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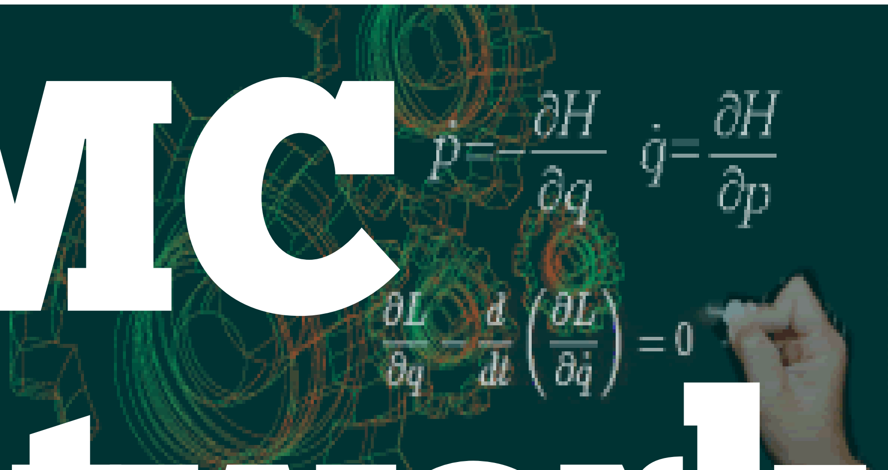

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Reduction in Jet Bundle Theory

based on a course by Marco Castrillón

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Reduction in Jet Bundle Theory

lectures by Marco Castrillón
notes by Cédric M. Campos and David Iglesias

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Reduction in Jet Bundle Theory

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Abstract

These notes follow the course given by M. Castrillón on covariant reduction of Field Theories when the configuration bundle is a principal bundle and the symmetry group is the structure group of the bundle. The preliminary notions needed to understand the geometry behind the construction are also given.

1 Bundles

In this first section, we introduce fiber bundles and associated notions, such as connections or sections. These are the objects in which we later develop our theory.

A fiber bundle is the generalization of the product of two manifolds, but in this case they are "glued" in a non trivial way. Roughly speaking, we could say that the space is locally the product of two manifolds. To be more precise,

Definition 1. A *fiber bundle* is a triple (E, π, M) where M and E are smooth manifolds (of dimension m and $m+n$, respectively) and $\pi \colon E \to M$ is a surjective submersion that satisfies the following condition: there is a smooth manifold F (of dimension n) such that

$$\forall x \in M \ \exists U \in \mathcal{N}(x) \ \exists \Psi \in \mathrm{Diff}(\pi^{-1}(U), U \times F) \ : \ \pi|_{\pi^{-1}(U)} \equiv pr_1 \circ \Psi \,.$$

In such a case, we call:

i) M, the *base space*;

ii) E, the *total space*;

iii) π, the *projection*;

iv) F, the *typical fiber*;

v) $E_x := \pi^{-1}(x)$, the *fiber over* $x \in M$;

vi) $E_u := \pi^{-1}(\pi(u))$, the *fiber through* $u \in E$;

vii) and $\{(U_\alpha, \Psi_\alpha)\}_{\alpha \in A}$, a *trivialization atlas.*

Note 2. A fiber bundle (E, π, M) it is also commonly denoted by the total space E itself or by the projection π.

Note 3. The existence of the typical fiber F could be stated locally, but the definition implies that a unique typical fiber (modulo diffeomorphisms) may be chosen.

Note 4. Any couple of local trivializations (U_α, Ψ_α) and (U_β, Ψ_β) induces, for each $x \in U_\alpha \cap U_\beta$, a diffeomorphism of the typical fiber F.

$$\begin{array}{rcl} (U_\alpha \cap U_\beta) \times F & \longrightarrow & (U_\alpha \cap U_\beta) \times F \\ (x, y) & \longmapsto & (x, \Psi_{\alpha\beta}(x, y)) \end{array} \quad \Longrightarrow \quad \Psi_{\alpha\beta}(x, \cdot) \in \mathrm{Diff}(F)$$

The bundle structure is encoded in two different parts: One is the way in which the fibers are glued together, which can be trivial or not. The other is the particular structure of the fiber itself.

Example 5. The simplest gluing technique would be the Cartesian one. If M and F are manifolds then $(M \times F, pr_1, M)$ is a fiber bundle, called the *trivial fiber bundle*.

Example 6. Depending on the structure of the fiber manifold F, we may have for instance:

i) Vector bundles ($F = V$), the fibers are endowed with a vector space structure: the tangent bundle TM, the cotangent bundle T^*M, the symmetric tensor product of the cotangent bundle S^2T^*M, etc.

ii) Affine bundles ($F = A$), the fibers are endowed with an affine structure. A particular example of this situation is the first-jet bundle $J^1\pi$ of a given fiber bundle. We will develop this notion in Section §3.

Example 7. Let (E, π, M) be a fiber bundle. Then, the set

$$\mathrm{Vert}(\pi) := \{X \in TE \,:\, T\pi(X) = 0\}$$

of vertical vectors with respect to π, together with the restriction of the canonical projection $\tau_{E|\mathrm{Vert}(\pi)} \colon \mathrm{Vert}(\pi) \to E$, is called the *vertical bundle of* π. It can be proved that it is in fact a vector bundle over E.

A particular class of fiber bundles, which is interesting for our purposes, is the family of principal bundles. In this case, the fiber manifold is a Lie group.

Definition 8. A *principal bundle* is a fivefold (G, Φ, P, π, M) such that

i) G is a Lie group;

ii) (P, π, M) is a fiber bundle with typical fiber G;

iii) $\Phi \colon G \times P \to P$ is a free[1] Lie group action of G on P; and

iv) $M = P/G$ and π is the quotient projection.

Note 9. One can check that condition i) to iv) above give as a consequence the existence of a trivialization atlas $\{(U_\alpha, \Psi_\alpha)\}_{\alpha\in A}$ of P, such that $\Psi_\alpha = \pi \times \psi_\alpha$, where $\psi_\alpha(\Phi(g, p)) = \Phi(g, \psi_\alpha(p))$, $p \in \pi^{-1}(U)$, $g \in G$, $\alpha \in A$.

Note 10. The notation $G : P \to M$ is also used and, when the action is a right action (resp. left action), then we note $\Phi(p, g) = R_g(p) = pg$ (resp. $\Phi(p, g) = L_g(p) = gp$). Finally, the induced action will be noted by the same symbol, that is: $\Phi(p, g)$ instead of $T_p\Phi(\,\cdot\,, g)$ and, similarly, R_g (resp. L_g) instead of TR_g (resp. TL_g). In what follows and if nothing else is stated, every action is assumed to be on the right.

[1] A Lie group action $\Phi \colon G \times P \to P$ is *free* if the only element $g \in G$ with fixed points is the identity.

Note 11. Given a principal bundle $\pi\colon P \to M$ with structure group G and a point $u \in P_x$, then $\pi^{-1}(x)$ is just the orbit of u, *i.e.*

$$\pi^{-1}(x) = \{ug \mid g \in G\} = \mathrm{Orb}(u).$$

Therefore, we clearly have that the fibers are diffeomorphic to G.

Example 12. Given a manifold M of dimension m and a point $x \in M$, a linear frame u at x is an ordered basis $v_1, \ldots, v_m$ of the tangent space T_xM. The *frame bundle* $\mathcal{F}M$ is the set formed by all linear frames u at all points of M. Given local coordinates $(x^1, \ldots, x^m)$ on a neighborhood U, we have that each element of any frame $u = \{v_1, \ldots, v_m\}$ can be written as $v_i = \sum_j A_i^j \partial/\partial x^j$. Thus, $\mathcal{F}M$ is locally of the form $U \times \mathrm{Gl}(\mathbb{R}^m)$, where $\mathrm{Gl}(\mathbb{R}^m)$ is the general linear group, with coordinates (x^i, A_i^j). These induce a manifold structure on $\mathcal{F}M$ and a surjective submersion $\mathcal{F}M \to M$ given by $u \longmapsto x$, where u is a linear frame at x. In addition, $\mathcal{F}M \to M$ is a principal bundle with structure group $\mathrm{Gl}(\mathbb{R}^m)$, where given a linear frame u and $A \in \mathrm{Gl}(\mathbb{R}^m)$, the linear frame $uA = \{w_1, \ldots, w_n\}$ is just $w_i = \sum_j A_i^j v_j$.

Given two manifolds M and F, a function between them $f\colon M \to F$ can be reinterpreted considering its graph, that is, the map $s_f\colon M \to M \times F$, $x \longmapsto (x, f(x))$. The map s_f satisfy that $pr_1 \circ s_f = \mathrm{id}_M$. Generalizing to the context of general fiber bundles, we have the notion of sections.

Definition 13. Given a fiber bundle $\pi\colon E \to M$, a *(local) section* of π is a function $s\colon M \to E$ such that $\pi \circ s = \mathrm{id}_M$. The set of sections of π is denoted $\mathrm{Sec}(\pi)$. The set of local sections around a point $x \in M$ is denoted $\mathrm{Sec}_x(\pi)$.

Example 14. As we have just mentioned, sections of the trivial fiber bundle $(M \times F, pr_1, M)$ are just the graph of functions from M to F. Other examples are the following ones.

i) Sections of $\tau_M\colon TM \to M$ are vector fields, $\mathrm{Sec}(\tau_M) = \mathfrak{X}(M)$.
ii) Sections of $\pi_M\colon T^*M \to M$ are differential forms, $\mathrm{Sec}(\pi_M) = \Omega(M)$.
iii) Sections of $S^2T^*M \to M$ are semi-Riemannian metrics (admitting singularities).
iv) Sections of $\mathcal{F}M$ are parallelizations.

There are situations in which the existence of global sections implies conditions on the fiber bundle. In this direction we have:

Proposition 15. *A principal bundle $\pi\colon P \to M$ with structure group G admits a global section if and only if the bundle P is trivializable, that is, $P \cong M \times G$.*

It is obvious that the bundle $M \times G$ admits the global section $s\colon m \in M \mapsto (m, \mathsf{e}) \in M \times G$, where e is the unit element of G. Conversely, if $s\colon M \to P$ is a global section, one can construct the diffeomorphism $\Psi_s\colon (m, g) \in M \times G \mapsto R_g(s(m)) \in P$, which is clearly smooth. Moreover, from the fact that the action is free, one can deduce that it is injective and, using that $\mathrm{Orb}(p) = \pi^{-1}(\pi(p))$, given two elements p, q of a fiber $\pi^{-1}(m)$ there exists a unique $g \in G$ such that $pg = q$, then Ψ_s is also surjective.

Definition 16. A *morphism* of fiber bundles $\pi_i\colon E_i \to M_i$, $i = 1, 2$, is a map $\Psi\colon E_1 \to E_2$ that maps fibers into fibers, *i.e.* it induces a map $\psi\colon M_1 \to M_2$ such that the following diagram commutes:

$$\begin{array}{ccc} E_1 & \xrightarrow{\Psi} & E_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ M_1 & \xrightarrow{\psi} & M_2 \end{array}$$

that is $\pi_2 \circ \Psi = \psi \circ (\pi_1)$. Furthermore,

i) if $\pi_i \colon E_i \to M_i$, $i = 1, 2$, are vector (resp. affine) bundles, Ψ is a *linear (resp. affine) morphism* if it is pointwise linear (resp. affine);

ii) if $\pi_i \colon P_i \to M_i$, $i = 1, 2$, are principal bundles with structure groups G_i, $i = 1, 2$, Ψ is a *principal bundle morphism* if Ψ is pointwise a group homomorphism, *i.e.* $\Psi(y \cdot g) = \Psi(y) \cdot \gamma(g)$ where $\gamma \colon G_1 \to G_2$ is a Lie group homomorphism;

iii) if in addition $\Psi \colon P \to P$ is a principal bundle automorphism over the identity, that is $\Psi \circ R_g = R_g \circ \Psi$ and $\psi = \mathrm{id}_M$, then Ψ is called a *gauge transformation*.

The previous notions of principal bundle automorphisms and gauge transformations can be recast from the infinitesimal point of view.

Definition 17. Let $\pi : P \to M$ be a principal fiber bundle with structure group G.

i) A vector field $X \in \mathfrak{X}(P)$ is said to be *invariant* when it is invariant under the action of G on P, *i.e.* $R_g(X) = X$.

ii) A vector field $X \in \mathfrak{X}(P)$ of a principal fiber bundle P is said to be *vertical* if $T\pi(X) = 0$

Note 18. It is clear that if X is an invariant vector field then its flow $\{\Phi_t\}$ commutes with the action, that is, $\Phi_t \circ R_g = R_g \circ \Phi_t$. Moreover, if X is π-vertical, the "induced flow" on the base is the identity map of M. Therefore, invariant vertical vector fields $X \in \mathfrak{X}(P)$ of a principal fiber bundle P are called *infinitesimal gauge transformations* since their flow are precisely 1-parameter groups of gauge transformations.

Consider a principal fiber bundle $G : P \to M$ and assume that G acts on some manifold F on the left. Then we may define a right action on the product $P \times F$ given by

$$\begin{array}{rcl} g \in G \colon P \times F & \longrightarrow & P \times F \\ (y, f) & \longmapsto & (R_g(y), L_{g^{-1}}(f)) = (y \cdot g, g^{-1} \cdot f) \end{array}$$

This action on the product space $P \times F$ allows us to construct a new fiber bundle over M with typical fiber F.

Definition 19. The fiber bundle *associated* to $G : P \to M$, is the bundle over M given by $P \times_G F := (P \times F)/G$ whose projection is $\pi_{P \times_G F}([(y, f)]_G) := \pi(y)$.

Example 20. Suppose that M has dimension m and set $P = \mathcal{F}M$, the frame bundle of M introduced in Example 12. If we consider the standard left action of the structure group $G = \mathrm{Gl}(\mathbb{R}^m)$ on $F = \mathbb{R}^m$, then the associated bundle $P \times_G F$ is just the tangent bundle TM. Given a class $[(u, (\lambda^1, \ldots, \lambda^m))]_G$ the corresponding tangent vector is $\sum_i \lambda^i v_i$, where $\{v_1, \ldots, v_m\}$ are the elements of the linear frame u.

Example 21. Given an arbitrary principal fiber bundle $G : P \to M$, consider the adjoint action of G on its lie algebra $\mathfrak{g}$, given by $\mathrm{Ad}_g := T_\mathfrak{e}(L_g \circ R_{g^{-1}})$. Then, the associated bundle $\mathrm{ad}\, P := P \times_G \mathfrak{g}$ is a vector bundle with typical fiber $\mathfrak{g}$ and it is called the *adjoint bundle* of P. Moreover, the fibers naturally carry a Lie algebra structure, inherited from the Lie bracket $[\cdot,\cdot]_\mathfrak{g}$ on $\mathfrak{g}$:

$$\big[[(y_1,\xi_1)]_G, [(y_1,\xi_2)]_G\big]_{\mathrm{ad}\, P} := [(y_1, -[\xi_1,\xi_2]_\mathfrak{g})]_G\,,$$

making the adjoint bundle into a bundle of Lie algebras over M.

There is a relation between sections of the adjoint bundle and infinitesimal gauge transformations in the sense of Definition 17.

Proposition 22. *Let $G : P \to M$ be an arbitrary principal fiber bundle. Then, there is a bijection between sections of the adjoint bundle $\mathrm{ad}\, P := P \times_G \mathfrak{g}$ and infinitesimal gauge transformations.*

In order to prove it, we just have to realize that given $[(y,\xi)]_G$ then one can associate the tangent vector $X_y \in T_yP$ by the formula $X_y = \xi_P(y)$, where ξ_P is the infinitesimal vector field associated to $\xi \in \mathfrak{g}$. The equivariance comes from the fact that $[(y,\xi)]_G = [(y\cdot g, \mathrm{Ad}_{g^{-1}}\,\xi)]_G$, $\forall g \in G$.

2 Connections

Given an $\mathbb{R}^k$-valued function $f\colon M \to \mathbb{R}^k$ on a manifold M, one can consider the notion of derivative (or, more generally, directional derivative along a tangent vector $v \in TM$). Now, given a fiber bundle $\pi\colon E \to M$, a section $s \in \mathrm{Sec}(\pi)$ and a curve γ on M with tangent vector $X = \dot\gamma(0)$, it is not possible to define the analogous operator $D_X s$ as

$$D_X s = \lim_{h\to 0} \frac{s(\gamma(h)) - s(\gamma(0))}{h}$$

because there is no way to go from the fiber $E_{s(\gamma(h))}$ to $E_{s(\gamma(0))}$. In order to do so, we introduce the notion of connection. An extended treatise on this subject is the classical book by Kobayashi and Nomizu [3]

Definition 23. An *Ehresmann connection* on a bundle $\pi\colon E \to M$ is a π-horizontal distribution, *i.e.* a distribution $\mathrm{Hor}\colon y \in E \longmapsto \mathrm{Hor}(y) \leq T_yE$ complementary to the vertical bundle:

$$\mathrm{Hor}(y) \oplus \mathrm{Vert}_y(\pi) = T_yE \quad \forall y \in E\,,$$

where $\mathrm{Vert}(\pi) = \ker(\pi)$.

In adapted coordinates (x^i, u^α),

$$\mathrm{Hor}(E) = \left\langle \left\{ \frac{\partial}{\partial x^i} + \mathcal{H}_i^\alpha \frac{\partial}{\partial u^\alpha} \right\} \right\rangle \quad \text{and} \quad \mathrm{Vert}(\pi) = \left\langle \left\{ \frac{\partial}{\partial u^\alpha} \right\} \right\rangle,$$

where the functions $\mathcal{H}_i^\alpha(x^i, u^\alpha)$ are the so-called Christoffel symbols of $\mathrm{Hor}(E)$.

Definition 24. A *principal connection* on a principal bundle $G : P \to M$ is a connection $\mathcal{H}$ invariant under the group action, *i.e.* $R_g(\mathcal{H}(y)) = \mathcal{H}(R_g(y))$.

Let $\xi \in \mathfrak{g}$ and consider the infinitesimal vector field ξ_P on P, whose flow $\{\phi_t\}$ is just $R_{\exp(t\xi)}$. Since $\pi \circ R_g = \pi$, then ξ_P is a vertical vector field, *i.e.* $\xi_P \in \mathrm{Vert}(\pi)$. Thus, we can define a 1-form on P with values on the Lie algebra $\mathfrak{g}$ as follows. Given any $X \in T_yP$, we decompose it in its vertical and horizontal parts, $X = X^{\mathsf{v}} + X^{\mathsf{h}}$. Then, $\omega(X) := \xi$ such that $\xi_P(y) = X^{\mathsf{v}}$. Clearly, a vector X is horizontal if and only if $\omega(X) = 0$. Reciprocally, given a vector valued 1-form $\omega \colon TP \to \mathfrak{g}$, one can define a horizontal distribution by

$$\mathrm{Hor}(y) = \{X \in T_yP \,|\, \omega(X) = 0\}.$$

We denote this horizontal distribution by $\mathrm{Hor}(\omega)$. Summing up,

Proposition 25. *A principal connection $\mathcal{H}$ on a principal bundle $G : P \to M$ is equivalent to an equivariant vector-valued 1-form $\omega \colon TP \to \mathfrak{g}$ ($\omega \circ R_g = \mathrm{Ad}_{g^{-1}} \circ \omega$) such that $\omega(\xi_P) = \xi$ for any $\xi \in \mathfrak{g}$. The 1-form ω is called the* connection 1-form *or the* canonical structure form *associated to $\mathcal{H}$. Moreover, $\mathcal{H} = \mathrm{Hor}(\omega)$.*

There is an object which can be canonically associated to any connection.

Definition 26. Given a principal connection $\mathcal{H}$ on a principal bundle $G : P \to M$, its *curvature* is the 2-form

$$\Omega(X,Y) = \mathsf{d}\omega(X,Y) + [\omega(X), \omega(Y)]\,,$$

where ω is the canonical structure form associated to $\mathcal{H}$.

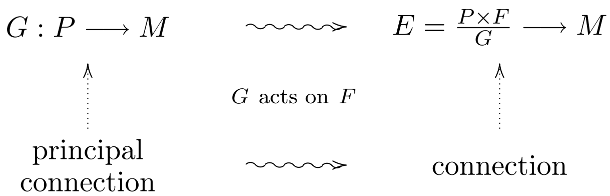
The curvature allows us to characterize the integrability of the horizontal distribution. More precisely,

Theorem 27. *The distribution associated with a principal connection is integrable if and only if it is* flat*, that is, if and only if its curvature vanishes identically, $\Omega = 0$.*

A principal connection $\mathcal{H}$ in a principal bundle $G : P \to M$ may be transferred to its associated bundles. Let F be a manifold such that G acts on it on the left. Given a fixed point $\xi_0 \in E = (P \times F)/G$, let $(p_0, f_0) \in P \times F$ be any class representative of ξ_0, that is $\xi_0 = [p_0, f_0]_G$. We define the horizontal subspace of TE at ξ_0 by

$$\mathrm{Hor}_{\xi_0}(E) := (T_{p_0}\Psi)(\mathcal{H}(p_0))\,,$$

where $\Psi \colon p \in P \mapsto [p, f_0] \in E$. It can be shown that, since $\mathcal{H}$ is invariant, $\mathrm{Hor}_{\xi_0}(E)$ does not depend on the class representative of ξ_0 and, therefore, it defines a connection $\mathrm{Hor}(E)$ on $E \to M$.



The covariant derivative. Consider the particular case where the associated bundle E of a principal bundle P is a vector bundle, that is, when the typical fiber F is a vector space. Let $\mathcal{H}$ be a fixed principal connection in P and lets denote by $\mathrm{Hor}(E)$ the associated G-invariant Ehresmann connection in E. Given a vector field $X \in \mathfrak{X}(M)$, its horizontal lift $X^{\mathsf{h}} \in \mathfrak{X}(E)$ induces a flow $\bar{\tau}_t \colon E \to E$ which is an automorphism over the flow $\tau_t \colon M \to M$ of X. We then define the *covariant derivative* of a section $s \colon M \to E$ along X at $x_0 \in M$ by the expression:

$$\nabla_{X(x_0)} s := \lim_{t\to 0} \frac{\bar{\tau}_{-t}(s(x_t)) - s(x_0)}{t}\,,$$

where $x_t := \tau_t(x_0)$. In adapted coordinates,

$$\nabla_X s = X^i \left(\frac{\partial s^\alpha}{\partial x^i} - \mathcal{H}_i^\alpha \circ s \right) e_\alpha\,,$$

where $X = X^i \partial/\partial x^i$, $s(x) = s^\alpha(x) e_\alpha(x)$, $\{e_\alpha(x)\}$ is a smooth local basis around $x_0 \in M$, and $\mathcal{H}_i^\alpha$ are the Christoffel symbols of $\mathrm{Hor}(E)$.

From the coordinate expression, it is clear that the covariant derivative ∇ is tensorial with respect to the vector field X. Therefore, it may be seen as a map $\nabla \colon \mathrm{Sec}(E) \to \mathrm{Sec}(T^*M \otimes_M E)$. Moreover, there is a unique natural extension of ∇ to an (exterior) covariant derivative $\nabla \colon \Omega^r_M(E) \to \Omega^{r+1}_M(E)$, for any $r \geq 0$, where $\Omega^r_M(E) := \mathrm{Sec}(\Lambda^r M \otimes_M E)$.

Example 28. If $P = \mathcal{F}M$ is the frame bundle of M, $E = TM$ is the tangent bundle of M and $\mathcal{H}_{ij}^k$ are the Christoffel symbols of a linear connection in M, *i.e.* a principal connection $\mathcal{H}$ in $\mathcal{F}M$. Then, the Christoffel symbols of the associated connection $\mathrm{Hor}(TM)$ are $\mathcal{H}_i^k(x^i, v^i) = -\mathcal{H}_{ij}^k(x^i) v^j$ (the Christoffel symbols are linear with respect to the fiber coordinates) and the covariant derivative we have just defined coincides with the usual covariant derivative associated to a linear connection:

$$\nabla_X Y = X^i \left(\frac{\partial Y^k}{\partial x^i} + \mathcal{H}_{ij}^k Y^j \right) \frac{\partial}{\partial x^k}\,.$$

The divergence operator. The usual divergence operator on the vector fields X of a manifold M is defined as the operator

$$\begin{array}{rcl} \mathrm{div}_\eta \colon \mathfrak{X}(M) & \longrightarrow & \mathcal{C}^\infty(M) \\ X & \longmapsto & \mathrm{div}_\eta X \quad \text{s.t. } \mathsf{d}(i_X \eta) = \mathrm{div}_\eta X \cdot \eta\,, \end{array}$$

where η is a fixed volume form over M. Besides of being $\mathbb{R}$-linear, this operator satisfies the Leibniz rule:

$$\mathrm{div}_\eta(f \cdot X) = \langle \mathsf{d}f, X \rangle + f \cdot \mathrm{div}_\eta X\,, \quad f \in \mathcal{C}^\infty(M),\ X \in \mathfrak{X}(M).$$

Similarly, one may define a divergence operator on the sections of $TM \otimes_M \mathrm{ad}^* P$ when a connection $\mathcal{H}$ in P has been given (in addition to a volume form η on M). This divergence operator is the map[2]

$$\begin{array}{rcl} \mathrm{div}_\eta^{\mathcal{H}} \colon \mathfrak{X}(M; \mathrm{ad}^* P) & \longrightarrow & \mathcal{C}^\infty(M; \mathrm{ad}^* P) \\ \mathcal{X} & \longmapsto & \mathrm{div}_\eta^{\mathcal{H}} \mathcal{X} \quad \text{s.t. } \nabla^{\mathcal{H}}(i_{\mathcal{X}} \eta) = \eta \otimes \mathrm{div}_\eta^{\mathcal{H}} \mathcal{X}\,. \end{array}$$

[2]We use here the notation $\mathcal{C}^\infty(M; \mathrm{ad}^* P) := \mathrm{Sec}(\mathrm{ad}^* P)$, $\mathfrak{X}(M; \mathrm{ad}^* P) := \mathrm{Sec}(TM \otimes_M \mathrm{ad}^* P)$ and $\Omega(M; \mathrm{ad}^* P) := \mathrm{Sec}(T^*M \otimes_M \mathrm{ad}^* P)$ since it is more suggestive.

Note that $i_{\mathcal{X}}\eta \in \Omega(M; \mathrm{ad}^* P)$, hence $\nabla^{\mathcal{H}}(i_{\mathcal{X}}\eta) \in \Omega^m(M; \mathrm{ad}^* P)$ and $\mathrm{div}^{\mathcal{H}}_\eta \mathcal{X}$ is well defined. As the usual divergence operator, this one is $\mathbb{R}$-linear and satisfies a Leibniz type rule too:

$$\mathrm{div}_\eta(\langle \xi, \mathcal{X}\rangle) = \langle \nabla^{\mathcal{H}}\xi, \mathcal{X}\rangle + \langle \xi, \mathrm{div}^{\mathcal{H}}_\eta \mathcal{X}\rangle\,, \quad \xi \in \mathcal{C}^\infty(M; \mathrm{ad}\, P),\ \mathcal{X} \in \mathfrak{X}(M; \mathrm{ad}^* P).$$

3 Jet bundles

One of the models of Classical Field Theory is to think of a classical field as a section of fiber bundle, where the base space represents the space-time and the total space represents the configuration space. In order to work in an invariant and geometric form with the derivatives of the fields, *i.e.* the sections of the fiber bundle, one goes through the theory of jet bundles. The reader is referred to the books of Saunders [4] and Binz *et al.* [1].

Two local sections $s_1, s_2 \in \mathrm{Sec}_x(\pi)$ of a bundle $\pi\colon E \to M$ are equivalent at $x \in M$ if $s_1(x) = s_2(x)$ and $T_x s_1 = T_x s_2$. This defines an equivalence relation in the set of local sections around x.

Definition 29. The *1st-jet* of a (local) section $s \in \mathrm{Sec}_x(\pi)$ at $x \in M$ is the equivalence class of the previous relation to which s belongs and it is denoted $j^1_x s$. The collection of such equivalence classes over M is called the *1st-jet bundle* of π and denoted

$$J^1\pi := \{ j^1_x s \,:\, x \in M,\ s \in \mathrm{Sec}_x(\pi)\}\,.$$

Proposition 30. *The 1st-jet manifold $J^1\pi$ of a fiber bundle $\pi\colon E \to M$ may be endowed with a smooth structure manifold. Moreover, adapted coordinates (x^i, u^α) on E induce coordinates $(x^i, u^\alpha, u^\alpha_i)$ on $J^1\pi$ such that*

$$x^i(j^1_x s) = x^i(x)\,, \quad u^\alpha(j^1_x s) = u^\alpha(s(x)) \quad \textit{and} \quad u^\alpha_i(j^1_x s) = \frac{\partial(u^\alpha \circ s)}{\partial x^i}\Big|_x\,.$$

Given a couple of adapted coordinates on E, (x^i, u^α) and (y^j, v^β), they induce coordinates $(x^i, u^\alpha, u^\alpha_i)$ and $(y^j, v^\beta, v^\beta_j)$ on $J^1\pi$. The change of coordinates is then given by the expression

$$v^\beta_j = \left(\frac{\partial v^\beta}{\partial x^i} + u^\alpha_i \frac{\partial v^\beta}{\partial u^\alpha}\right)\frac{\partial x^i}{\partial y^j}\,.$$

Corollary 31. *The 1st-jet manifold $J^1\pi$ of a fiber bundle $\pi\colon E \to M$ is a fiber bundle over M and an affine fiber bundle over E. The projections are given by*

$$\begin{array}{ccc} J^1\pi & \xrightarrow{\pi_{1,0}} & E \\ & \searrow^{\pi_1} & \downarrow{\scriptstyle \pi} \\ & & M \end{array} \qquad \begin{array}{ll} \pi_{1,0}(j^1_x s) = s(x) & (\ \pi_{1,0}(x^i, u^\alpha, u^\alpha_i) = (x^i, u^\alpha)\ ) \\ \pi_1(j^1_x s) = x & (\ \pi_1(x^i, u^\alpha, u^\alpha_i) = (x^i)\ ) \end{array}$$

π_1 and $\pi_{1,0}$ are called respectively the source *and the* target *projections.*

Note 32. Whenever $\pi\colon E \to M$ is a vector bundle, so is $\pi_1\colon J^1\pi \to M$, even though $\pi_{1,0}\colon J^1\pi \to E$ remains affine. But, whenever π is trivial, that is whenever $E = M \times F$ (and whatever the fiber structure be), then $\pi_{1,0}\colon J^1\pi \to E$ may be endowed with a vector bundle structure by using the constant sections of $\pi\colon E = M \times F \to M$. More precisely, $J^1\, pr_1 \cong T^*M \times TF$ where $pr_1\colon M \times F \to M$ is the projection into the first component.

Let us consider now sections of the jet bundle and its relation with connections.

Definition 33. The sections of the target projection $\pi_{1,0}$ are called *jet fields.*

A jet field generalizes the concept of a vector field. Instead of depending on a single parameter, like does the flow of a vector field, the "flow" of a jet field would depend on multiple parameters (in the base manifold). Moreover, they permit to give a new interpretation of Ehresmann connections.

Proposition 34. *Given an arbitrary fiber bundle $\pi\colon E \to M$, consider its 1st-jet manifold $J^1\pi$. The set of jet fields,* $\mathrm{Sec}(\pi_{1,0})$*, and the set of Ehresmann connections in $\pi\colon E \to M$ are in bijective correspondence.*

The proof is quite simple. The idea is based on the following: Given a jet field $\sigma \in \mathrm{Sec}(\pi_{1,0})$, at each point $y \in E$, $\sigma(y)$ is seen as a linear map from $T_{\pi(y)}M$ to T_yE. Using this linear map, we define $\mathrm{Hor}(y) \subseteq T_yE$ to be the image of $T_{\pi(x)}M$ by $\sigma(y)$.

Conversely, given an Ehresmann connection with horizontal distribution $\mathrm{Hor}(E)$, we define a jet field $\sigma_{\mathcal{H}} \in \mathrm{Sec}(\pi_{1,0})$ in such a way that, for any $y \in E$, $\sigma_{\mathcal{H}}(y)$ is the linear map from $T_{\pi(x)}M$ to T_yE whose image is $\mathrm{Hor}(y)$ and projects to the identity.

In the same way that a jet generalizes the concept of a tangent vector, we may generalize the concept of a tangent map, although it is a bit more restrictive.

Definition 35. Given two fiber bundles (E, π, M) and (F, ρ, N), let $\Phi\colon E \to F$ be a fiber bundle morphism such that the base transformation $\phi\colon M \to N$ is a diffeomorphism. The *1st prolongation* of Φ is the map

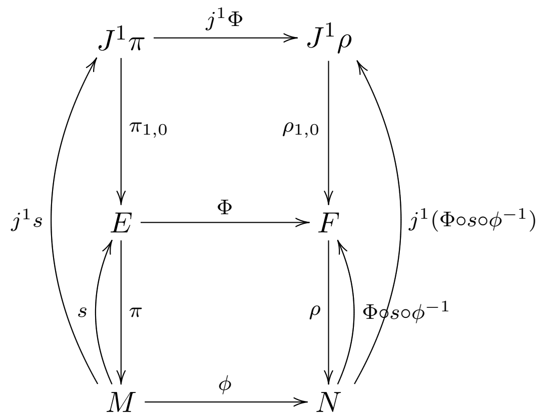
$$j^1\Phi\colon j^1_x s \in J^1\pi \longmapsto j^1_{\phi(x)}(\Phi \circ s \circ \phi^{-1}) \in J^1\rho\,.$$

Note 36. Note that the first prolongation $j^1\Phi$ of a morphism Φ is both, an affine morphism between $(J^1\pi, \pi_{1,0}, E)$ and $(J^1\rho, \rho_{1,0}, F)$, and a morphism between $(J^1\pi, \pi_1, M)$ and $(J^1\rho, \rho_1, N)$. In each case, the induced functions between the base spaces are Φ and ϕ, respectively.

If $(x^i, u^\alpha, u^\alpha_i)$ and $(y^j, v^\beta, v^\beta_j)$ denote adapted coordinates in $J^1\pi$ and $J^1\rho$, respectively, then we have

$$v^\beta_j((j^1\Phi)(x^i, u^\alpha, u^\alpha_i)) = \frac{\mathsf{d}\Phi^\beta}{\mathsf{d}x^i} \cdot \frac{\partial \phi^{-i}}{\partial y^j} = \left(\frac{\partial \Phi^\beta}{\partial x^i} + u^\alpha_i \frac{\partial \Phi^\beta}{\partial u^\alpha}\right) \cdot \frac{\partial \phi^{-i}}{\partial y^j}\,.$$

The expression $\mathsf{d}/\,\mathsf{d}x^i = \partial/\partial x^i + u^\alpha_i \partial/\partial u^\alpha$ is know as the *total coordinate derivative.*



Definition 37. Let (E, π, M) be a fiber bundle. Given a vector field on the total space, $X \in \mathfrak{X}(E)$, we defines its *1st-jet prolongation* to $J^1\pi$ as the vector field $X^{(1)} \in \mathfrak{X}(J^1\pi)$ locally given by the expression

$$X^{(1)} = X^i \frac{\partial}{\partial x^i} + X^\alpha \frac{\partial}{\partial u^\alpha} + \left(\frac{\partial X^\alpha}{\partial x^i} - u_j^\alpha \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial u_i^\alpha},$$

where $X = X^i \partial/\partial x^i + X^\alpha \partial/\partial u^\alpha$.

Note 38. It is easy to show that, if X is π-projectable (or in particular vertical) and Φ_t is its flow, which is a family of bundle automorphisms, then the flow of $X^{(1)}$ is just the 1st-jet prolongation $j^1\Phi_t$.

Jet bundles and principal bundles. Now, let $G : P \to M$ be a principal fiber bundle. The action of G on P induces an action of G on $J^1\pi$: For each $g \in G$, R_g is an automorphism of P over the identity. We define the (left) action of $g \in G$ on $J^1\pi$ as the first prolongation of R_g and, by abuse of notation, we denote it with the same symbol, that is

$$R_g(j_x^1 s) := j_x^1(R_g \circ s), \; s \in \mathrm{Sec}_x(\pi).$$

The quotient of $J^1\pi$ by this induced action will be of particular interest. In fact, we have that

Proposition 39. *The set* $\mathrm{Conn}(P) := J^1\pi/G$ *is an affine bundle over* $M = P/G$*, called the* bundle of connections*. Moreover, it is modeled over the vector bundle* $T^*M \otimes_M \mathrm{ad}\, P \to M$*.*

Note 40. Fixed a connection $\mathcal{H}$ on the bundle P, let $\sigma_{\mathcal{H}} \colon M \to \mathrm{Conn}(P)$ be the associated section of the connection bundle. Recall that $\mathrm{Conn}(P)$ is an affine bundle over M, therefore $\sigma_{\mathcal{H}}$ allows us to identify it with the vector bundle it is modeled over, that is, with $T^*M \otimes_M \mathrm{ad}\, P$. The identification works as follows: To each $\sigma \in \mathrm{Conn}_x(P)$ corresponds the unique element $\alpha \otimes \xi \in T^*M \otimes_M \mathrm{ad}\, P$ such that $\sigma = \sigma_{\mathcal{H}}(x) + \alpha \otimes \xi$, *i.e.* the element $\overrightarrow{\sigma_{\mathcal{H}}(x)\sigma} = \alpha \otimes \xi$.

Note 41. Let $\Phi \colon P \to P$ be a principal bundle automorphism (or a gauge transformation). Since $\Phi \circ R_g = R_g \circ \Phi$, its first prolongation $j^1\Phi$ passes to the quotient inducing an automorphism $\Phi^c \colon \mathrm{Conn}(P) \to \mathrm{Conn}(P)$ (over the same base diffeomorphism).

This can be reproduced infinitesimally. If $X \in \mathfrak{X}(P)$ is an projectable invariant vector field (or an infinitesimal gauge transformation), then its $\{\Phi_t\}_{t\in\mathbb{R}}$ is a 1-parameter group of principal bundle automorphisms. Therefore, the induced family $\{\Phi_t^c\}_{t\in\mathbb{R}}$ in $\mathrm{Conn}(P)$ defines a vector field $X^c \in \mathfrak{X}(\mathrm{Conn}(P))$, which projects over the same vector on M.

Corollary 42. *Let* $G : P \to M$ *be a principal fiber bundle. The set of equivariant jet fields, that is the set of sections* $\sigma \in \mathrm{Sec}(\pi_{1,0})$ *such that* $R_g \circ \sigma = \sigma \circ R_g$ *for any* $g \in G$*, and the set of principal connections in* $\pi \colon P \to M$ *are in bijective correspondence.*

Note 43. By passing to the quotient, to give a section of $J^1\pi/G \to M$ would be equivalent to give an equivariant section of $J^1\pi \to P$ and, thus, a principal connection.

$$\begin{array}{ccc} J^1\pi & & J^1\pi/G = \mathrm{Conn}(P) \\ \downarrow & \dashrightarrow & \downarrow \\ P & & P/G = M \end{array}$$

Moreover, if Φ is a gauge transformation, then Φ^c maps principal connections into principal connection: For instance, if $\omega\colon TP \to \mathfrak{g}$ is a connection 1-form and we define a new connection 1-form $\omega' = (\Phi^{-1})^*\omega$ then, considered as sections of $\mathrm{Conn}(P) \to M$, we have that $\omega' = \Phi^c \circ \omega$.

4 Calculus of variations

Classically, the theory of mechanics seeks for curves $c\colon \mathbb{R} \to Q$ in a configuration space Q that optimize, for a given function $L\colon \mathbb{R} \times TQ \to \mathbb{R}$, an integral functional of the type

$$\int_{t_0}^{t_1} L(t, c(t), \dot{c}(t))\, \mathsf{d}t\,.$$

Recall that, if we consider the trivial bundle $pr_1\colon \mathbb{R} \times Q \to \mathbb{R}$, we have $J^1\, pr_1 \cong \mathbb{R} \times TQ$. If we think of the curves $c\colon \mathbb{R} \to Q$ as sections of the trivial bundle $\mathbb{R} \times Q$, then L may rather be seen as a function $L\colon J^1 pr_1 \to \mathbb{R}$. Now we extend this idea to general fiber bundles $\pi\colon E \to M$.

Definition 44. Given a fiber bundle $\pi\colon E \to M$, a *Lagrangian function* is a function $L\colon J^1\pi \to \mathbb{R}$. The associated *integral action* is the functional

$$\mathcal{A}_L(s) := \int_M L(j^1 s)\eta\,,$$

where η is a fixed volume form on M.

A *critical section* of $\mathcal{A}_L$ is a section $s \in \mathrm{Sec}(\pi)$ for which the derivative of $\mathcal{A}_L$ is null, that is, a section such that

$$\frac{\mathsf{d}}{\mathsf{d}t}\left[\mathcal{A}_L(s_t)\right]\Big|_{t=0} = \frac{\mathsf{d}}{\mathsf{d}t}\left[\int_M L(j^1 s_t)\eta\right]\Big|_{t=0} = 0$$

for any variation s_t of s.

Note 45. In classical field theory, it is usually assumed that the base manifold M is an oriented smooth manifold endowed with a fixed volume form $\eta \in \Omega^m(M)$. In addition and for simplicity, we will also assume that M is a smooth compact manifold without boundary.

Theorem 46 (Euler-Lagrange equations, [1, 4])**.** *Given a Lagrangian function $L\colon J^1\pi \to \mathbb{R}$, critical sections of the associated integral action $\mathcal{A}_L$ are characterized by the* Euler-Lagrange *equations:*

$$\frac{\partial L}{\partial u^\alpha} - \frac{\mathsf{d}}{\mathsf{d}x^i}\left(\frac{\partial L}{\partial u^\alpha_i}\right) = 0\,,$$

where $\mathsf{d}/\,\mathsf{d}x^i$ is the total coordinate derivative

$$\frac{\mathsf{d}}{\mathsf{d}x^j} = \frac{\partial}{\partial x^j} + u^\alpha_j \frac{\partial}{\partial u^\alpha} + u^\alpha_{ij}\frac{\partial}{\partial u^\alpha_i}\,.$$

Note 47. It is important to point out that the Euler-Lagrange equations are of second order. In fact, they are defined in the 2nd-jet manifold $J^2\pi$, which is defined similarly to the 1st-jet manifold $J^1\pi$ and which is an affine subbundle of the iterated 1st-jet manifold $J^1\pi_1$. Even though it would be interesting to define and study with detail these objects, it is not the purpose of these notes. The reader should refer to [4] for further details.

Definition 48. Given a Lagrangian function $L\colon J^1\pi \to \mathbb{R}$, a vertical vector field $X \in \mathfrak{X}(E)$ is said to be a *symmetry* of L if the Lagrangian is constant along $X^{(1)}$, that is if

$$X^{(1)}(L) = 0 \quad \text{or} \quad L \circ j^1\Phi_t = L\,,$$

where Φ_t is the flow of X.

Theorem 49 (Noether)**.** *There is an m-form Θ_L in $J^1\pi$ such that if X if a symmetry of L, then*

$$\mathsf{d}[(j^1 s)^*(i_{X^{(1)}}\Theta_L)] = 0\,,$$

for any critical section $s \in \mathrm{Sec}(\pi)$ of $\mathcal{A}_L$.

Euler-Poincaré reduction. We now consider the case of a field theory where the configuration bundle space is a principal fiber bundle $\pi\colon P \to M$ with structure group G. In such a case, symmetries of a Lagrangian $L\colon J^1\pi \to M$ are assumed to be infinitesimal gauge transformations. Therefore, L will be invariant along any ξ_P, with $\xi \in \mathfrak{g}$ if and only if it is invariant under the induced action of the structure group G, sort of speaking G is the group of symmetries of L (assuming G is connected).

$$\xi_P^{(1)}(L) = 0 \ \forall \xi \in \mathfrak{g} \quad \Longleftrightarrow \quad L \circ j^1 R_g = L \ \forall g \in G$$

Therefore, if we assume that the group of symmetries of L is the whole of the structure group G, we then may define a function $l\colon \mathrm{Conn}(P) = J^1\pi/G \to \mathbb{R}$, which is called the *reduced Lagrangian function.*

The following theorem relates critical sections for G-invariant Lagrangian function L and critical sections for the reduced Lagrangian function l. Some of the technical aspects of the theorem's assertions are explained afterward.

Theorem 50 (Euler-Poincaré reduction [2])**.** *Let $L\colon J^1\pi \to \mathbb{R}$ be a G-invariant Lagrangian. Let $s\colon M \to P$ be a (local) section and $\sigma\colon M \to \mathrm{Conn}(P)$ be the induced reduced (local) section, $\sigma = \mu \circ j^1 s$. Hence, the following points are equivalent:*

i) s_0 *is critical for any infinitesimal variation* δs.

ii) s_0 *satisfies the Euler-Lagrange equations.*

iii) σ_0 *is critical for any infinitesimal variation* $\delta\sigma = \nabla^\sigma \eta$, $\eta \in \mathrm{Sec}(\mathrm{ad}\, P)$.

iv) σ_0 *satisfies the Euler-Poincaré equations:*

$$\mathrm{div}_\eta^{\sigma_0}\left[\frac{\delta l}{\delta \sigma} \circ \sigma_0\right] = 0\,.$$

Note 51. Thanks to the G-invariance of L, in order to compute its infinitesimal variation along a section, one only needs to consider infinitesimal gauge transformations $\xi_E \in \mathfrak{X}(E)$, with $\xi \in \mathfrak{g}$, instead of arbitrary vertical vector fields $X \in \mathfrak{X}(E)$. It turns out that, then, $T\mu(\xi_E^{(1)}) \in \mathfrak{X}(\mathrm{Conn}(P))$ is of the form $\nabla^\sigma \eta$ for some $\eta \in \mathrm{Sec}(\mathrm{ad}\, P)$, where $\mu\colon P \to \mathrm{Conn}(P)$ is the canonical projection. That is, to consider variations $\delta(j^1 s)$ for L of the form $\xi_E^{(1)}$, where $\xi_E \in \mathfrak{X}(E)$ is an infinitesimal gauge transformation, is equivalent to consider variations $\delta\sigma$ for l of the form $\nabla^\sigma \eta$ for some $\eta \in \mathrm{Sec}(\mathrm{ad}\, P)$.

Note 52. The variation of the reduced Lagrangian l with respect to the variation of a fixed connection $\sigma_x \in \mathrm{Conn}(P)$ is defined as its fiber derivative at σ_x. Recall that the connection bundle is an affine bundle modelled over the vector bundle $T^*M \otimes_M \mathrm{ad}\, P$. Therefore, $\delta l/\delta\sigma$ is the fibered map

$$\begin{array}{rcl} \dfrac{\delta l}{\delta \sigma}\colon \mathrm{Conn}(P) & \longrightarrow & TM \otimes_M \mathrm{ad}^* P \\ \sigma_x & \longmapsto & \left.\dfrac{\delta l}{\delta\sigma}\right|_{\sigma_x} \end{array}$$

where

$$\begin{array}{rcl} \left.\dfrac{\delta l}{\delta \sigma}\right|_{\sigma_x}\colon T_x^*M \otimes \mathrm{ad}_x P & \longrightarrow & \mathbb{R} \\ A & \longmapsto & \dfrac{\mathsf{d}}{\mathsf{d}t}\big[l(\sigma_x + t\cdot A)\big]\Big|_{t=0} \end{array}$$

Note that $\frac{\delta l}{\delta\sigma} \circ \sigma_0 \in \mathfrak{X}(M; \mathrm{ad}^* P)$ for a fixed connection $\sigma_0\colon M \to \mathrm{Conn}(P)$.

Note 53. The Euler-Poincaré equation is of first-order.

Note 54. If $\sigma_0\colon M \to \mathrm{Conn}(P)$ is an arbitrary fixed connection, then

$$\mathrm{div}_\eta^\sigma = \mathrm{div}_\eta^{\sigma_0} + \mathrm{ad}^*_{\sigma-\sigma_0}\,.$$

Hence, the Euler-Poincaré equations may be rewritten in the form

$$\mathrm{div}_\eta^{\sigma_0}\left[\frac{\delta l}{\delta\sigma}\right] + \mathrm{ad}^*_{\sigma-\sigma_0}\left[\frac{\delta l}{\delta\sigma}\right] = 0\,.$$

Whenever P is trivial, *i.e.* $P = M \times G$, the bundle of connections $\mathrm{Conn}(P)$ has a canonical vector bundle structure and a distinguished connection may be chosen "$\sigma_0 = 0$". Then, the Euler-Poincaré equation simplifies to

$$\mathrm{div}_\eta\left[\frac{\delta l}{\delta\sigma}\right] + \mathrm{ad}^*_{\sigma}\left[\frac{\delta l}{\delta\sigma}\right] = 0\,,$$

where div_η is the usual divergence operator on M. If in addition we assume that $M = \mathbb{R}$, which corresponds to the case of Classical Mechanics, then we recover the classical Euler-Poincaré equations of mechanics

$$\frac{\mathsf{d}}{\mathsf{d}t}\left[\frac{\delta l}{\delta\sigma}\right] + \mathrm{ad}^*_{\sigma}\left[\frac{\delta l}{\delta\sigma}\right] = 0\,.$$

Corollary 55 (Reconstruction)**.** *The Euler-Lagrange equations of a G-invariant Lagrangian are equivalent to the Euler-Poincaré equation plus the flatness of its solutions (compatibility condition). More precisely, if $\sigma\colon M \to \mathrm{Conn}(P)$ is a flat connection ($\mathrm{Curv}(\sigma) = 0$) that satisfies the Euler-Poincaré equation, then its integral sections $s\colon M \to P$ ($\mu(j^1 s) = \sigma$) satisfies the Euler-Lagrange equations.*

Proposition 56. *Let $L\colon J^1\pi \to \mathbb{R}$ be a G-invariant Lagrangian. Let $s\colon M \to P$ be a (local) section and $\sigma\colon M \to \mathrm{Conn}(P)$ be the induced reduced (local) section, $\sigma = \mu \circ j^1 s$. Hence, the following points are equivalent:*

i) s satisfies the Noether conservation law for any $\xi \in \mathfrak{g}$*:*

$$\mathsf{d}[(j^1 s)^*(i_{\xi_P^{(1)}}\Theta_L)] = 0\,.$$

ii) σ *satisfies the Euler-Poincaré equations:*

$$\mathrm{div}^\sigma_\eta\left[\frac{\delta l}{\delta\sigma}\right] = 0\,.$$

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