3. Discrete symmetries

In the model under study, apart from the global $SO(3)$ symmetry, there are some other discrete symmetries that can simplify a little more the problem. This symmetries are going to be introduced only for technical reasons. With them some assumptions concerning the relative equilibria can be made, simplifying the computations.

Definition 8.3. The time reversal involution of the symplectic manifold T^*Q, ω is the map:

$$\tau : T^*Q \to T^*Q$$
$$(q,p) \mapsto (q,-p)$$

it's an antisymplectic involution, that is $\tau^2 = Id$ and $\tau^*\omega = -\omega$.

All the simple mechanical systems are time-reversible, in particular the rigid body in the gravitational field, that is $\tau^* H = H$. In terms of the augmented potential we can write

$$H_{-\xi}(\tau(z)) = H_\xi(z)$$

Let

$$\sigma_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \sigma_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \sigma_3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

be reflections around each of the principal axes, This 3 matrices induce involutions of the configuration space:

$$\sigma_i : Q \to Q$$

$$(B, r) \mapsto (B\sigma_i, r)$$

or in body coordinates $(B, R) \mapsto (B\sigma_i, \sigma_i R)$. This involutions leave fixed the second order potential. And we have a relation:

$$H_\xi(\sigma_i z) = H_\xi(z)$$

For example if $B = Id, R = (R_1, R_2, R_3), \xi = (\xi_1, \xi_2, \xi_3)$ is a relative equilibrium then:

- Using τ, $B = Id, R = (R_1, R_2, R_3), \xi = (-\xi_1, -\xi_2, -\xi_3)$ is another relative equilibrium with the same stability properties of the first one.
- Using σ_1, $B = Id, R = (R_1, -R_2, -R_3), \xi = (\xi_1, -\xi_2, -\xi_3)$ is a relative equilibrium.
- Using Proposition 3.16. $B = M, R = (R_1, R_2, R_3), \xi = M(\xi_1, \xi_2, \xi_3)$ is also a relative equilibrium.

Using this symmetries and Proposition 3.16 to study the family of orthogonal relative equilibria we may assume:

$$B = Id,\ \xi = (\xi_1, 0, 0),\ R = (0, R_2, 0)$$

where $\xi_1 > 0$ and $R_2 > 0$.

8.4. Orthogonal equilibria

We will assume without loss of generality that $\mathbf{I} = diag(I_1, I_2, I_3)$, $\xi = (\xi, 0, 0)$, $R = (0, R_2, 0)$. $\xi > 0$ and $R_2 > 0$.

The modulus of the angular velocity and the radius are related by the corrected Kepler formula:

$$\xi^2 = \frac{1}{2}\frac{2R_2^2 + 3 - 9I_2}{|R_2|^5}$$

For the second order potential:

$$\nabla^2 V = \begin{bmatrix} \frac{1}{2}\frac{2R_2^2+3-15I_2+6I_1}{|R_2|^5} & 0 & 0 \\ 0 & -2\frac{R_2^2+3-9I_2}{|R_2|^5} & 0 \\ 0 & 0 & \frac{1}{2}\frac{2R_2^2+3-15I_2+6I_3}{|R_2|^5} \end{bmatrix}$$

in a more compact way, using the Kepler's relation:

$$\nabla^2 V = \xi^2 \begin{bmatrix} 1 + 3\frac{I_1 - I_2}{|R_2|^5\xi^2} & 0 & 0 \\ 0 & -4 + 2\frac{1}{|R_2|^3\xi^2} & 0 \\ 0 & 0 & 1 + 3\frac{I_3 - I_2}{|R_2|^5\xi^2} \end{bmatrix}$$

The momentum map at this relative equilibrium is

$$\mu = \mathbb{I}\xi = \begin{bmatrix} \left(I_1 + {R_2}^2\right)\xi \\ 0 \\ 0 \end{bmatrix} \tag{41}$$

The metric at this point and the momentum variables at the equilibrium are:

$$\langle\!\langle , \rangle\!\rangle = \begin{bmatrix} I_1 + R_2^2 & 0 & 0 & 0 & 0 & R_2 \\ 0 & I_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_3 + R_2^2 & -R_2 & 0 & 0 \\ 0 & 0 & -R_2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ R_2 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Pi = \begin{bmatrix} (I_1 + R_2^2)\xi \\ 0 \\ 0 \end{bmatrix} \qquad P = \begin{bmatrix} 0 \\ 0 \\ R_2 \end{bmatrix}$$

The locked inertia tensor, because of the coordinates chosen, is a block of the metric matrix, that is:

$$\mathbb{I}(q_{eq}) = \begin{bmatrix} I_1 + R_2^2 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 + R_2^2 \end{bmatrix}$$

Therefore:

$$\mathfrak{g}_\mu = \left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\rangle \implies \mathfrak{g}_\mu^\perp = \left\langle \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle$$

and one possible basis of admissible variations is:

$$\text{basis of } \mathcal{V} = \begin{bmatrix} 0 & 0 & 0 & 0 & -R_2 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & I_1 + {R_2}^2 \end{bmatrix} \tag{42}$$

The restriction of $\delta^2 V_\mu$ to $\mathcal{V}_{\text{RIG}}$ is the Arnold form (Proposition 4.16), and can be computed as follows:

$$\begin{aligned} d^2 V_\mu(\eta_Q, \nu_Q) &= ad^*_\eta \mu \cdot \mathbb{I}^{-1} ad^*_\nu \mu + ad^*_\eta \mu \cdot [\nu, \xi] = -\hat{\eta}\mu \cdot \mathbb{I}^{-1}(-\hat{\nu}\mu) - \hat{\eta}\mu \cdot \hat{\nu}\xi \\ &= \eta \cdot \left(-\hat{\mu}\mathbb{I}^{-1}\hat{\mu} + \hat{\mu}\hat{\xi} \right) \nu \end{aligned}$$

leading to the diagonal Arnold form:

$$d^2 V_\mu | \mathcal{V}_{\text{RIG}} \times \mathcal{V}_{\text{RIG}} = \xi^2 \begin{bmatrix} \frac{(I_1 - I_3)(I_1 + R_2^2)}{I_3 + R_2^2} & 0 \\ 0 & \frac{(I_1 - I_2 + R_2^2)(I_1 + R^2)}{I_2} \end{bmatrix}$$

The variation of the locked inertia tensor (40) in the equilibrium point is:

$$\delta(\mathbb{I}\xi) = \begin{bmatrix} 0 & 0 & 0 & 0 & 2R_2\xi & 0 \\ 0 & 0 & \xi(I_1 - I_2 + R_2^2) & -R_2\xi_1 & 0 & 0 \\ 0 & \xi(I_3 - I_1) & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta\theta \\ \delta R \end{bmatrix}$$

one possible election for the reduced energy momentum splitting is (see Proposition 4.17):

$$\text{basis of } \mathcal{V}_{\text{RIG}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{basis of } \mathcal{V}_{\text{INT}} = \begin{bmatrix} 0 & -R_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & R_2 \\ 0 & 0 & I_1 + R_2^2 - I_2 \\ 1 & 0 & 0 \\ 0 & I_1 + R_2^2 & 0 \end{bmatrix} \tag{43}$$

The second variation of the augmented potential $\delta^2 V_\xi$ plus the correction term $(\delta \mathbb{I}\xi)^T \mathbb{I}_{eq}^{-1} (\delta \mathbb{I}\xi)$ expressed in terms of the basis of $\mathcal{V}_{INT}$ is:

$$\xi^2 \begin{bmatrix} \frac{3R_2^2 - I_1}{R_2^2 + I_1} - 4 + \frac{2}{R_2^3 \xi^2} & 0 & 0 \\ 0 & \frac{3(I_1+R_2^2)^2(I_3-I_2)}{R_2^5\xi^2} & 0 \\ 0 & 0 & \frac{(I_1-I_2+R_2^2)(I_1-I_2)(6I_1+8R_2^2+3-15I_2)}{R_2^5\xi^2} \end{bmatrix}$$

that is, Smale's form is diagonal.

In order to linearise properly the system we still need a basis for the space $\mathcal{W}_{INT}^*$ of pure momenta variations, we can choose:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{I_1 - I_2 + R_2^2} \\ 1 & 0 & 0 \\ 0 & \frac{1}{I_1 + R_2^2} & 0 \end{bmatrix}$$

this rather complicated expression comes from imposing to be "conjugate" to $\mathcal{W}_{INT}$.

The Hessian of the Hamiltonian in this splitted basis takes a diagonal form:

$$d^2 H_\xi = \text{diag}([h_1, h_2], [h_3, h_4, h_5], [h_6, h_7, h_8])$$

where the first block corresponds to Arnold's form, the second one to Smale's form and the last three values are the restriction to the norm to $\mathcal{W}_{INT}^*$, that is:

$$\begin{aligned} h_1 &= \xi^2 \frac{(I_1 - I_3)(I_1 + R_2^2)}{I_3 + R_2^2} & h_5 &= \frac{(I_1 - I_2 + R_2^2)(I_1 - I_2)(6I_1 + 8R_2^2 + 3 - 15I_2)}{R_2^5} \\ h_2 &= \xi^2 \frac{(I_1 - I_2 + R_2^2)(I_1 + R^2)}{I_2} & h_6 &= 1 \\ h_3 &= \left(\frac{3R_2^2 - I_1}{R_2^2 + I_1} - 4 + \frac{2}{R_2^3 \xi^2} \right) \xi^2 & h_7 &= \frac{1}{I_1(I_1 + R_2^2)} \\ h_4 &= \frac{3(I_1 + R_2^2)^2 (I_3 - I_2)}{R_2^5} & h_8 &= \frac{I_3 + R_2^2}{(I_1 - I_2 + R_2^2)^2 I_3} \end{aligned} \tag{44}$$

Recall that h_6, h_7, h_8 are always positive, they are the metric applied to the pure momenta block. See Section 4.3 for details. This three values will play no role in the stability analysis.

8.5. Linearisation of orthogonal equilibria

Using Proposition 4.20 we can derive the block structure of the symplectic form at the equilibrium.

The first block, of rigid-rigid interactions is given by the coadjoint orbit symplectic form: if $\Delta_1 z = T\alpha_\mu(\eta_{1Q})$ and $\Delta_2 z = T\alpha_\mu(\eta_{2Q})$, then:

$$\omega(z_{eq})(\Delta_1 z, \Delta_2 z) = -\langle \mu, [\eta_1, \eta_2] \rangle = \eta_1 \cdot \hat{\mu}\eta_2$$

The rigid-internal interactions $\mathcal{S}_{\mathrm{RIG}}$, $\mathcal{S}_{\mathrm{INT}}$ are determined by the metric (if $\Delta z = T\alpha_\mu(\zeta_Q)$):

$$\omega(z_{eq})(\Delta z, \delta z) = -\langle\!\langle \zeta_Q, \delta q \rangle\!\rangle$$

The magnetic (or Coriollis) block is determined by the exterior differential of the mechanical connection. The exterior differential of the mechanical connection can be expressed in coordinates as $A - A^T$ where A is the "matrix" of equation (27).

$$\omega(z_{eq})(\delta_1 z, \delta_2 z) = -d\alpha_\mu(\delta_1 q, \delta_2 q) = -\delta_1 q \cdot (A - A^T)\delta_2 q$$

Putting all the blocks together:

$$\omega(z_{eq}) = \begin{bmatrix} \begin{bmatrix} 0 & -(I_1 + R_2^2)\xi_1 \\ (I_1 + R_2^2)\xi_1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & \omega_{1,4} \\ 0 & 0 & 0 \end{bmatrix} & 0 \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -\omega_{1,4} & 0 \end{bmatrix} & \begin{bmatrix} 0 & -2\xi I_1 & 0 \\ 2\xi I_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ 0 & \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} & 0 \end{bmatrix}$$

where $\omega_{1,4} = -\frac{R_2\left(I_1+R_2^2\right)\xi_1(-I_3+I_1-I_2)}{I_3+R_2^2}$.

The inverse matrix is given by:

$$\mathbb{J}(z_{eq}) = \begin{bmatrix} 0 & \frac{1}{(I_1+R_2^2)\xi_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{(I_1+R_2^2)\xi_1} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{R_2(I_2+I_3-I_1)}{I_3+R_2^2} \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -2\xi_1 I_1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2\xi_1 I_1 & 0 & 0 \\ 0 & -\frac{R_2(I_2+I_3-I_1)}{I_3+R_2^2} & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

The linearisation is (using the notation of (44)):

$$dX_H(z_{eq}) = \begin{bmatrix} 0 & \frac{h_2}{(I_1+R_2^2)\xi_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{h_1}{(I_1+R_2^2)\xi_1} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{h_8 R_2(I_2+I_3-I_1)}{I_3+R_2^2} \\ 0 & 0 & 0 & 0 & 0 & -h_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -h_7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -h_8 \\ 0 & 0 & h_3 & 0 & 0 & 0 & -2\xi_1 I_1 h_7 & 0 \\ 0 & 0 & 0 & h_4 & 0 & 2\xi_1 I_1 h_6 & 0 & 0 \\ 0 & -\frac{h_2 R_2(I_2+I_3-I_1)}{I_3+R_2^2} & 0 & 0 & h_5 & 0 & 0 & 0 \end{bmatrix} \tag{45}$$

After a basis permutation it can be put in block form:

$$\begin{bmatrix} \begin{bmatrix} 0 & \frac{h_2}{(I_1+R_2^2)\xi_1} & 0 & 0 \\ -\frac{h_1}{(I_1+R_2^2)\xi_1} & 0 & 0 & \frac{h_8 R_2(I_2+I_3-I_1)}{I_3+R_2^2} \\ 0 & 0 & 0 & -h_8 \\ 0 & -\frac{h_2 R_2(I_2+I_3-I_1)}{I_3+R_2^2} & h_5 & 0 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} 0 & 0 & -h_6 & 0 \\ 0 & 0 & 0 & -h_7 \\ h_3 & 0 & 0 & -2\xi_1 I_1 h_7 \\ 0 & h_4 & 2\xi_1 I_1 h_6 & 0 \end{bmatrix} \end{bmatrix}$$

The characteristic polynomial of this two blocks is easily computed and is equivalent to test the stability of two second order polynomials (70).

8.6. Large radii dynamics

In the previous section the Reduced Energy Momentum method was applied to the family of orthogonal equilibria. The 5 eigenvalues of the augmented Hamiltonian were, after eliminating positive factors:

$$\begin{aligned} \tilde{h}_1 &= I_1 - I_3 \\ \tilde{h}_2 &= 1 + \frac{I_1 - I_2}{R_2^2} \\ h_3 &= \frac{3R_2^2 - I_1}{R_2^2 + I_1} - 4 + \frac{2}{R_2^3 \xi^2} = \frac{3 - I_1 R_2^{-2}}{1 + I_1 R_2^{-2}} - 4 + \frac{2}{1 + \frac{3-9I_2}{2R_2^2}} \\ \tilde{h}_4 &= I_3 - I_2 \\ \tilde{h}_5 &= (I_1 - I_2)(1 + \frac{I_1 - I_2}{R_2^2})(8 + \frac{6I_1 + 3 - 15I_2}{R_2^2}) \end{aligned}$$

In practical situations the radius in this normalized coordinates is large. At first approximation we can neglect terms of order R_2^{-1} or higher, thus the 5 eigenvalues are (up to positive factors):

$$\{I_1 - I_3, 1, 1, I_3 - I_2, I_1 - I_2\}$$

Applying the stability criteria, and the previous results we can obtain the same result as [**WKM90**]:

Proposition 8.4. *For large orbits all the equilibria are orthogonal and if the* Lagrange stability conditions *are met:*

$$I_1 > I_3 > I_2$$

then the equilibria is non-linearly stable.

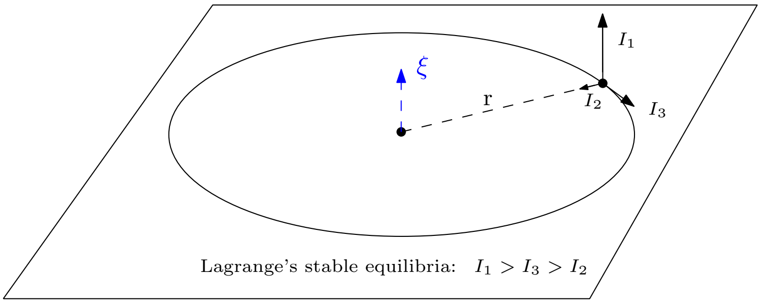


FIG. 5. Lagrange's stable relative equilibria

With Proposition 2.8 we can establish also unstable zones in the parameter space. But for some regions the method is going to be inconclusive (Figure 6).

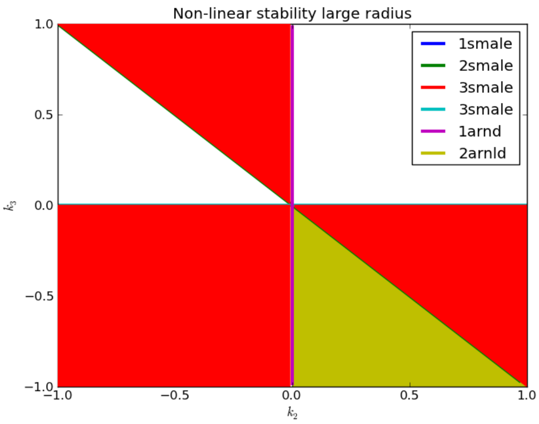


FIG. 6. Stabilities in the Smelt plane for large R. In red unstable zone, in yellow formally stable and in white inconclusive zone.

The linearisation of the relative equilibria can decide if the inconclusive zones are actually linearly unstable. Using the polynomials (70) that give the eigenvalues of the linearised system we can represent the stabilities in the Smelt plane (see Section A.4), see Figure 7.

Most of the inconclusive zone is unstable but there is a region where the system is linearly stable this region is the analogue of the he DeBra-Delp region [**Hal02**], [**BH98**] of the restricted system.

8.7. Evolutions of h_i

Actually R has not to be very large to have this stability behaviour. A careful study of the eigenvalues gives that for $R > 2$ the previous approximation remains valid.

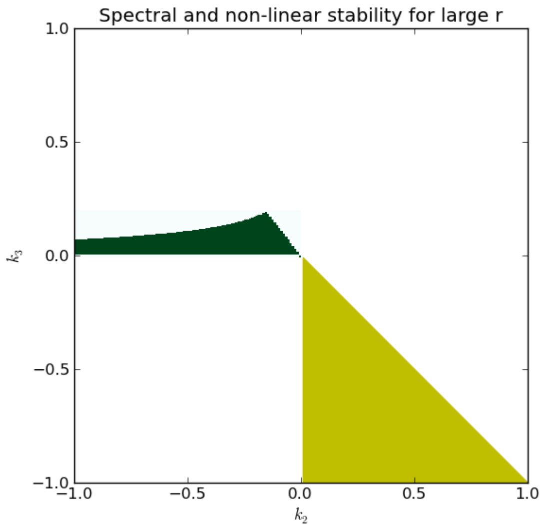


FIG. 7. Non-linear and spectral stabilities in the Smelt plane for large R. In green linearly stable zone, yellow formal stability and white instability.

Instead of assuming R_2 large enough we can compute for each R the values of d^2H for all the Smelt plane. In Figure 8 and 9 the vanishing of the values of d^2H is represented. The formally stable region is painted in yellow.

In view of the plots there are two different main regimes. For R in the interval between 1 and 2 the stability region is reduced progressively until it disappears completely. The value that causes this change is the first element of the Smale form.

In the range from 0 to 1 there are some regions in the Smelt plane which are forbidden. They are in the repulsive area, in those points the potential becomes repulsive. In this points parallel equilibria and oblique equilibria are going to appear.

8.8. Stability loss

According to the previous plots it seems that when R is small enough (in the 1 to 2 interval) the bodies which were in equilibrium for some configuration suddenly become unstable.

First of all we will compute the evolution of the values of d^2H for a rigid body which satisfies Lagrange conditions. Let $I_1 = 0.4, I_2 = 0.25, I_3 = 0.35$, this values give Smelt parameters $k_2 = 0.2, k_3 = -0.43$, this point for large R is in the Lagrange region, it's a non-linearly stable equilibrium.

We can track the values of the 5 first elements of d^2H, see Figure 10.

We see how exactly one value, the first element of Smale's form, crosses 0 an becomes negative. The equilibria has turned unstable. In the other graphic the value of the momentum squared μ^2 is plotted with the same scale. The momentum attains its minimum at exactly the same point as the first element of Smale's form crosses 0. The phenomenon is the same as the one in the example in Section 5.2.

Note that, although, the momentum attains a minimum the modulus of the angular velocity ξ has no critical point. Also the total energy of the equilibria, the value of the Hamiltonian attains a minimum at that point.

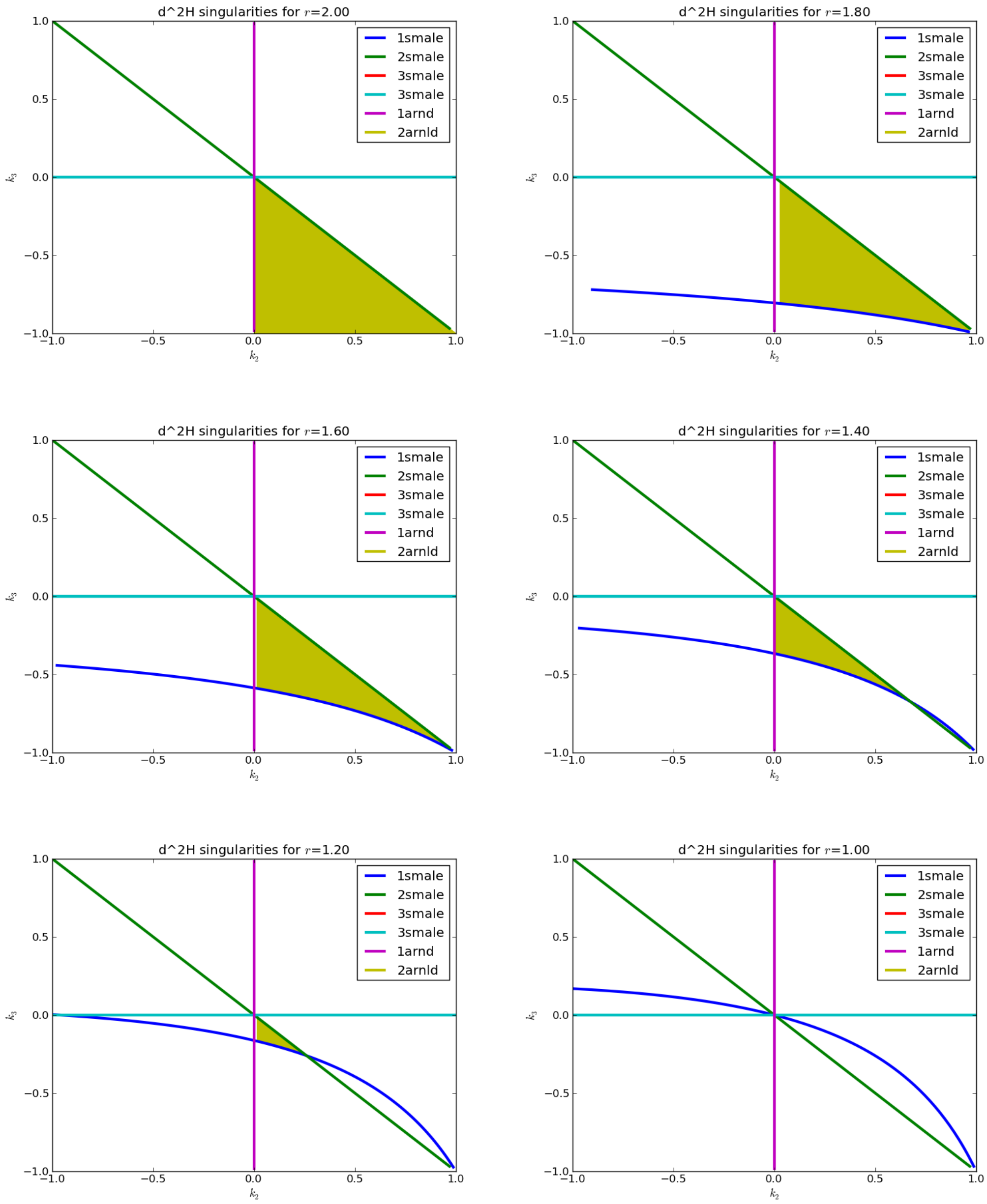


FIG. 8. Values of d^2H for $1 \leq R \leq 2$

This situation is not particular of this example, in fact we can along the orthogonal family we can compute the momentum as a function of R:

$$|\mu|^2 = (\mathbb{I}\xi)^2 = (I_1 + R_2^2)^2\xi^2 = (I_1 + R_2^2)\frac{2R_2^2 + 3 - 9I_2}{2R_2^5}$$

then the first value of Smale's form is equal to:

$$\frac{3R_2^2 - I_1}{R_2^2 + I_1} - 4 + \frac{2}{R_2^3\xi^2} = \frac{R_2}{|\mu|^2}\frac{d|\mu|^2}{dR_2}$$

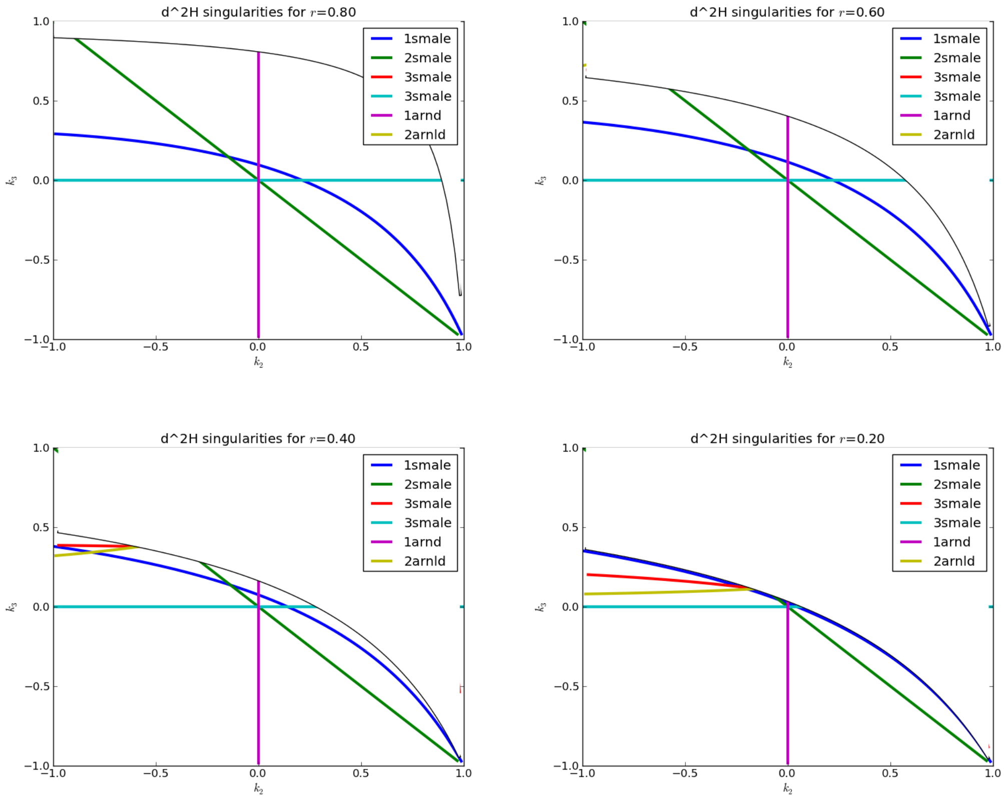


FIG. 9. Values of d^2H for $0.2 \leq R \leq 0.8$

so vanishing of this value implies that the momentum map attains a critical value.

8.9. Bifurcations

As we can see from Figure 9 the orthogonal equilibria disappear for certain values of the moments of inertia and R small enough (compare with Remark 8.1). We will study one example and after that checking that the behaviour is general, not specific of an example.

Take:

$$I_1 = 0.24, I_2 = 0.4, I_3 = 0.35$$

Equations (35) can be thought as defining R and ξ as functions of the parameter α (that is $F(R, \xi, \alpha) = 0$).

Starting at known orthogonal equilibria the implicit function theorem shows that the solution will be unique and smooth if the matrix $\frac{\partial F}{\partial(R,\xi)}$ is invertible. By computing the rank at singular values we can check if new branches appear. This is the basic concept of numerical continuation [**DCF$^+$97**].

The bifurcations, in this case, are difficult to visualize due to the high dimensionality of the space.

The initial point of the form $R = (0, R_2, 0)$ and $\xi = (\xi_1, 0, 0)$ where $\xi_1 > 0, R_2 > 0$ lead to the branch (in the Figure 11 notation) number 2. This branch was decreasing the value of R_2, until it arrived to

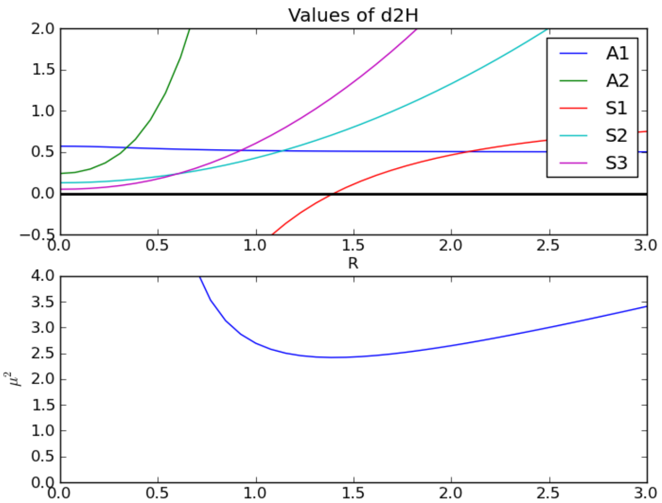


FIG. 10. Evolution of the values of d^2H as R decreases

the critical point $R = (0, R^*, 0)$ where the angular velocity attained the zero value. Nevertheless the branch can be continued after that point with $\xi_2 < 0$.

At the critical point $R = (0, R^*, 0), \xi = (0, 0, 0)$ we have arrived with $\alpha_1 = I_1 + (R_2^*)^2$; the value of α at this point is arbitrary, this gives rise to branch number 0, which in the figure is only one point because only R and ξ are represented.

This branch bifurcates at only three points, $\alpha_3 = (I_3 + R_2^2)$ and $\alpha_2 = I_2$. α_1 gives rise to the orthogonal equilibria with $\xi = (*, 0, 0)$, α_3 bifurcates to the orthogonal equilibria with $\xi = (0, 0, *)$. The point α_2 bifurcates to $\xi = (0, *, 0)$, that is, a parallel equilibria.

This parallel equilibria was also continued as a function of α and it bifurcates later to the branches 4 and 5 of oblique equilibria. Note that ξ and R in any case move in a principal plane, as it was shown in (39).

In Figure 12 there is a schematic diagram of the different relationships between the found relative equilibria.

Each line represents a family of relative equilibria, that is, according to the persistence theorem, each line is locally a 4 dimensional manifold of relative equilibria according to Theorem 4.21.

8.10. Parallel equilibria

Algebraically we can prove that in fact, the behaviour of the example is the same one for all the bodies which stop their orthogonal equilibria motion.

Consider a family of orthogonal relative equilibria given by the points

$$\xi = (\xi_1, 0, 0), R = (0, R_2, 0) \quad \xi_1^2 = \frac{2R_2^2 + 3 - 9I_2}{2R_2^5}$$

if $3 - 9I_2 < 0$ for R_2 small enough the equilibrium will disappear, exactly at the point $R = (0, R^*, 0), \xi = (0, 0, 0)$ where $(R^*)^2 = (9I_2 - 3)\frac{1}{2}$.

The relative equilibrium conditions are the 6 equations:

$$(\mathbf{I} - RR^T + |R|^2)\xi - \alpha\xi = 0$$
$$\nabla_R V(R) + \xi(R\cdot\xi) - R|\xi|^2 = 0$$

this can be thought as a map $F:\mathbb{R}^7 \to \mathbb{R}^6$. This set of equations has as family of solutions:

$$\xi^* = (0,0,0) \quad R^* = (0,R^*,0) \quad \alpha\in\mathbb{R}$$

the implicit function theorem can be used to see if this family bifurcates, the matrix of partial derivatives with respect to $\xi_1, \xi_2, \xi_3, R_1, R_2, R_3$ is:

$$\frac{\partial F}{\partial(\xi,R)}|_{\xi^*,R^*} = \begin{bmatrix} 0 & 0 & 0 & I_1+(R^*)^2-\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & I_2-\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & I_3+(R^*)^2-\alpha \\ * & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 \end{bmatrix}$$

where $*$ represents non-zero terms independent of α. In view of this matrix this family of solutions can bifurcate at three points:

$$\alpha_1 = I_1+(R^*)^2 \quad \alpha_2 = I_2 \quad \alpha_3 = I_3+(R^*)^2$$

Remark 8.5. The degeneracy of the Hessian at the relative equilibrium, in view of results of Section 4.6, only gives necessary conditions for bifurcation. A local study around each "bifurcation candidate" has to be done to check if actually another branch bifurcates at that point. In all the cases in the present section this bifurcating branch is found.

The first bifurcation point corresponds to an orthogonal family spinning about e_1 or e_2 and with R aligned with e_2, check that the value of the multiplier α is the same as the given in formula (41). The third bifurcation point corresponds to an orthogonal family spinning about e_3 and with R aligned with e_2.

The second bifurcation value $\alpha_2 = I_2$ corresponds to a parallel equilibria, where R and ξ are aligned. Relation (38a) gives the same value for the multiplier α.

This family of parallel equilibria can be studied using the Reduced Energy-Momentum method.

The equilibrium point is given by:

$$R = \begin{bmatrix} 0 \\ R^* \\ 0 \end{bmatrix} \quad \xi = \begin{bmatrix} 0 \\ \xi_2 \\ 0 \end{bmatrix}$$

note that ξ_2 is free, it can take any value. The momentum map and the cotangent coordinates are:

$$\mathbf{J}(z) = \begin{bmatrix} 0 \\ I_2\xi_2 \\ 0 \end{bmatrix} \quad \Pi = \begin{bmatrix} 0 \\ I_2\xi_2 \\ 0 \end{bmatrix} \quad P = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

the basis adapted to the reduced energy momentum splitting can be:

$$\text{basis of } \mathcal{V}_{\text{RIG}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{basis of } \mathcal{V}_{\text{INT}} = \begin{bmatrix} -R^* & 0 & 0 \\ 0 & 0 & 0 \\ 0 & R^* & 0 \\ 0 & (R^*)^2+I_1-I_2 & 0 \\ 0 & 0 & 1 \\ I_3-I_2+(R^*)^2 & 0 & 0 \end{bmatrix} \tag{46}$$

The Arnold form is diagonal with values:

$$A_1 = -\xi_2^2 \frac{I_2(7I_2 + 2I_3 - 3)}{9I_2 + 2I_3 - 3}$$
$$A_2 = -\xi_2^2 \frac{I_2(7I_2 + 2I_1 - 3)}{2I_1 + 9I_2 - 3}$$

Smale's form is diagonal with values:

$$S_1 = \frac{(\xi_2^2(R^*)^5 - 3(R^*)^2 + 3I_2 - 3I_3)(I_2 - I_3)((R^*)^2 - I_2 + I_3)}{(R^*)^5}$$
$$S_2 = \frac{(\xi_2^2(R^*)^5 - 3(R^*)^2 + 3I_2 - 3I_1)(I_2 - I_1)((R^*)^2 - I_2 + I_1)}{(R^*)^5}$$
$$S_3 = \frac{2}{(R^*)^3}$$

Given that values there are two values of ξ_2 that makes the Hessian degenerate (4 counted with sign). This corresponds to the two values where bifurcation to oblique equilibria may occur.

Remark 8.6. Arnold's form in this case can't be positive definite, so parallel equilibria will never be formally stable. Suppose the Arnold form is positive definite then:

$$7I_2 + 2I_3 - 3 < 0 \text{ and } 7I_2 + 2I_1 - 3 < 0$$

adding them:

$$14I_2 + 2I_3 + 2I_1 < 6 \implies 12I_2 < 4 \implies I_2 < \frac{1}{3}$$

but I_2 has to be greater than $\frac{1}{3}$ for the existence of parallel equilibria in the R_2 direction, contradiction.

The linearisation around this equilibria could be computed in the same way as it has been done in the orthogonal case but it hasn't been done in this work. Numerical experiments done suggest that parellel equilibria always have a unstable linearization.

8.11. Oblique equilibria

Two oblique equilibria appear, one of them confined in the R_1, R_2 plane and the other one in the R_2, R_3 plane.

In the example presented before, the branch number 5 lies in the R_1, R_2 plane, in this case the branch is open, because as $I_1 < 1/3$ the body can get as closer as we go to the attractive centre and spinning faster and faster. Thus in the ξ gets larger and the orbit can't be closed. Note that this branch gets closer and closer to the branch of orthogonal equilibria spinning around I_1.

The branch number 4 lies in the R_2, R_3 plane where the potential is repulsive for R small enough so the branch can't "escape" to the origin and it loops. Note that this branch passes through the point $R = (*, 0, 0)$ where it bifurcates with the orthogonal family spinning around I_3.

A schematic bifurcation diagram is sketched in Figure 13, where the R variables are represented. O represents the centre of attraction, the singularity in the potential.

The reduced Energy Momentum has not been applied to the branch of oblique equilibria. Due to the complexity of the expressions involved, no algebraic results were found. Numerically the computations of the splittings and the linearisation are easier and in all the oblique equilibria studied an instability was found. A more careful algebraic computation can lead to a rigorous proof of this result.

The existence conditions for the relative equilibria found before always gave, up to symmetries, one unique solution. This solution is the one that bifurcates to the parallel/orthogonal family. All relative equilibria for this problem are related through a bifurcation diagram like the one in Figure 13. Differences may appear if the number of principal inertias greater than $\frac{1}{3}$ is one or two.

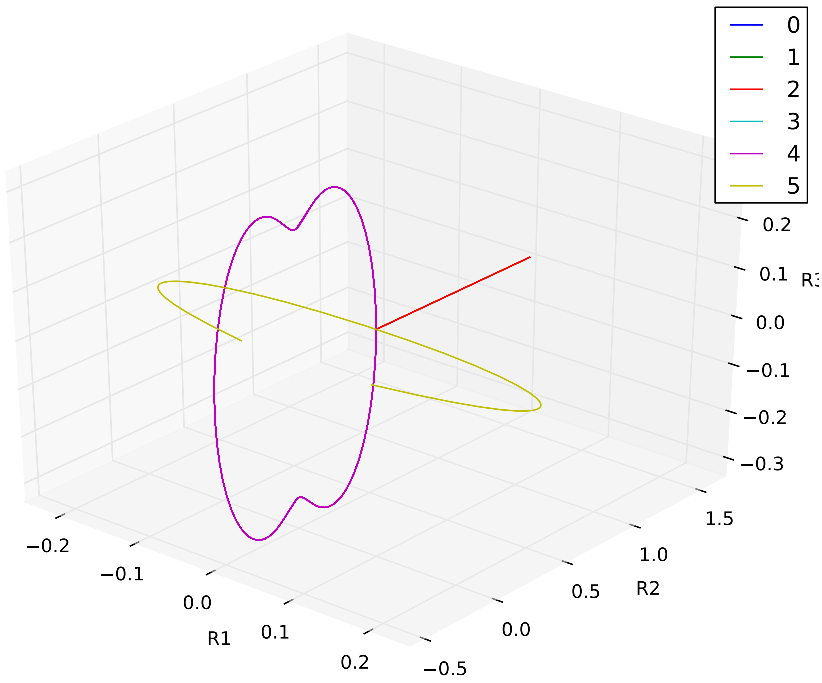
8.12. Conclusions

All the relative equilibria for this model have been found. We have new conditions that can be used to explicitly find all the relative equilibria.

For large orbits, only the classical Lagrange equilibria can exist. The stability configuration is the classical one, with the body spinning along the major axis of inertia and with the minor one pointing toward the centre of attraction.

For small orbits, the counter-intuitive oblique equilibria appear. Also, some strange phenomena like the repulsive barrier appear. Most of them can be justified by a misuse of an approximation which is correct for large distances. Nevertheless, we will show in the next chapter that oblique equilibria can be physically realizable and even non-linearly stable. It was known that the oblique equilibria, if they exist, can be physically realistic, for example [**Ste69**] gives an explanation of how the forces and torques can balance to give oblique equilibria. In the next chapter, a new explicit construction of oblique equilibria for large orbits will be done for the axisymmetric case.

A detailed description of the different bifurcation phenomena for low orbits has been done. Most of the behaviour observed has an analogue in the numerical continuation research done in [**OT04**] for the axisymmetric case. Up to our knowledge, no analytical treatment has been done in the low orbit regime.



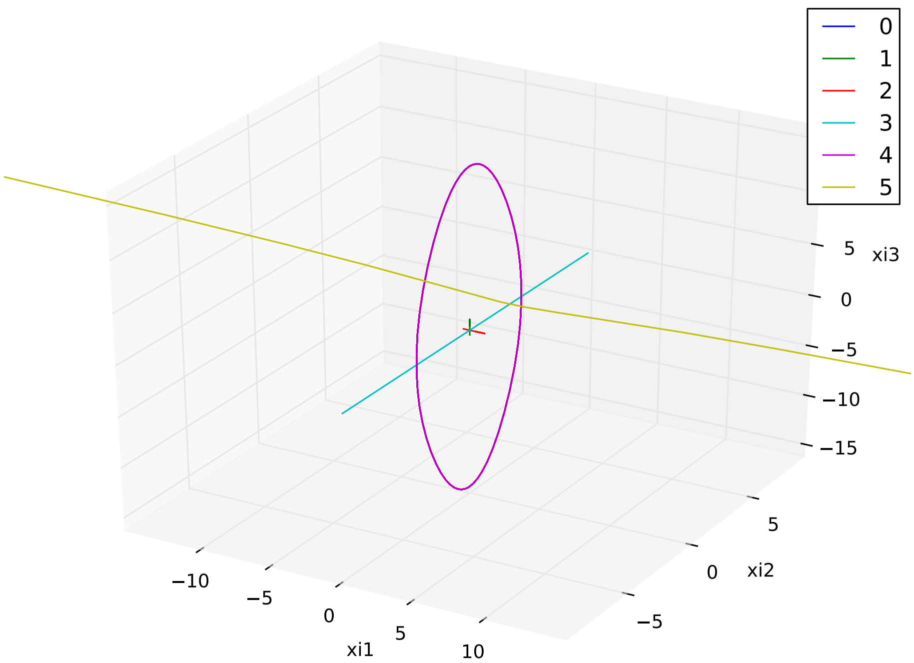


FIG. 11. Continuation of orthogonal equilibria. In the first plot, evolution in the R space, in the second, evolution of the ξ components.

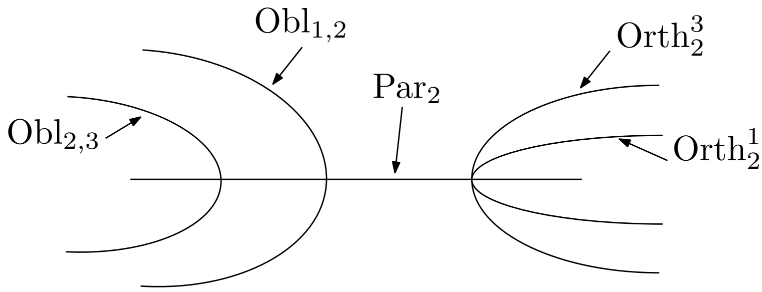


FIG. 12. Sketch of local bifurcations for the example.

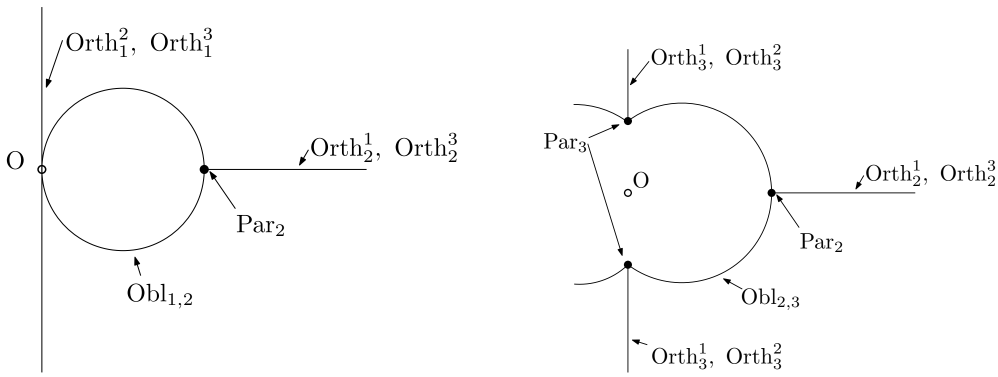


FIG. 13. Motions in the 12 plane (left), and in the 23 plane (right). Compare with Figure 11.

Chapter 9
Axisymmetric second order approximation

The system of interest in this section is the particular case of a satellite with an axis of symmetry which is orbiting a fixed spherically symmetric rigid body. [**Tho62**] is one of the seminal works for the system of interest. In that paper, the steady motion of an axially symmetric satellite was investigated assuming that the satellite's centre of mass described a prescribed circular Keplerian orbit. During this motion, the axis of symmetry of the satellite was parallel to the normal to the orbital plane. Thomson showed that it was possible to arbitrarily spin the satellite about its axis of symmetry while still preserving the circular orbit. Assuming that the orbit and attitude are uncoupled, his stability analysis of these motions only considered the attitude stability of the satellite. In other words, he considered the restricted problem. Later, [**Pri64**] and [**LR66**] showed that other steady motions were present in the restricted problem (see Appendix B).

The unrestricted problem, where the attitude–orbit coupling is incorporated, was examined, by [**Ste69**]. Among other items, he established stability criteria for some of the unrestricted problem's counterparts to the Thomson equilibria and the Pringle–Likins hyperbolic equilibria. He subsequently suggested that the unrestricted problem's counterparts to the Pringle–Likins conical equilibria could feature instances where the orbital plane of the satellite's centre of mass does not contain the centre of mass of the primary.

The four coupled equations found were not solved explicitly. Consequently, several issues pertaining to the steady motions of the unrestricted problem have remained open.

[**Bec97**] suggests the existence of a bound for the conical equilibria similar to the inequality (36) for the non-symmetric case, but he doesn't succeed. [**OT04**] use numerical continuation techniques to follow the cylindrical equilibria family until it bifurcates to other equilibria. The only bifurcations they found are hyperbolic and conical equilibria for very small radius, like the ones found in the non-symmetric case. This leads them to affirm that something similar to Proposition 8.2 should be also true in the axisymmetric case.

We will show that the conical equilibria of Pringle–Likins have a analogue in the non-restricted case and it is not bound to small orbit as it was suggested. For certain cases this motions are shown to be stable.

9.1. Relative equilibria

Using the action defined in Section 6.6, the augmented potential is:

$$V_{(\xi,\eta)} = V(R) - \frac{1}{2}(\xi,\eta)\cdot\mathbb{I}(\xi,\eta) = V(R) - \frac{1}{2}(\xi\cdot B(\mathbf{I}-\hat{R}\hat{R})B^T\xi) - \xi\cdot Be_1 I_1\eta - \frac{1}{2}\eta^2 I_1$$

therefore relative equilibria conditions are:

$$\widehat{B^T\xi}(\mathbf{I}-\widehat{R}\widehat{R})B^T\xi+\widehat{B^T\xi}I_1e_1\eta=0 \tag{47a}$$
$$\nabla_R V(R)+B^T\xi(R^TB^T\xi)-R\xi^T\xi=0 \tag{47b}$$

without loss of generality $B=Id$ can be assumed, then,

$$\widehat{\xi}(\mathbf{I}-\widehat{R}\widehat{R})\xi+\widehat{\xi}I_1e_1\eta=0 \tag{48a}$$
$$\nabla_R V(R)+\xi(R^T\xi)-R\xi^T\xi=0 \tag{48b}$$

Recall that, for the second order approximation:

$$\nabla_R V(R)=\frac{R}{|R|^3}+\frac{3R}{2|R|^5}+\frac{3\mathbf{I}R}{|R|^5}-\frac{15R(R\cdot\mathbf{I}R)}{2|R|^7}$$

Orthogonal equilibria: suppose first that $\xi\cdot R=0$ then:

$$\widehat{\xi}\mathbf{I}\xi-\widehat{\xi}I_1e_1\eta=0 \tag{49a}$$
$$\nabla_R V(R)-R|\xi|^2=0 \tag{49b}$$

The first condition is

$$\mathbf{I}\xi+I_1e_1\eta=\lambda\xi$$

that is:

$$\begin{bmatrix} I_1-\lambda & 0 & 0\\ 0 & I_2-\lambda & 0\\ 0 & 0 & I_2-\lambda\end{bmatrix}\begin{bmatrix}\xi_1\\ \xi_2\\ \xi_3\end{bmatrix}=\begin{bmatrix}-I_1\eta\\ 0\\ 0\end{bmatrix}$$

This linear system is easy to discuss:

- If $\lambda\neq I_1,I_2$ then the system has only one solution $\xi=((\lambda-I_1)^{-1}I_1\eta,0,0)$, whereas the second condition forces R to be an eigenvector of the inertia matrix. R can't be aligned with ξ by the orthogonality constraint. So R is not aligned with ξ and $R=(0,R\cos\alpha,R\sin\alpha)$ for some angle α. Using the S^1 action we can suppose $R=(0,R,0)$.
  Thus one representative point is:

  $$\xi=\begin{bmatrix}(\lambda-I_1)^{-1}I_1\eta\\ 0\\ 0\end{bmatrix}\qquad R=\begin{bmatrix}0\\ R\\ 0\end{bmatrix}\qquad |\xi|^2=\frac{2R^5+3-9I_2}{2R^5}$$

  this family of equilibria will be called *cylindrical equilibria.*
- If $\lambda=I_2$:

  $$\xi=\begin{bmatrix}(I_2-I_1)^{-1}I_1\eta\\ 0\\ \beta\end{bmatrix}\qquad R=\begin{bmatrix}0\\ R\\ 0\end{bmatrix}\qquad |\xi|^2=\frac{2R^5+3-9I_2}{2R^5}$$

  this case is called *hyperbolic equilibria.*
- If $\lambda=I_1$ the only solution is $\eta=0$ $\xi=(\xi_1,0,0)$. This is a solution without axis rotation ($\eta=0$). Using the symmetry we can rotate R such that $R=(0,R_2,0)$. The second equation of the relative equilibria is the usual Kepler's condition:

  $$\xi=\begin{bmatrix}\xi_1\\ 0\\ 0\end{bmatrix}\qquad R=\begin{bmatrix}0\\ R\\ 0\end{bmatrix}\qquad |\xi|^2=\frac{2R^5+3-9I_2}{2R^5}\qquad \eta=0$$

  it can be though as a limiting case of the cylindrical family with $\lambda\to I_1$.

The names are based on the surface that describes the symmetry axis of the body as it travels along the orbit, see Figure 1, the dotted line represents the symmetry axis of the body.

The conical case will be described in the following sections; note that for the conical case the centre of the orbit is not the same spatial point as the centre of attraction.

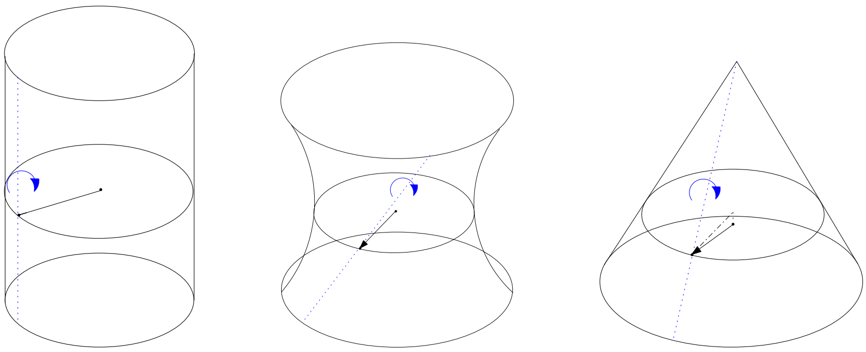


FIG. 1. Families of equilibria for the axisymmetric body.

Remark 9.1. In the classification introduced used in [**OT04**], cylindrical equilibria, hyperbolic and conic equilibria are called type I,III and IV respectively. The type III motion has no analogue in large orbits, it appears when the radius is so small that the satellite arrives to the repulsive surface, exactly in the same way as the parallel equilibria appear in the non-symmetric case for very small radius.

9.2. Cylindrical equilibria

As a first case consider the first family of solutions to the axisymmetric problem, where the body follows a circular orbit and in addition it spins with angular velocity parallel to the orbital angular velocity.

$$\xi = \begin{bmatrix} (\lambda - I_1)^{-1} I_1 \eta \\ 0 \\ 0 \end{bmatrix} \qquad R = \begin{bmatrix} 0 \\ R \\ 0 \end{bmatrix} \qquad |\xi|^2 = \frac{2R^5 + 3 - 9I_2}{2R^5}$$

for notational reasons we will set $\eta = \alpha\xi_1$, that is $\alpha = \frac{\lambda - I_1}{I_1}$.

The metric at the equilibrium point is

$$\begin{bmatrix} I_1 + R_2^2 & 0 & 0 & 0 & 0 & R_2 \\ 0 & I_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_2 + R_2^2 & -R_2 & 0 & 0 \\ 0 & 0 & -R_2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ R_0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The moments at the equilibrium point are:

$$\Pi = \begin{bmatrix} (I_1(1+\alpha) + R_2^2)\xi_1 \\ 0 \\ 0 \end{bmatrix} \qquad P = \begin{bmatrix} 0 \\ 0 \\ R_2\xi_1 \end{bmatrix}$$

the momentum map at the equilibrium point is $\mathbf{J}(z_e q) = ((I_1(1+\alpha) + R_2^2)\xi_1, 0, 0; (1+\alpha)I_1\xi_1)$, the stabilizer of the momentum is therefore:

$$\mathfrak{g}_\mu = \left\langle \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \right\rangle \implies \mathfrak{g}_\mu^\perp = \left\langle \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \right\rangle$$

The space of admissible variations $\mathcal{V}$ is spanned by e_2, e_3, e_4, e_5 where e_i is the ith vector of the canonical basis (the fundamental fields are written in Section 6.6).

The Arnold form is:

$$\begin{bmatrix} \xi_1^2 \frac{((1+\alpha)I_1+R_2^2)(I_1(1+\alpha)-I_2)}{I_2+R_2^2} & 0 \\ 0 & \xi_1^2 \frac{((1+\alpha)I_1+R_2^2)(I_1(1+\alpha)+R_2^2-I_2)}{I_2+R_2^2} \end{bmatrix} \tag{50}$$

A basis for the internal variations can be:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ R_2 & 0 \\ I_1(1+\alpha)+R_2^2-I_2 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Smale's form in this basis is diagonal and has values:

$$S_1 = \frac{1}{R_2^5}(I_1(1+\alpha)+R_2^2-I_2)(I_1(1+\alpha)-I_2)\xi_1^2 R_2^5 + 3(I_1-I_2)(I_1(1+\alpha)-I_2+R_2^2) \tag{51a}$$

$$S_2 = \frac{2}{R_2^3} - \xi_1^2 \tag{51b}$$

If R_2 is large, $\xi_1 = R_2^{-\frac{3}{2}} + O(R_2^{-\frac{7}{2}})$ and the 4 eigenvalues are:

$$\begin{aligned} A_1 &= (I_1(1+\alpha)-I_2)R_2^3 + O(R_2^{-5}) \\ A_2 &= \frac{R_2}{I_2} + O(R_2^{-1}) \\ S_1 &= (I_1(4+\alpha)-4I_2)R_2^{-1} + O(R_2^{-3}) \\ S_2 &= R_2^{-3} + O(R_2^{-5}) \end{aligned}$$

The self-spinning coefficient α stabilizes the motion if it is large enough. According to the order two model the motion with $I_1 < I_2$ is unstable but if α is large enough this motion is "stabilized".

Proposition 9.2. *For large orbits, the cylindrical equilibria are non-linearly stable if the self spinning quotient α is large enough, that is, satisfies*

$$I_1(1+\alpha) > I_2 \text{ and } I_1(4+\alpha) > I_2$$

in particular for all oblate bodies ($I_1 > I_2$) self-spinning in the same direction as the orbital motion implies stability.

Remark 9.3. This proposition is the analogue of the fast top conditions for the heavy top problem.

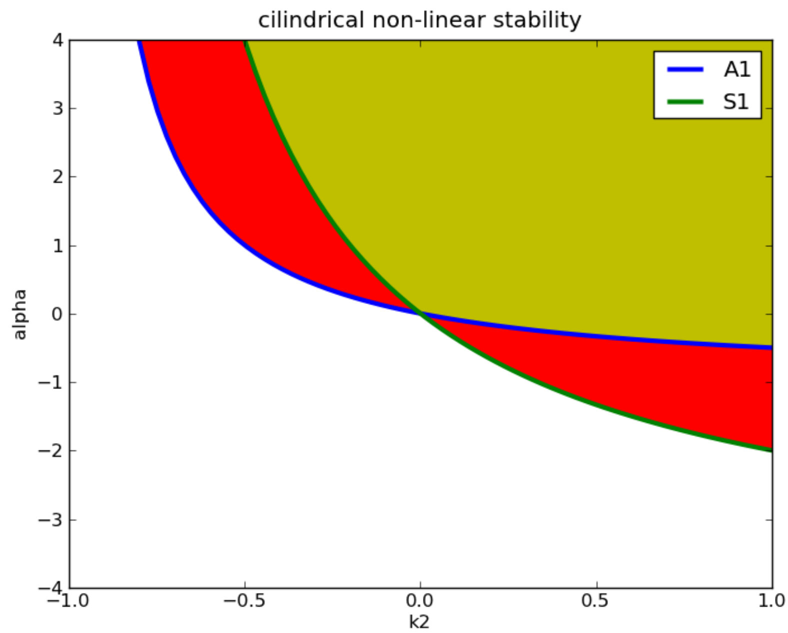


FIG. 2. Stability regions for the cylindrical case. $A1$ establishes the bifurcation points with the hyperbolic family and $S1$ the bifurcations with the conical family.

Note that for $\alpha = \frac{I_2 - I_1}{I_1}$ the vanishing of one eigenvalue can induce a bifurcation, an in this case it does, it evolves to the hyperbolic equilibria point parametrized with $\psi = 0$.

Another eigenvalue vanishing occur when $I_1(4+\alpha) - I_2 = 0$ for this spinning rate α the system bifurcates to an conical equilibria. The angular velocity will be:

$$\xi = \begin{bmatrix} r^{-\frac{3}{2}} \\ 0 \\ 0 \end{bmatrix} + O(r^{-\frac{5}{2}}) \quad \eta = 4\frac{I_2 - I_1}{I_1}\frac{1}{r^{\frac{3}{2}}}$$

a point of the conical family parametrized with $\psi = \frac{\pi}{2}$.

Detailed explanation of this transition between relative equilibria is found in Section 9.5.

The linearisation of cylindrical equilibria is computed in Section C.2. The spectral stability, graphically, in the k_2, α plane is represented on Figure 3. A large region in the plane is spectrally stable but not conclusive with the reduced energy momentum method, in this region the stability analysis done here is inconclusive.

In the linearisation (see Section C.2) there is a 2x2 block :

$$\begin{bmatrix} 0 & -1 \\ \frac{2}{R_2^3} - \xi_1^2 & 0 \end{bmatrix}$$

when $\frac{2}{R_2^3} - \xi_1^2 < 0$ the eigenvalues of this block become real passing from two purely imaginary eigenvalues to two real eigenvalues implying instability. This behaviour seems is very similar to the stability-loss mechanism studied in the non-axially symmetric case. As in that case this will only happen for small R, a unrealistic case.

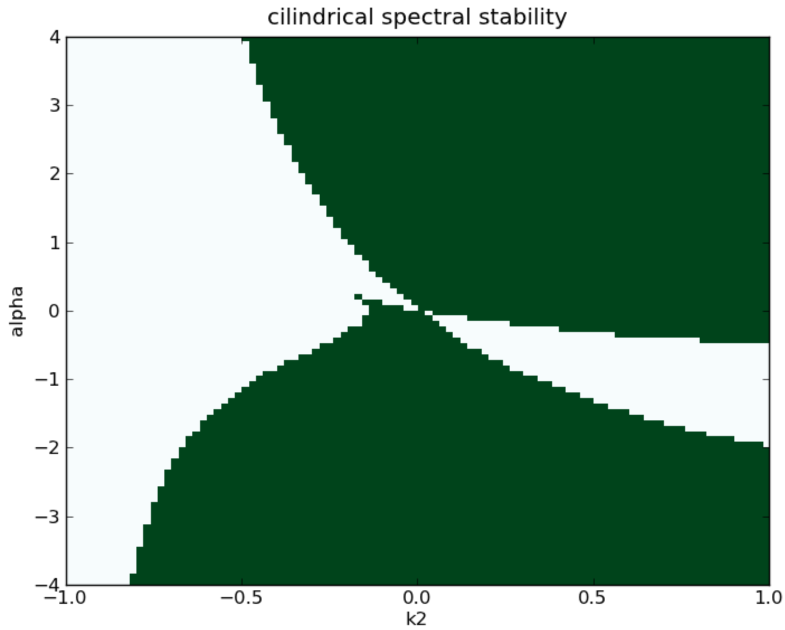


FIG. 3. Spectral stability regions for the cylindrical case.

9.3. Hyperbolic equilibria

According to the previous sections, the hyperbolic equilibria satisfy:

$$\xi = \begin{bmatrix} (I_2 - I_1)^{-1} I_1 \eta \\ 0 \\ \beta \end{bmatrix} \qquad R = \begin{bmatrix} 0 \\ R \\ 0 \end{bmatrix} \qquad |\xi|^2 = \frac{2R^5 + 3 - 9I_2}{2R^5}$$

for notational reasons set $\xi = \xi(\cos\psi, 0 \sin\psi)$ then $\eta = \alpha\xi\cos\psi$ where $\alpha = \frac{I_2 - I_1}{I_1}$.

The metric at the equilibrium point is

$$\begin{bmatrix} I_1 + R_2^2 & 0 & 0 & 0 & 0 & R_2 \\ 0 & I_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_2 + R_2^2 & -R_2 & 0 & 0 \\ 0 & 0 & -R_2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ R_0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The moments at the equilibrium point are:

$$\Pi = \begin{bmatrix} (I_2 + R_2^2)\xi\cos\psi \\ 0 \\ (I_2 + R_2^2)\xi\sin\psi \end{bmatrix} \qquad P = \begin{bmatrix} -R_2\xi\sin\psi \\ 0 \\ R_2\xi\cos\psi \end{bmatrix}$$

the momentum map at the equilibrium point is $\mathbf{J}(z_{eq}) = ((I_2 + R_2^2)\xi\cos\psi, 0, (I_2 + R_2^2)\xi\sin\psi; I_2\xi\cos\psi)$, the stabilizer of the momentum is therefore:

$$\mathfrak{g}_\mu = \left\langle \begin{bmatrix} \cos\psi \\ 0 \\ \sin\psi \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle \implies \mathfrak{g}_\mu^\perp = \left\langle \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\sin\psi(I_2 + R_2^2) \\ 0 \\ \cos\psi R_2^2 \\ \sin\psi(I_2 + R_2^2) \end{bmatrix} \right\rangle$$

because the locked inertia tensor is:

$$\begin{bmatrix} I_1 + R_2^2 & 0 & 0 & I_1 \\ 0 & I_2 & 0 & 0 \\ 0 & 0 & I_2 + R_2^2 & 0 \\ I_1 & 0 & 0 & I_1 \end{bmatrix}$$

The space of admissible variations $\mathcal{V}$ is

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & R_2 & 0 & -R_2\cos\psi \\ 0 & I_2 + R_2^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & (I_2 + R_2^2)\sin\psi \end{bmatrix}$$

After some computations d^2V_μ takes diagonal form, the values expanded in powers of R_2 are:

$$\begin{aligned} A_1 &= \sin^2\psi \frac{I_2}{R_2^3} + O(R_2^{-5}) \\ A_2 &= \frac{R_2^5}{I_2} + O(R_2^3) \\ S_1 &= \frac{1}{R_2^3} + O(R_2^{-5}) \\ S_2 &= (I_1 - I_2)\frac{R_2^3}{I_2^2\cos^2\psi} + O(R_2) \end{aligned}$$

Proposition 9.4. *For all oblate bodies ($I_1 > I_2$) the hyperbolic equilibria is stable, whereas for all prolate bodies the hyperbolic equilibria is unstable, independently of the orientation angle ψ.*

In this case only one eigenvalue changes sign in the parameter space. There is no inconclusive zone and the linearization at the relative equilibria was not needed.

9.4. Conical equilibria

We will drop the hypothesis of $R \cdot \xi = 0$ allowing oblique orbits. From the equations (48) and the expression of the second order gradient (34), take the cross product of the second one with R and subtracts it from the first one, we have the equation:

$$\hat{\xi}\mathbf{I}\xi + \hat{\xi}I_1 e_1 \eta - \hat{R}\frac{3\mathbf{I}R}{r^5} = 0$$

in coordinates this is:

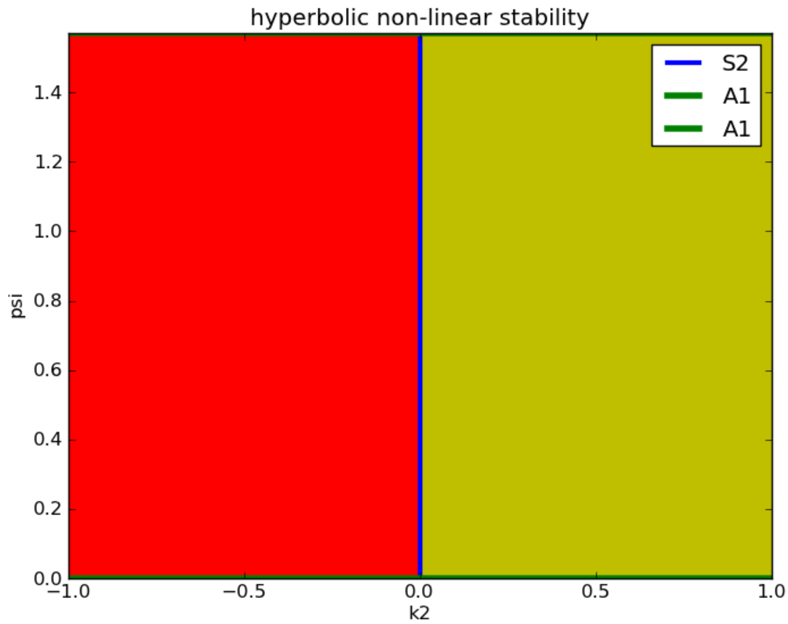


FIG. 4. Stability regions for the hyperbolic case

$$\begin{bmatrix} 0 \\ \xi_3 I_1 \xi_1 - \xi_3 I_2 \xi_1 + \eta \xi_3 I_1 - 3\frac{R_3}{|R|^5}(I_1 - I_2)R_1 \\ -\xi_2 I_1 \xi_1 + \xi_2 I_2 \xi_1 - \eta \xi_2 I_1 + 3\frac{R_2}{|R|^5}(I_1 - I_2)R_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

the two non-linear equations can be written as the linear system:

$$\begin{bmatrix} \xi_3 & \frac{-3R_3}{|R|^5} \\ -\xi_2 & \frac{3R_2}{|R|^5} \end{bmatrix} \begin{bmatrix} \xi_1 \\ R_1 \end{bmatrix} = \frac{I_1 \eta}{I_1 - I_2} \begin{bmatrix} -\xi_3 \\ \xi_2 \end{bmatrix}$$

if the matrix is invertible then the solution is given by:

$$\xi_1 = \frac{I_1}{I_2 - I_1}\eta, \quad R_1 = 0$$

this conditions and the second condition of (48):

$$\frac{R}{|R|^3} + \frac{3R}{2|R|^5} + \frac{3\mathbf{I}R}{|R|^5} - \frac{15R(R \cdot \mathbf{I}R)}{2|R|^7} + \xi(R^T\xi) - R\xi^T\xi = 0 \tag{52}$$

forces $R \cdot \xi = 0$ because $\xi_1 \neq 0$ and $R_1 = 0$, this solution corresponds to the hyperbolic equilibria studied above.

Suppose now that the matrix has not full rank, that is, there is some $\lambda \in \mathbb{R}$ such that $\xi_2 = \lambda R_2$ and $\xi_3 = \lambda R_3$. If $\lambda = 0$ then the only solutions are $\xi = (*, 0, 0)$ and $R = (0, *, *)$, this is the cylindrical case.

Suppose now that $\lambda \neq 0$; using the group action we can rotate R such that it has a zero in the third component: $R = (R_1, R_2, 0)$, by the degeneracy assumption $\xi_2 = \lambda R_2$ and $\xi_3 = \lambda R_3$, $\xi_3 = 0$. The

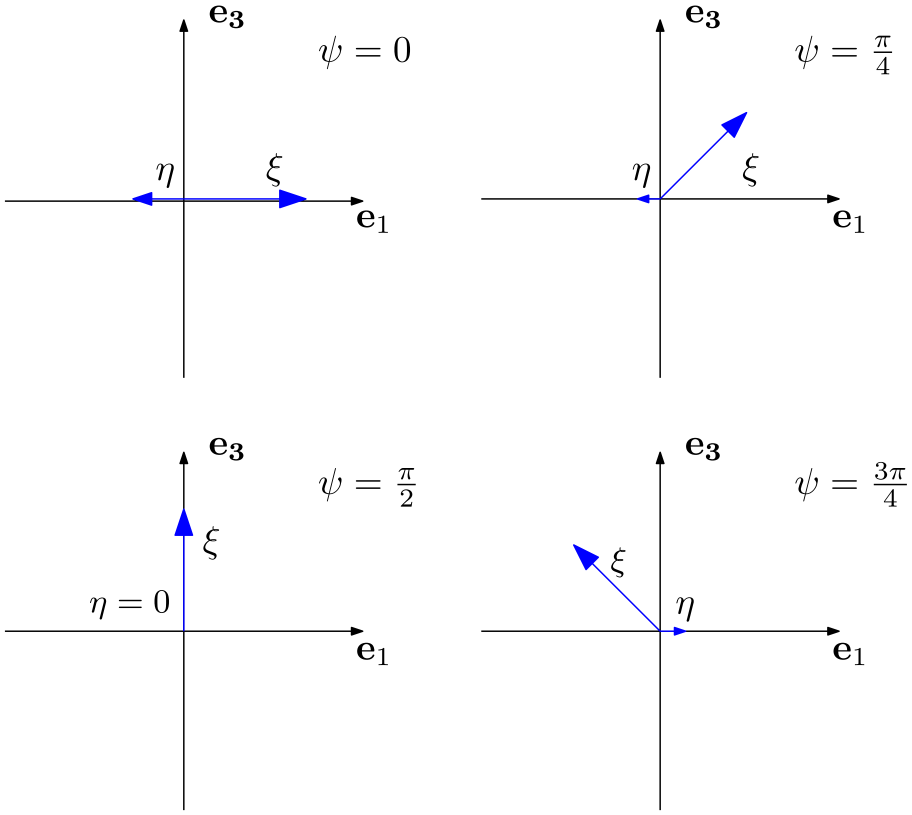


FIG. 5. Transition from cylindrical to hyperbolic equilibria as seen from a body fixed frame. Oblate case. e_1, e_2, e_3 is an orthonormal basis aligned with the principal axes.

solution of the linear system is now:

$$\xi_1 = 3\frac{R_1}{\lambda|R|^5} + \frac{I_1}{I_2 - I_1}\eta$$

defining the angle ψ as $R = (r\cos\psi, r\sin\psi, 0)$, substitution of this relations the second component of (52) gives:

$$\begin{aligned} -\lambda^2(\sin^2\psi\cos\psi)r^3 - \frac{I_1}{I_1 - I_2}(\sin^2\psi)\lambda\eta r^2 - \frac{1}{r^2}\left(3\cos^2\psi - 4\right)\cos\psi \\ -\frac{3}{2}\frac{\cos\psi}{r^4}\left((I_1 - I_2)5\cos^2\psi - 1 - 2I_1 + 5I_2\right) = 0 \end{aligned} \tag{53}$$

a linear relation in η from which η can be solved. Substituting this η a the third component of (52) gives $A + B\lambda^{-2} = 0$ from which λ can be easily found.

The exact expressions for both variables are given in equations (72) (73).

Remark 9.5. The equation for λ for each set of parameters has two solutions $+\lambda_0$ and $-\lambda_0$, this change of sign in λ leads in (53) to two possible spin rates η_0 and $-\eta_0$. Finally this gives the equilibrium $(Id, R_0; \xi_0, \eta_0)$ and $(Id, R_0; -\xi_0, -\eta_0)$, but this equilibria are related trough the time reversal symmetry. So without loss of generality we can choose any of the two solutions for λ and use the time reversal symmetry to find the other one.

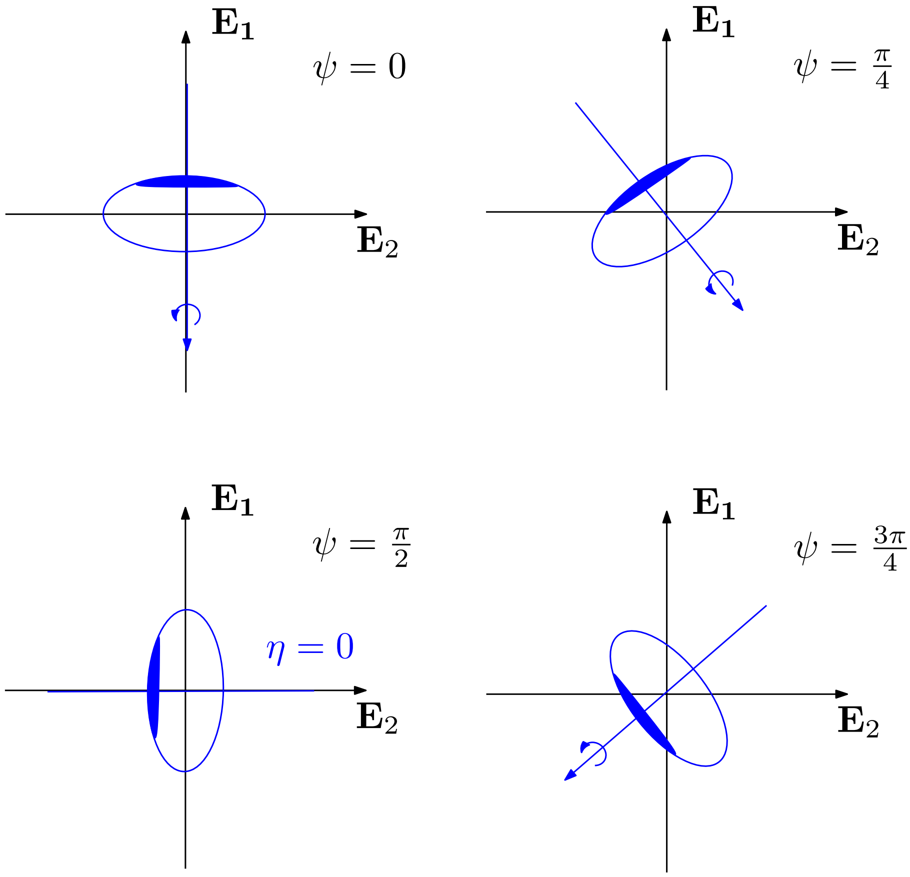


FIG. 6. Transition from cylindrical to hyperbolic equilibria as seen from a orbital frame. Oblate case. E_1, E_2, E_3 is an orthonormal frame rotating with the orbit.

For each given radius r (large enough) and each given $\psi \in [0, 2\pi)$ there exists a conical equilibria described by those equations. The series expansions of those formulas for large r are more useful:

$$\lambda = \frac{\cos\psi}{\sin\psi} r^{-\frac{5}{2}} + O(r^{-\frac{9}{2}})$$
$$\eta = -4\frac{I_2 - I_1}{I_1} \sin\psi\, r^{-\frac{3}{2}} + O(r^{-\frac{7}{2}})$$

this leads to the equilibrium configuration:

$$\xi_1 = -\sin\psi \frac{1}{r^{\frac{3}{2}}} + O(r^{-\frac{7}{2}})$$
$$\xi_2 = \cos\psi \frac{1}{r^{\frac{3}{2}}} + O(r^{-\frac{7}{2}})$$
$$\xi_3 = 0$$
$$R = (R\cos\psi, R\sin\psi, 0)$$

Note that with this approximation the orbit is *orthogonal* $R \cdot \xi \cong 0$, but this is only and approximation, if more terms are taken into account then:

$$\sin^2 \varkappa = \frac{(\xi \cdot R)^2}{\xi^2 R^2} = 9(I_2 - I_1)^2 \sin^2 \psi \cos^2 \psi \frac{1}{r^4} + O\left(\frac{1}{r^6}\right)$$

the *conical equilibria are not orthogonal.*

Remark 9.6. Conical equilibria are orbits in a plane that does not contain the centre of attraction, although the offset is very small (the offset angle $\varkappa$ decays like r^{-2}) this small effect allows the existence of this family of equilibria.

Remark 9.7. In the case of three different moments of inertia (previous chapter) the oblique orbits appear for only extremely small radius where the approximation looses most of the physical meaning. This case is completely different, the family of conical equilibria exists for r as large as we want.

Remark 9.8. All the three kinds of equilibria for the axisymmetric case have an analogue for the restricted problem, the one in which the satellite's centre of mass is forced to follow a circular orbit and the Hamiltonian equations are solved for the attitude, see Section B

The Reduced Energy Momentum method was applied to the conical equilibria. After calculations omitted here. The Arnold form is diagonal with values:

$$A_1 = \frac{r}{\sin^2 \psi I_2} + O(r^{-1})$$
$$A_2 = (I_2 - I_1)\frac{3\sin^2 \psi}{r^3 \cos^2 \psi}$$

Smale's form was not diagonal, it is a full 2x2 symmetric matrix with entries:

$$S_1 = \frac{\sin^2 \psi}{\cos^2 \psi \; r} + O(r^{-3})$$
$$S_2 = \frac{1}{\cos^2 \psi r^3} + O(r^{-5})$$
$$S_{12} = \frac{-\sin \psi}{\cos^2 \psi r^2} + O(r^{-4})$$

to check for the positive definiteness of this block we can check if $S_1 > 0$ and the determinant $S_1 S_2 - S_{12}^2 > 0$, taking more terms than in the expansions given above:

$$S_1 S_2 - S_{12}^2 = \frac{4I_2 - 3I_1}{r^6} + O(r^{-8})$$

For $I_1 < I_2$, that is for prolate bodies, the equilibrium is formally stable for $I_1 > I_2$ and $4I_2 > 3I_1$ it's unstable and for bigger I_1 we can't conclude nothing.

No algebraic results concerning the linearisation in the inconclusive regions were found, numerical investigations shows that the linear stability regions are in correspondence with the regions for the restricted problem (see [**Bec97**]).

In Figure 7 the numerical tests in the linearised system are also shown in the inconclusive region. In green the spectrally stable points are represented, in white the linearised system is unstable.

In Figure 8 and 10 the transition between cylindrical and and conical equilibria is shown. For $\psi = \frac{\pi}{2}$ the equilibrium point belongs to both families. As ψ becomes larger, the body starts to orbit in a

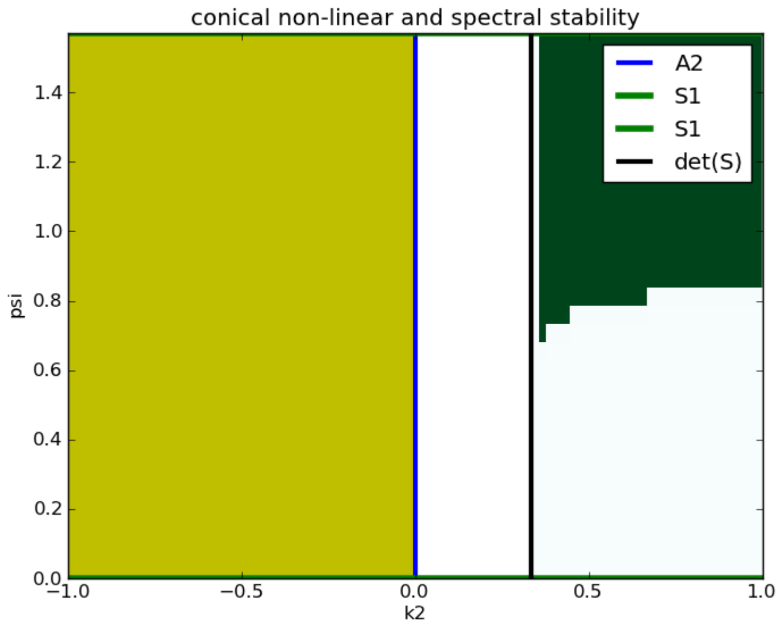


FIG. 7. Stability regions for the conical case . In yellow non-linear stability and in green spectra stability.

"higher" plane, the centre of the orbit now doesn't contain the centre of attraction. The angular velocity η decreases.

This offset achieves a maximum for $\psi = 3\pi/4$, after that the orbit starts to "fall" until for $\psi = \pi$ the angular spin η vanishes and the orbit is again coplanar.

If ψ is increased further, the orbit lowers down and the body describes again a cone. For $\psi = \frac{3\pi}{2}$ the orbit is again a cylindrical equilibria like the original point but with the body turned upside down, or with angular velocities $(-\xi, -\eta)$.

9.5. Transitions between equilibria

The transitions between the different families for large radii are going to be studied.

Oblate bodies. For positive spin rates α, that is spinning in the same direction as the orbital motion, the cylindrical equilibria is non-linearly stable. If α is decreased, $I_1(1+\alpha)-I_2$ becomes negative making the cylindrical equilibrium unstable and bifurcating into the hyperbolic family, which is non-linearly stable.

If α is further reduced $I_1(4+\alpha) - I_2$ becomes negative, the conical family appears. The conical family near the bifurcation point ($\psi = \frac{\pi}{2}$) is spectrally stable but the non-linear stability test is inconclusive.

Schematically this process is described in Figure 11. Non-linearly stable equilibria are represented with a thick solid line, unstable equilibria with a dotted line and spectrally stable equilibria with a thin solid line. Note that each line, according to the persistence theorem represents a 6-dimensional manifold, $SO(3) \times S^1$ action and variation of $|R|$ give the additional dimensions.

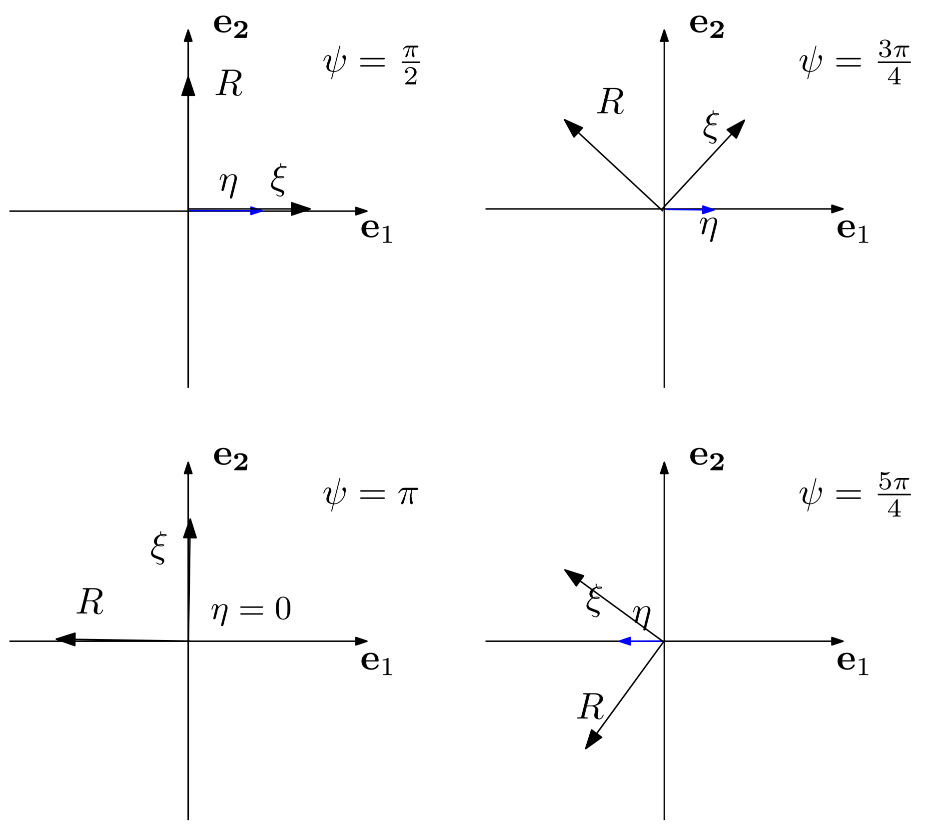


FIG. 8. Transition from cylindrical to conical equilibria as seen from a body fixed frame. Prolate case.

Prolate bodies. For prolate bodies the situation is very similar but now the destabilization of the cylindrical equilibria occurs for positive spin rate α.

The first bifurcation to appear is now the conical family which is always non-linearly stable. Later, for α smaller, the hyperbolic equilibria appears, being always unstable. The final segment of the cylindrical equilibria is spectrally unstable but no conclusion about non-linear stability can be made.

9.6. Conclusions

We have proved that in the unrestricted problem the three families of relative equilibria for the restricted problem (cylindrical, hyperbolic and conical) have an analogue.

The stability results are very similar to the restricted analogues (see [**Bec97**] for a complete study of the restricted problem). This justifies the use of the restricted problem as an approximation.

The conical equilibria are an example of the counter-intuitive non-orthogonal equilibria. In the previous chapter, one can think that the oblique equilibria are possible –only because an approximation is incorrectly used at very low orbits–, but in this axisymmetric case, one can see that the oblique equilibria are actually possible. It is not product of a misused approximation.

Note that Proposition 9.2 states that for a fast enough spinning all the bodies can stabilize the cylindrical equilibria. In some sense, the self-spinning acts as a stabilization mechanism. Bodies which should be in unstable equilibrium according to the previous chapter can be stabilized by self-spinning.

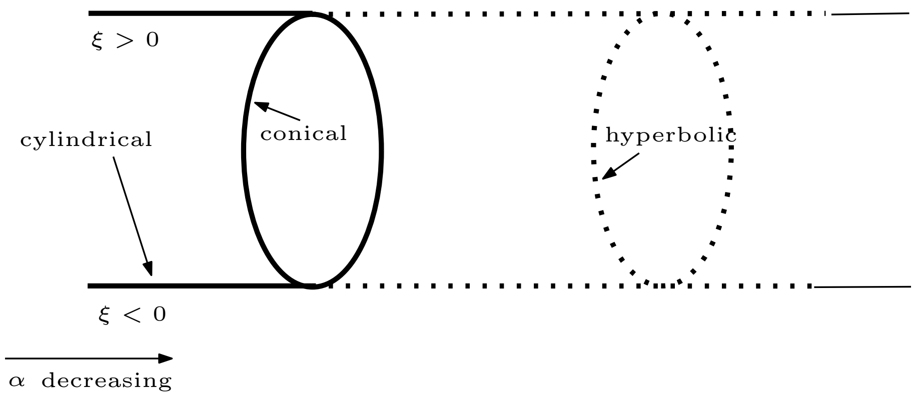


FIG. 9. Bifurcation diagram for an prolate body.

This is the basis of the dual-spin stabilization used in real satellites: a rotor inside the satellite can stabilize its motion [**Hal02**].

A large region of the parameter space is in some cases spectrally stable, but linearly inconclusive. For real satellite stabilization problems, linear stability is not enough. For example, [**JvdH10**], explores the non-linear instability but linear stability that was the cause of many problems on ESA's GEOS-I satellite in 1979. A deeper study has to be made in order to conclude the non-linear behaviour of the equilibria in the inconclusive zone.

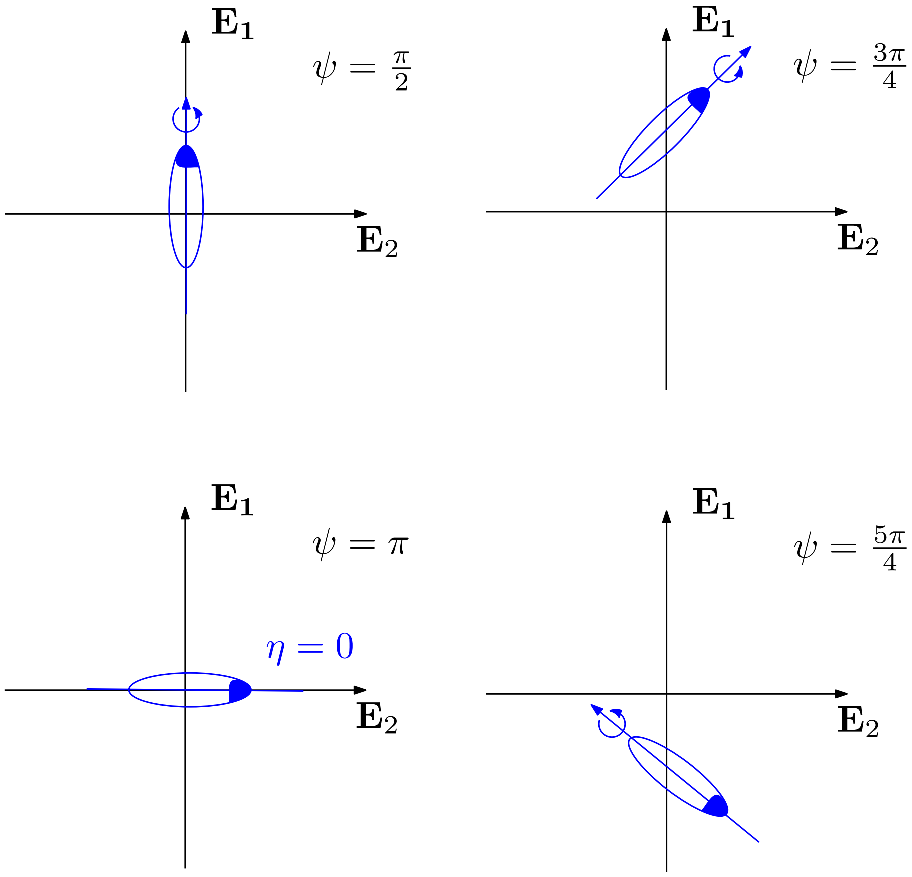


FIG. 10. Transition from cylindrical to conical equilibria as seen from a orbital frame. Prolate case.

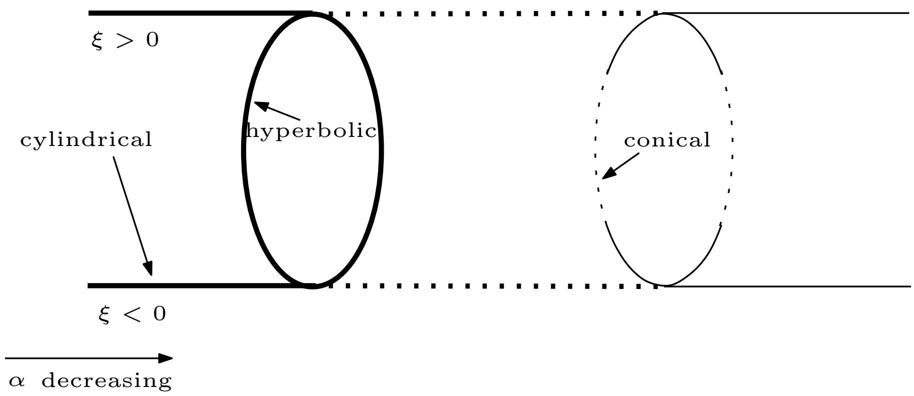


FIG. 11. Bifurcation diagram for a oblate body.

Chapter 10
Relative equilibria for the exact potential

In this section existence results for relative equilibria will be given. No approximate expression for the potential is going to be assumed, in all this section the potential will be given by (15) after adimensionalization, that is:

$$V = \int_{m_2} \frac{dm}{r}$$

The relative equilibria can be obtained as the critical points of the augmented potential, they were described in (29). Assuming $B = Id$, the equations are:

$$\widehat{\xi}(\mathbf{I} - \widehat{R}\widehat{R})\xi = 0 \tag{54a}$$

$$\nabla_R V(R) + \xi(R^T\xi) - R\xi^T\xi = 0 \tag{54b}$$

this 6 equations in ξ, R are algebraically difficult, especially if the potential is written is terms of a integral. Instead of using algebraic manipulations topological considerations are going to be applied to proof the existence of solutions in many cases.

To use topological tools [**WMK91**] writes the equations as solutions to variational problems.

10.1. Orthogonal equilibria

Following [**WMK91**] the first step to study the solutions of (54) is assuming that $\xi \cdot R = 0$, the usual orthogonality condition always present in the classical treatment. Under this assumption equations (54) are reduced to:

$$(\mathbf{I} + |R|^2)\xi = \beta\xi \implies \mathbf{I}\xi = (\beta - |R|^2)\xi \tag{55a}$$

$$\nabla_R V(R) - R|\xi|^2 = 0 \tag{55b}$$

for some $\beta \in \mathbb{R}$. That is, ξ is an eigenvector of the rigid body inertia tensor. By assumption, as $\xi \cdot R = 0$, R has to be in a principal plane.

$|\xi|^2$ can be though as the Lagrange multiplier; (55) are the critical point conditions for the variational problem:

$$\text{critical points of } V(R) \text{ such that } \frac{1}{2}|R|^2 = c \tag{56}$$

That is, if ξ, R are a orthogonal relative equilibria, i.e. solutions of (54):

- R solves (56)
- ξ lies in a principal axis of the rigid body.
- $R \cdot \xi = 0$

On the other hand this argument is reversible. If a vector R_0 solves (56), then there exists a multiplier κ such that:

$$\nabla_R V(R_0) = \kappa R_0 \tag{57}$$

For physically relevant situations, the centre of attraction doesn't lie inside the rigid body, therefore:

$$|R|^2 + R \cdot Q \geq |R|^2 - |R||Q| = |R|(|R| - |Q|) > 0$$

as $\nabla_R V(R) = \int \frac{R+Q}{|R+Q|^3} dm(Q)$ taking the scalar product of (57) with R_0:

$$\kappa |R_0|^2 = \int \frac{|R_0|^2 + Q \cdot R_0}{|R_0 + Q|^3} dm(Q) > 0 \tag{58}$$

this implies $\kappa > 0$.

If R_0 lies in a principal plane, there is ξ such that $\kappa = |\xi|^2$ and ξ is aligned with a principal axis such that $\xi \cdot R_0 = 0$. Actually, both ξ and $-\xi$ are solution of the problem.

Proposition 10.1. *If* (56) *has a solution R_0 in a principal plane, then there are, at least, two orthogonal relative equilibria.*

If the rigid body has a symmetry plane, it has to be a principal plane. So $V(R)$ has at least two critical points in the symmetry plane (one maximum and one minimum), that is:

Proposition 10.2. *If the rigid body possess a symmetry plane, there are, at least, 4 orthogonal relative equilibria, if there are two symmetry planes, there are 8 orthogonal relative equilibria and if the body possess 3 symmetry planes there are 24 orthogonal relative equilibria for each value of $|R|$. For R such that the rigid body doesn't contain the centre of attraction.*

The second order potential has implicitly 3 planes of symmetry (the three principal planes), so in this approximation this proposition gives, without calculations, the 24 classical equilibria of Lagrange.

10.2. Asymmetric molecule

To show the existence of non-orthogonal orbits, in [**WKM90**] a simple body consisting of 6 point masses is introduced, called the asymmetric molecule.

The parameters, expressed in units introduced in Section 6.2 are:

$$\begin{array}{ccc}
m_{x_1} = \dfrac{101}{303} & m_{y_1} = \dfrac{100}{303} & m_{z_1} = \dfrac{99}{303} \\
m_{x_2} = \dfrac{1}{303} & m_{y_2} = \dfrac{1}{303} & m_{z_2} = \dfrac{1}{303} \\
x_1 = 0.1 & y_1 = 0.1 & z_1 = 0.1 \\
x_2 = 10.1 & y_2 = 10.0 & z_2 = 9.90
\end{array}$$

This highly asymmetric body (Figure 1), note that the mass on each axis is 100 times bigger in one end than in the other. Also the three inertias are nearly equal to make the second order approximation inappropriate (see Section 6.1).

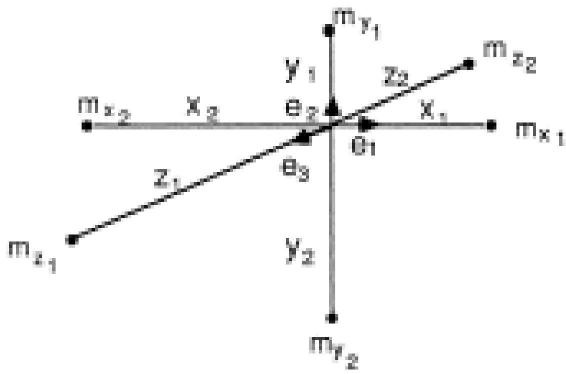


FIG. 1. Asymmetric molecule of [**WKM90**]

The asymmetric molecule has no orthogonal relative equilibria. The proof in [**WKM90**] is of numeric nature, the critical points of (56) are computed on with the condition $|R| = 4000$, none of the critical points lies in a principal plane so no orthogonal relative equilibria can exist for this body at that orbit.

10.3. Oblique equilibria

In [**WMK91**] the equivalence between (54) and the following variational problem is proved:

$$\text{critical points of } V_\xi(R,\xi) = V(R) - \frac{1}{2}\xi \cdot \mathbf{I}\xi - \frac{1}{2}|\hat{\xi}R|^2 \text{ such that } \frac{1}{2}|\xi|^2 = c \tag{59}$$

The problem of this formulation is that "critical points at the infinity" appear, that is, solutions of (59) with ξ and R aligned and $|R| \to \infty$. To solve this [**WMK91**] states another variational problem where a compacity result can be applied:

$$\text{critical points of } \frac{1}{2}\xi \cdot \mathbf{I}\xi + \frac{1}{2}|\hat{\xi}R|^2 \text{ such that } \frac{1}{2}|\xi|^2 = c_1 \text{ and } V(R) = c_2 \tag{60}$$

The equivalence of (59) and (60) is not trivial. If (ξ_0, R_0) is solution of (60), using Lagrange's multipliers $\exists\beta$ such that (ξ_0, R_0) is solution of:

$$\text{critical points of } \beta V(R) - \frac{1}{2}\xi \cdot \mathbf{I}\xi - \frac{1}{2}|\hat{\xi}R|^2 \text{ such that } \frac{1}{2}|\xi|^2 = c_1 \tag{61}$$

If $\beta > 0$ then, rescaling $(\xi_0/\sqrt{\beta}, R_0)$ is solution of:

$$\text{critical points of } V(R) - \frac{1}{2}\xi \cdot \mathbf{I}\xi - \frac{1}{2}|\hat{\xi}R|^2 \text{ such that } \frac{1}{2}|\xi|^2 = c_1/\beta \tag{62}$$

That is, solutions of (59) are in correspondence with solutions of (60) with multiplier $\beta > 0$.

The equations for (60) are:

$$(\mathbf{I} - \hat{R}\hat{R})\xi = \beta'\xi \tag{63a}$$
$$-\hat{\xi}\hat{\xi}R - \beta\nabla_R V(R) = 0 \tag{63b}$$
$$\frac{1}{2}|\xi|^2 = c_1 \tag{63c}$$
$$V(R) = c_2 \tag{63d}$$

The second one can be written as, after scalar multiplication with R:

$$\beta\int \frac{|R|^2 + R\cdot Q}{|R+Q|^3}dm(Q) = (|\xi|^2|R|^2 - (\xi\cdot R)^2)$$

Using Cauchy-Schwartz and condition (58), $\beta \geq 0$ with equality if and only if ξ and R are parallel.

10.4. Solutions with $\beta = 0$

If $\beta = 0$ then ξ, R are parallel, let $a \in \mathbb{R}$ such that $\xi = aR$. The conditions (63) are:

$$\mathbf{I}R = \beta' R \tag{64a}$$
$$\frac{1}{2}|aR|^2 = c_1 \tag{64b}$$
$$V(R) = c_2 \tag{64c}$$

that is, R has to be in a principal axes and a is arbitrary. It can be checked that the function in (60) doesn't have a minimum in this family of points, the Hessian is:

$$\begin{bmatrix} \mathbf{I} - \hat{R}\hat{R} & 2\hat{\xi}\hat{R} - \hat{R}\hat{\xi} \\ 2\hat{R}\hat{\xi} - \hat{\xi}\hat{R} & -\hat{\xi}\hat{\xi} \end{bmatrix} = \begin{bmatrix} \mathbf{I} - \hat{R}\hat{R} & a\hat{R}\hat{R} \\ a\hat{R}\hat{R} & -a^2\hat{R}\hat{R} \end{bmatrix}$$

Taking a basis aligned with principal axis, $\mathbf{I}$ diagonalizes, and $R = (R_1, 0, 0)$ can be assumed. In this case the Hessian is:

$$\begin{bmatrix} I_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_2 + R_1^2 & 0 & 0 & -aR_1^2 & 0 \\ 0 & 0 & I_3 + R_1^2 & 0 & 0 & -aR_1^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -aR_1^2 & 0 & 0 & a^2R_1^2 & 0 \\ 0 & 0 & -aR_1^2 & 0 & 0 & a^2R_1^2 \end{bmatrix}$$

The matrix is positive semi-definite with kernel generated by $(0,0,0,1,0,0)$ is $a \neq 0$. But the character of the critical point is given by the restriction of the Hessian to the linearised constrains: $\mathcal{V} = \{(y_1, y_2)| R\cdot y_1 = 0, \nabla_R V(R)\cdot y_2 = 0\}$, but by (58)

$$(1,0,0)\cdot\nabla V_R(R_0) = bR_0\cdot\nabla_R V(R_0) \neq 0$$

so all the critical points of the variational problem with $\beta = 0$ are local minima. For each choice of restrictions c_1, c_2 there are 6 possibles critical points with $\beta = 0$.

10.5. Existence of oblique equilibria

The solutions of (60) with $\beta = 0$ are local minima. The variational problem (60) is finding the critical points of a function defined over $S^2 \times S^2$, so by compactness there has to be a maximum and a minimum; the maximum has multiplier $\beta > 0$, so the existence of one relative equilibria can always be assured.

Proposition 10.3. *For all value of $|R| > 0$ such that the rigid body doesn't contain the centre of attraction there is, at least, one relative equilibria.*

With this result and the non-existence of orthogonal orbits for the asymmetric molecule, the existence of oblique equilibria for the asymmetric molecule is proved. In [**WMK91**] some numerical simulations are done, they show that although not orthogonal, the offset is extremely small.

As is noted in [**WMK91**] Morse inequalities can be used to find more critical points and discrete symmetry considerations can give more equilibria.

Appendix A
$SO(3)$ and the rigid body

Usual identities in $\mathfrak{so}(3)$ and identifications with $\mathbb{R}^3$ are reviewed.

Some classical results concerning the dynamics of the free rigid body are stated (using [**Mar92**]) as the main reference). Smelt's inertia parameters are introduced.

A.1. $SO(3)$ and $\mathfrak{so}(3)$

The rotation group $SO(3)$ consists of all orthogonal linear transformations of the Euclidean space which have determinant one. The Lie algebra consists of all 3x3 skew matrices[1]. This algebra is 3 dimensional and can be identified with $\mathbb{R}^3$ via the "hat" isomorphism:

$$\Omega = (\Omega^1, \Omega^2, \Omega^3) \mapsto \hat{\Omega} = \begin{bmatrix} 0 & -\Omega^3 & \Omega^2 \\ \Omega^3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix}$$

It can be checked that for any vector $v \in \mathbb{R}^3$: $\hat{\Omega} v = \Omega \times v$, the hat map is the linear map associated with the cross product. The adjoint action of $SO(3)$ into $\mathfrak{so}(3)$ is given by the linear action of $SO(3)$ in $\mathbb{R}^3$, that is, if $M \in SO(3)$:

$$\mathrm{Ad}_M \hat{\Theta} = \widehat{M\Theta}$$

or $\widehat{M\Theta} = M\widehat{\Theta}M^{-1}$.

The left action gives the left invariant vector field $M \mapsto \widehat{\Theta}_M = (M, M\widehat{\Theta}) \in T_M SO(3) \subset T_M GL(3)$

$\mathbb{R}^3$ and its dual can be identified using the dot product. In the same way $\mathfrak{so}(3)^*$ is identified with $\mathfrak{so}(3)$ using the dot product: $\Theta_1 \cdot \Theta_2 = \frac{1}{2}\mathrm{tr}[\widehat{\Theta}_1^T \widehat{\Theta}_2]$

Using left trivialization (body coordinates) the symplectic form on $T^*SO(3)$ is written as (see [**AM78**]) :

$$\omega_{\mathrm{body}}(B, \Pi) = \begin{bmatrix} \hat{\Pi} & Id_3 \\ -Id_3 & 0 \end{bmatrix}$$

A.2. Useful calculations

Many times we will need to compute the first and second variations respect to the $SO(3)$ variable of expressions like:

$$F = v \cdot A\mathbf{I}A^T v$$

[1]This identification makes sense if $SO(3) \subset GL(3)$ and tangent vectors to $GL(3)$ and matrices in $GL(3)$ are identified.

If the variations like $A_\varepsilon = Ae^{\varepsilon\widehat{\delta\theta}}$ the we have:

$$\begin{aligned}\delta F &= v \cdot A\widehat{\delta\theta}\mathbf{I}A^T v - v \cdot A\mathbf{I}\widehat{\delta\theta}A^T v = -A^T v \cdot \widehat{\mathbf{I}A^T v}\delta\theta + A^T v \cdot I\widehat{A^T v}\delta\theta \\ &= (\widehat{\mathbf{I}A^T v} - \widehat{A^T v}\mathbf{I})A^T v \cdot \delta\theta = 2\widehat{\mathbf{I}A^T v}A^T v \cdot \delta\theta \end{aligned} \tag{65}$$

The second variation is computed in the same way:

$$\delta^2 F = 2\delta\theta \cdot \left(-\mathbf{I}\widehat{\widehat{\delta\theta}A^T v}A^T v - \widehat{\mathbf{I}A^T v}\widehat{\delta\theta}A^T v\right) = 2\delta\theta \cdot \left(\widehat{\boldsymbol{I}A^T v}\widehat{A^T v} - \widehat{A^T v}\boldsymbol{I}\widehat{A^T v}\right)\delta\theta \tag{66}$$

A.3. Relative equilibria

The relative equilibria for the rigid body are completely described by the following result:

Proposition A.1. *For a rigid body with three different moments of inertia, the relative equilibria are motions spinning with constant angular speed along one of the three principal axes.*

If the rotation is along the major or minor axis the equilibria is formally stable. If the equilibria is along the middle axis it's unstable.

This result can be easily proved with the relative equilibria characterization using the augmented potential and checking definiteness of Arnold's form.

A.4. Smelt inertia parameters

Instead of working directly with the three moments of inertia of the body it will be more useful to work with Smelt's inertia parameters [**Hal02**], defined as follows:

$$\begin{aligned} k_1 &= \frac{I_2 - I_3}{I_1} \\ k_2 &= \frac{I_1 - I_3}{I_2} \\ k_3 &= \frac{I_2 - I_1}{I_3} \end{aligned}$$

two of them determine the third, for example there is a relation expressing k_1 in terms of k_2 and k_3:

$$k_1 = \frac{k_3 + k_2}{1 + k_2 k_3}$$

the triangular inequalities between the I_i are expressed easily in terms of the Smelt parameters: $|k_i| < 1$. So the space of all possible rigid bodies can be parametrized as the square $[-1, 1] \times [-1, 1]$ in the k_2, k_3 plane. This representation will be used in most of the graphical representations done.

Also if we know some condition like $I_1 + I_2 + I_3 = 1$ we can express the moments of inertia in terms of the Smelt inertia parameters, for example:

$$I_1 = \frac{k_3 k_2 + 1}{k_3 - k_2 + k_3 k_2 + 3} \qquad I_2 = \frac{k_3 + 1}{k_3 - k_2 + k_2 k_2 + 3}$$

Appendix B
Restricted problem

Classically instead of writing the Hamiltonian for the second order potential and deriving the equations of motion it is supposed that the satellite is so small that it has no effect on the orbit determination. Thus the evolution of the centre of the satellite is a solution to the classical Kepler problem for a punctual mass. This problem is called *restricted problem* the one studied in the previous chapters is called the *unrestricted problem*.

Once the evolution of $r(t), p(t)$ has been determined this "solution" is put into the second order equations and solved for $B(t), \Pi(t)$. This approach is the usual one to model the satellite dynamics.

This method of solution can be put in a Hamiltonian formalism and we can study it's stability following methods similar to the methods used before.

B.1. Relative equilibria

As we have explained in Section 6.3, in (B, r) coordinates the kinetic energy is given by:

$$T = \frac{1}{2}\Omega \cdot \mathbf{I}\Omega + \frac{|p|^2}{2m}$$

The second order potential has the expression

$$V = -\frac{GMm}{|r|} - \frac{GM\mathrm{tr}(\mathbf{I})}{2|r|^3} + \frac{3GMB^T r \cdot \mathbf{I}B^T r}{2|r|^5}$$

The relative equilibria for the Keplerian particle are $B(t) = e^{t\hat{\xi}}, r(t) = e^{t\hat{\xi}}r_0$ we can introduce the time dependent coordinates $B = e^{t\hat{\xi}}A$ and angular velocity Ω_A such that $\dot{A} = A\hat{\Omega}_A$.

The kinetic energy and potential become:

$$T = \frac{1}{2}(\Omega_A + A^T e^{-t\hat{\xi}}\xi) \cdot \mathbf{I}(\Omega_A + A^T e^{-t\hat{\xi}}\xi) + \frac{|\hat{\xi}r_0|^2}{2m} = \frac{1}{2}(\Omega_A + A^T\xi) \cdot \mathbf{I}(\Omega_A + A^T\xi) + \frac{|\hat{\xi}r_0|^2}{2m}$$

$$V = -\frac{GMm}{|r_0|} - \frac{GM\mathrm{tr}(\mathbf{I})}{2|r_0|^3} + \frac{3GMA^T r_0 \cdot \mathbf{I}A^T r_0}{2|r_0|^5}$$

both expressions being time-independent.

The Lagrangian for this system in (A, Ω_A) is, after removing constant terms:

$$L = T - V = \frac{1}{2}(\Omega_A + A^T e^{-t\hat{\xi}}\xi) \cdot \mathbf{I}(\Omega_A + A^T e^{-t\hat{\xi}}\xi) - \frac{3GMA^T r_0 \cdot \mathbf{I}A^T r_0}{2|r_0|^5}$$

Taking the Legendre transformation we get the Hamiltonian:

$$H = \frac{1}{2}\Omega_A \cdot \mathbf{I}\Omega_A - \frac{1}{2}A^T\xi \cdot \mathbf{I}A^T\xi + \frac{3GMA^T r_0 \cdot \mathbf{I}A^T r_0}{2|r_0|^5}$$

If $\Pi_A = \mathbf{I}\Omega_A$ then the symplectic form is not the usual one, a momentum shift has to be done.The Hamiltonian is written in the (A, Π_A) coordinates:

$$H = \frac{1}{2}\Pi_A \cdot \mathbf{I}^{-1}\Pi_A - \frac{1}{2}\xi \cdot A\mathbf{I}A^T\xi + \frac{3GMr_0 \cdot A\mathbf{I}A^T r_0}{2|r_0|^5}$$

Using the expression (65) we have:

$$\delta H = \delta\Pi_A \cdot \mathbf{I}^{-1}\Pi_A - \widehat{\mathbf{I}A^T\xi}A^T\xi \cdot \delta\theta + \frac{3GM}{2|r_0|^5}\widehat{\mathbf{I}A^T r_0}A^T r_0 \cdot \delta\theta$$

The condition of critical point for the Hamiltonian gives $\Pi_A = 0$, the angular condition is:

$$-\widehat{\mathbf{I}A^T\xi}A^T\xi + \frac{3GM}{2|r_0|^5}\widehat{\mathbf{I}A^T r_0}A^T r_0 = 0$$

If $\mathbf{i}, \mathbf{j}$ are unit vectors in the direction of ξ, r_0 respectively as we know that $\xi^2 = \frac{GM}{|r_0|^3}$ the equilibrium condition is:

$$-\widehat{\mathbf{I}A^T\mathbf{i}}A^T\mathbf{i} + 3\widehat{\mathbf{I}A^T\mathbf{j}}A^T\mathbf{j} = 0 \tag{67}$$

this condition, to be precise a very similar one with one extra term for the axisymmetric case, was first obtained in [**LR66**].

Among the several possible solutions of (67) we are interested in those in which $A^T\mathbf{i}$ and $A^T\mathbf{j}$ are eigenvectors of $\mathbf{I}$.

We can nondimensionalize the system in such a way that $\xi = \mathbf{i}$ and $r_0 = \mathbf{j}$ so that the Hamiltonian is:

$$H = \frac{1}{2}\Pi_A \cdot \mathbf{I}^{-1}\Pi_A - \frac{1}{2}\mathbf{i} \cdot A\mathbf{I}A^T\mathbf{i} + 3\mathbf{j} \cdot A\mathbf{I}A^T\mathbf{j}$$

Using (66) we get:

$$\delta^2 H = \begin{pmatrix} -\widehat{\mathbf{I}\mathbf{i}}\hat{\mathbf{i}} + \hat{\mathbf{i}}\widehat{\mathbf{I}\mathbf{i}} + 3\widehat{\mathbf{I}\mathbf{j}}\hat{\mathbf{j}} - 3\hat{\mathbf{j}}\widehat{\mathbf{I}\mathbf{j}} & 0 \\ 0 & \mathbf{I}^{-1} \end{pmatrix}$$

Explicitly if $\mathbf{i}$ and $\mathbf{j}$ are the first vectors of the orthogonal basis:

$$\delta^2 H = \begin{pmatrix} 3(I_3 - I_2) & 0 & 0 & 0 & 0 & 0 \\ 0 & I_1 - I_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4(I_1 - I_2) & 0 & 0 & 0 \\ 0 & 0 & 0 & I_1^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_2^{-1} & 0 \\ 0 & 0 & 0 & & 0 & I_3^{-1} \end{pmatrix} \tag{68}$$

Therefore as H is conserved on motions H serves as a Lyapunov function and the equilibria is non-linearly stable if $I_1 > I_3 > I_2$.

The symplectic form has matrix (after the correct momentum shift):

$$\omega = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -I_1 + I_2 + I_3 & 0 & -1 & 0 \\ 0 & I_1 - I_2 - I_3 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

And the linearised system has matrix:

$$A = \begin{pmatrix} 0 & 0 & 0 & I_1^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_2^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & I_3^{-1} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_3 - I_1 & 0 & 0 & 0 & \frac{I_3+I_2-I_1}{I_3} \\ 0 & 0 & I_2 - I_1 & 0 & \frac{I_1-I_3-I_2}{I_2} & 0 \end{pmatrix}$$

The characteristic polynomial for this matrix is:

$$p(s) = (s^2 + 3k_1)(s^4 + (1 - 3k_3 - k_2k_3)s^2 - 4k_2k_3) \tag{69}$$

Where k_i are the Smelt inertia parameters [**Hal02**], [**BH98**]

B.2. Spectral stability

With the factorization (69) of the characteristic polynomial we can easily study the linear stability of the equilibria.

Plotting the conditions on the k_2, k_3 plane we get the following diagram:

The red region is the Lagrange region which is non-linearly stable (68). The other spectrally stable region is the DeBra-Delp region [**Hal02**], [**BH98**].

The detailed analysis of the axisymmetric body in this restricted problem can be found in [**Bec97**], where instead of using the symplectic formulation given here a Poisson formulation is given.

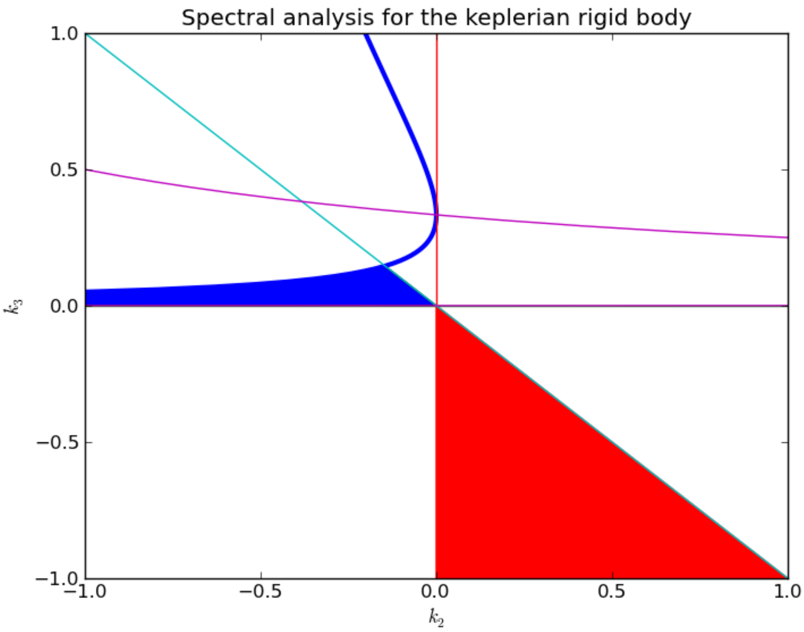


FIG. 1. Spectral stability of the Keplerian system. In red Lagrange's region and in blue, the DeBra-Delp region.

Appendix C
Detailed computations

C.1. Characteristic polynomials for the second order linearisation

The block structure of the linearised system factors the characteristic polynomial as:

$$p(\lambda) = (\lambda^4 + A_2\lambda^2 + A_0)(\lambda^4 + B_2\lambda^2 + B_0) \tag{70}$$

where after manipulations and factorizations:

$$\begin{aligned}
A_2 &= -\frac{1}{2}\frac{-4I_2I_3 + 6I_2^2 - 2I_1^2 + 2I_3I_1 - 4I_1I_2}{I_3I_2R_2^3} \\
&\quad - \frac{1}{2}\frac{-15I_1I_2I_3 + 3I_1I_2 - 6I_2I_3 + 24I_2^2I_3 + 3I_3I_1 - 3I_1^2 + 9I_1^2I_2 - 9I_1I_2^2}{I_3I_2R_2^5} \\
A_0 &= \frac{1}{4}\frac{(2R_2^2 + 3 - 9I_2)(-6I_1 - 8R_2^2 + 15I_2 - 3)(I_1 - I_2)(-I_1 + I_3)}{R_2^{10}I_3I_2} \\
B_2 &= -\frac{1}{2}\frac{6I_2 - 2I_1 - 6I_3}{I_1R_2^3} - \frac{1}{2}\frac{3 - 6I_3 - 3I_2}{R_2^5} \\
B_0 &= 3\left(-\frac{1}{I_1R_2^6} - \frac{9I_2 - 6I_1 - 3}{2I_1R_2^8} - \frac{45I_2 - 15}{2I_1R_2^{10}}\right)(I_2 - I_3)
\end{aligned}$$

C.2. Linearisation of the cylindrical equilibria

After a permutation of the basis given by:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

the linearised system has as matrix:

$$\begin{bmatrix} 0 & \frac{A2}{\xi_1(R_2^2+I_1(1-\alpha))} & 0 & 0 & 0 & 0 \\ -\frac{A1}{\xi_1(R_2^2+I_1(1-\alpha))} & 0 & 0 & C_1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{R_2^2+I_2^2}{(R_2^2+I_1(1+\alpha))^2 I_2} & 0 & 0 \\ 0 & C_2 & S1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & S2 & \end{bmatrix}$$

where

$$C_1 = \frac{R_2\left(I_2I_1\alpha + 2\,{R_2}^2I_1\alpha + {I_1}^2 + {I_1}^2\alpha + {R_2}^2I_1 - 2\,I_2I_1 - 2\,{R_2}^2I_2\right)}{\left(I_1\alpha - I_1 - {R_2}^2\right)\left(-I_1 - I_1\alpha - {R_2}^2 + I_2\right)^2 I_2}$$

$$C_2 = -\frac{R_2\left(I_2I_1\alpha + 2\,{R_2}^2I_1\alpha + {I_1}^2 + {I_1}^2\alpha + {R_2}^2I_1 - 2\,I_2I_1 - 2\,{R_2}^2I_2\right)A2}{\left(I_2 + {R_2}^2\right)\left(I_1\alpha - I_1 - {R_2}^2\right)}$$

and the values of Arnold's and Smale's form are given in (50), (51) respectively.

From this linearised system the characteristic polynomial is:

$$p(\lambda) = (\lambda^4 + D_2\lambda^2 + D_0)(\lambda^2 + S_2) \tag{71}$$

where, in series expansion in R_2:

$$D_2 = \frac{4I_1^2\alpha + I_2I_1 - 6I_2I_1\alpha - I_2^2 + 4I_1^2\alpha^2 + I_1^2}{I_2^2R_2^3} + O(R_2^{-5})$$

$$D_0 = \frac{(I_1(1+\alpha) - I_2)(I_1(4+\alpha) - 4I_2)}{I_2^2R_2^6} + O(R_2^{-8})$$

C.3. Resolution of the conical equilibria

$$\lambda^2 = \frac{1}{2}\,\frac{(\cos(\psi))^2\left(2\,r^2 + \left(9 - 15\,(\cos(\psi))^2\right)(I_1 - I_2)\right)^2}{r^7(\sin(\psi))^2\left(2\,r^2 + \left(3 - 9\,(\cos(\psi))^2\right)(I_1 - I_2)\right)} \tag{72}$$

$$\begin{aligned}\eta = -\frac{1}{2}\,\frac{\cos(\psi)\,(I_1 - I_2)}{(\sin(\psi))^2\,\lambda I_1 r^6}\Big((15\,(\cos(\psi))^2 - 9)(I_1 - I_2) - 8\,r^2 + 6\,r^2\,(\cos(\psi))^2 \\ +2\lambda^2 r^7\,(\sin(\psi))^2\Big)\end{aligned} \tag{73}$$

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