INTRODUCTION TO GRADED BUNDLES III: ALGEBROIDS AND LINEARIZATION

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Groupoids

Definition

A groupoid over a set Γ_0 is a set Γ equipped with source and target mappings $\alpha, \beta : \Gamma \to \Gamma_0$, a multiplication map m from $\Gamma_2 \stackrel{\text{def}}{=} \{(g,h) \in \Gamma \times \Gamma |\ \beta(g) = \alpha(h)\}$ to Γ, an injective unit mapping $\epsilon : \Gamma_0 \to \Gamma$, and an inversion mapping $\iota : \Gamma \to \Gamma$, satisfying the following properties (where we write gh for $m(g,h)$ and g^{-1} for $\iota(g)$):

- (associativity) $g(hk) = (gh)k$ in the sense that, if one side of the equation is defined, so is the other, and then they are equal;
- (identities) $\epsilon(\alpha(g))g = g = g\epsilon(\beta(g))$;
- (inverses) $gg^{-1} = \epsilon(\alpha(g))$ and $g^{-1}g = \epsilon(\beta(g))$.

The elements of Γ_2 are sometimes referred to as composable (or admissible) pairs.

A groupoid Γ over a set Γ_0 will be denoted $\Gamma \rightrightarrows \Gamma_0$.

Groupoids: α- and β-fibers

- We can regard Γ_0 as a subset in Γ, and thus ϵ as the identity, that simplifies the picture, since α, β become just projections in Γ.
- The inverse images of points under the source and target maps we call α- and β-fibres. The fibres through a point g, will be denoted by $\mathcal{F}^\alpha(g)$ and $\mathcal{F}^\beta(g)$, respectively.

- Another approach to groupoids is that of Zakrzewski: in the definition of a group just replace maps with relations.

Groupoid as a small category

- The full information about the groupoid is contained in the multiplication relation:

$$\Gamma_3 = \{(x,y,z) \in \Gamma \times \Gamma \times \Gamma \mid (x,y) \in \Gamma_2 \text{ and } z = xy\} \ .$$

- Alternatively, a groupoid $\Gamma \rightrightarrows \Gamma_0$ is defined as a small category, i.e. a category whose objects form a set Γ_0, in which every morphism is invertible. Elements of Γ represent morphisms in this category.
- Any group G is a groupoid over its neutral element, $G \rightrightarrows \{e\}$. Here, any morphism is an automorphism.

Lie groupoids

- In differential geometry we consider differentiable (Lie) groupoids introduced by Ehresmann, i.e. groupoids $G \rightrightarrows M$, where G, G_2, G_3, M are smooth manifolds, α, β are smooth submersions, ϵ is an immersion and ι is a diffeomorphism.
- The unities associativity assumption implies that each element g of G determines the left and right translation maps

$$l_g : \mathcal{F}^\alpha(\beta(g)) \to \mathcal{F}^\alpha(\alpha(g))\,, \quad r_g : \mathcal{F}^\beta(\alpha(g)) \to \mathcal{F}^\beta(\beta(g))\,,$$

- We consider the vector bundle $\tau : A(G) \to M$, whose fiber at a point $x \in M$ is $A_x G = V_{\epsilon(x)}\alpha = Ker(T_{\epsilon(x)}\alpha)$.
- With any sections X of τ, $X \in \mathrm{Sec}(\tau)$, there is canonically associated a left-invariant vector field $\overleftarrow{X}$ on G, the left prolongation of X, namely,

$$\overleftarrow{X}(g) = (T_{\epsilon(\beta(g))} l_g)(X(\beta(g)))\,.$$

for $g \in G$. It is, by definition, tangent to α-fibers.

Lie algebroid of a Lie groupoid

- We can now introduce a Lie algebroid structure $([\cdot,\cdot],\rho)$ on $A(G)$, which is defined by

$$\overleftarrow{[X,Y]} = [\overleftarrow{X},\overleftarrow{Y}], \quad \rho(X)(x) = (T_{\epsilon(x)}\beta)(X(x)), \tag{1}$$

for $X, Y \in \Gamma(\tau)$ and $x \in M$.

- We recall that a Lie algebroid A over a manifold M is a real vector bundle $\tau : A \to M$ together with a skew-symmetric bracket $[\cdot,\cdot]$ on the space $\Gamma(\tau)$ of sections of $\tau : A \to M$ and a bundle map, called the anchor map, such that

$$[X, fY] = f[X,Y] + \rho(X)(f)Y,$$

for $X, Y \in \Gamma(\tau)$ and $f \in C^\infty(M)$. Here we denoted by ρ also the map induced by ρ on sections.

Theorem

For any groupoid $G \rightrightarrows M$, the formulae (1) define on $\tau : A(G) \to M$ the structure of a Lie algebroid.

Pair groupoid

Example

Let M be a set and $\Gamma = M \times M$. define the source and target maps as

$$\alpha(u,v) = u\,, \quad \beta(u,v) = v\,.$$

Then, $M \times M$ is a groupoid over M with the units mapping $\epsilon(u) = (u,u)$, and the partial composition by $(u,v)(v,z) = (u,z)$. In other words,

$$\Gamma_3 = \{(u,v,v,z,u,z) \in \Gamma \times \Gamma \times \Gamma \mid u,v,z \in M\}\,.$$

Note that M can be viewed as embedded into $M \times M$ as the diagonal. If M is a manifold, we deal with a Lie groupoid. We can identify α-fibers as

$$\mathcal{F}^{\alpha}(u,u) = \{(u,v) \,|\, v \in M\} \simeq M\,,$$

so $A(\Gamma)$ with $\mathsf{T}M$. The left invariant vector field $\overleftarrow{X}$ tangent to α-fibers in Γ and corresponding to $X \in \mathcal{X}(M)$ is, under this identification, $\overleftarrow{X}(u,v) \simeq X(v)$. In consequence, the Lie algebroid of Γ is $\mathsf{T}M$ with the bracket of vector fields.

Ehresmann gauge groupoid

Example

For$p: P \to M$ being a principal bundle with the structure group G, consider the set $\Gamma = (P \times P)/G$ of G-orbits, where G acts on $P \times P$ diagonally, $(v, u)g = (vg, ug)$.
For the coset $\langle v|u\rangle$ of (v, u), define the source and target maps

$$\alpha\langle v|u\rangle = p(u) \quad \beta\langle v|u\rangle = p(v)\,,$$

and the (partial) multiplication $\langle w|v\rangle\langle v|u\rangle = \langle w|u\rangle$. It is well defined, as

$$\alpha\langle w|v\rangle = \beta\langle v'|u'\rangle \Leftrightarrow v' = vg$$

and

$$\langle w|v\rangle\langle vg|ug\rangle = \langle wg|vg\rangle\langle vg|ug\rangle = \langle wg|ug\rangle = \langle w|u\rangle = \langle w|v\rangle\langle v|u\rangle\,.$$

In this way we obtained a Lie groupoid $\Gamma = (P \times P)/G \rightrightarrows M = M$, the Ehresmann gauge groupoid of P.
The Lie algebroid of Γ is the Atyiah algebroid $\mathsf{T}P/G$.

Weighted Lie groupoids and algebroids

Besides the compatibility of graded bundle structures, we can consider a compatibility of a graded bundle structure with some other geometric structures, e.g. a Lie algebroid or a Lie groupoid structure.
Thanks to the fact that a graded bundle structure can be expressed in terms of an $(\mathbb{R}, \cdot)$-action, there is an obvious natural concept of such a compatibility.

Definition

A weighted Lie groupoid (resp., a weighted Lie algebroid) of degree k is a Lie groupoid (resp., Lie algebroid) equipped with a homogeneity structure h of degree k such that homotheties h_t act as Lie groupoid (resp., Lie algebroid) morphisms.

We use the name 'weighted', as the term graded Lie algebroids is already used in various meanings.
Note that weighted Lie groupoids (algebroids) of degree 1 have already appeared in the literature under the name VB-groupoids (VB-algebroids).

Weighted Lie theory

Example. If $\mathcal{G}$ is a Lie groupoid (algebroid), then $\mathsf{T}^k\mathcal{G}$ is canonically a weighted Lie groupoid (algebroid) of degree k.

Note that the compatibility condition between the extra homogeneity structure on $\mathcal{G}$ and its Lie algebroid structure we use in applications for mechanics is that the double vector bundle morphism associated with the Lie algebroid structure $\varepsilon : \mathsf{T}^*\mathcal{G} \simeq \mathsf{T}^*\mathcal{G}^* \to \mathsf{T}\mathcal{G}^*$ is a morphisms of triple graded bundles.

Theorem (Bruce-Grabowska-Grabowski)

There is a one-to-one correspondence between weighted Lie groupoids of degree k with simple-connected source fibers and integrable weighted Lie algebroids of degree k, i.e. compatible homogeneity structures can be differentiated and integrated.

Example. Let G be a Lie groupoid with the Lie algebroid $\mathcal{G}$. The weighted Lie algebroid for T^kG is $\mathsf{T}^k\mathcal{G}$.

Holonomic vectors and linearization

- It is well know that $\mathrm{T}^k M$ is cononically embedded in $\mathrm{T}(\mathrm{T}^{k-1}M)$ as the set of "holonomic vectors". Obviously, $\mathrm{T}(\mathrm{T}^{k-1}M) \to \mathrm{T}^{k-1}M$ is a vector bundle and these features we wish to generalise to arbitrary graded bundles.
- Consider F_k equipped with local coordinates (x^A, y^a_w, z^i_k), where the weights are assigned as $\mathrm{w}(x) = 0$, $\mathrm{w}(y_w) = w$ $(1 \leq w < k)$ and $\mathrm{w}(z) = k$.
- Consider the vertical bundle VF_k as a bi-graded subbundle of the tangent bundle $\mathrm{T}F_k$ and employ homogeneous local coordinates with the shifted bi-weight

$$(\underbrace{x^A}_{(0,0)}, \underbrace{y^a_w}_{(w,0)}, \underbrace{z^i_k}_{(k,0)}; \underbrace{\dot{y}^b_w}_{(w-1,1)}, \underbrace{\dot{z}^j_k}_{(k-1,1)}),$$

  where we have shifted the first component of the weight of the linear fibre coordinates so that the vertical bundle itself a double graded bundle of bi-degree $(k-1, 1)$.

Linearization of graded bundles

Definition

The *linearisation of a graded bundle* F_k is the double graded bundle defined as

$$\mathsf{l}(F_k) := VF_k[\nabla^1_{VF_k} \leq k-1]\,,$$

where $\nabla^1_{VF_k} = \mathrm{d}_{\mathsf{T}}\nabla_{F_k} - \nabla_{VF_k}$ and ∇_{VF_k} is the Euler vector field of the vector bundle $VF_k \to F_k$.

Thus on $\mathsf{l}(F_k)$ we have local homogeneous coordinates

$$(\underbrace{x^A}_{(0,0)}, \underbrace{y^a_w}_{(w,0)}; \underbrace{\cancel{z^i_k}}_{(k,0)}, \underbrace{\dot{y}^b_w}_{(w-1,1)}, \underbrace{\dot{z}^j_k}_{(k-1,1)}).$$

The natural projection $p^{VF_k}_{\mathsf{l}(F_k)} : VF_k \to \mathsf{l}(F_k)$ is just 'forgetting' the coordinates z^i_k. The embedding $\iota_{F_k} : F_k \hookrightarrow \mathsf{l}(F_k)$ is given by $\iota_{F_k} = p^{VF_k}_{\mathsf{l}(F_k)} \circ \nabla_F k$. In coordinates,

$$\iota^*_{F_k}(x^A, y^a_w, \dot{y}^b_w, \dot{z}^j_k) = (x^A, y^a_w, w\, y^b_w, k\, z^j_k)\,.$$

Linearization via 'time-derivative'

One can also understand the linearization as adding the 'time-derivative' of variables of non-zero degree. If (x^a, y^A, z^j) are coordinates on a graded bundle F_2 of degrees $0, 1, 2$, respectively. Then, the induced coordinate system on $\mathsf{l}(F_2)$ is

$$(x^a, y^A, \dot{y}^B, \dot{z}^j)\,,$$

where x^a, y^A, $\dot{y}^B$, and $\dot{z}^j$ are of bi-degree $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$, respectively. The transformation laws for the extra coordinates are obtained by differentiation;

$$\begin{aligned}\dot{y}^A &= \dot{y}^B T_B^A(x),\\ \dot{z}^i &= \dot{z}^j T_j{}^i(x) + \dot{y}^B y^A T_{AB}^i(x)\,.\end{aligned}$$

Thus,

$$(x^a, y^A, \dot{y}^B, \dot{z}^j) \mapsto (x^a, y^A)$$

is a linear fibration over F_1.

The embedding $\iota : F_2 \hookrightarrow \mathsf{l}(F_2)$ reads

$$\iota(x^a, y^A, z^j) = (x^a, y^A, y^A, 2z^j)\,.$$

Functor of linearization

The described linearization procedure gives rise to a functor from the category of graded bundles into the category of GrL-bundles.

Theorem (Bruce-Grabowska-Grabowski)

There is a canonical linearization functor $\mathsf{l}:\mathsf{GrB}\to\mathsf{GrL}$ *from the category of graded bundles into the category of* GrL*-bundleswhich assigns, for an arbitrary graded bundle* F_k *of degree* k*, a canonical* GrL*-bundle* $\mathsf{l}(F_k)$ *of bi-degree* $(k-1,1)$ *which is linear over* F_{k-1}*, called the linearization of* F_k*, together with a graded embedding* $\iota: F_k \hookrightarrow \mathsf{l}(F_k)$ *of* F_k *as an affine subbundle of the vector bundle* $\mathsf{l}(F_k)\to F_{k-1}$*.*

Elements of $F_k\subset \mathsf{l}(F_k)$ may be viewed as 'holonomic vectors' in the linear-graded bundle $\mathsf{l}(F_k)$.

Example. We have $\mathsf{l}(\mathsf{T}^kM)\simeq \mathsf{T}\mathsf{T}^{k-1}M$ and

$$\iota:\mathsf{T}^kM\hookrightarrow \mathsf{l}(\mathsf{T}^kM)\simeq \mathsf{T}\mathsf{T}^{k-1}M$$

is the canonical embedding of T^kM as holonomic vectors in $\mathsf{T}\mathsf{T}^{k-1}M$.

Lie algebroid structures on graded bundles

Definition

The *linear dual of a graded bundle* F_k is the dual of the vector bundle $\mathsf{l}(F_k) \to F_{k-1}$, and we will denote this $\mathsf{l}^*(F_k)$.

Definition

We will say that a graded bundle F_k carries the structure of a weighted Lie algebroid if its linearization $\mathsf{l}(F_k)$ is equipped with a weighted Lie algebroid structure, i.e. if there exists a graded morphism

$$\varepsilon : \mathsf{T}^* \mathsf{l}(F_k) \to \mathsf{T}\,\mathsf{l}^*(F_k)\,,$$

such that $(\mathsf{l}(F_k), \varepsilon)$ is a weighted Lie algebroid.

In the above we view $\mathsf{T}^* \mathsf{l}(F_k)$ and $\mathsf{T}\,\mathsf{l}^*(F_k)$ as triple graded bundles. Note that F_k is canonically an affine subbundle in the vector bundle $\mathsf{l}(F_k) \to F_{k-1}$, so in an obvious sense a double affine-linear bundle.

Total linerization of graded bundles

Applying the linearization functor consecutively to a graded bundle of degree k, we arrive at a k-fold graded bundle od degree $(1, \ldots, 1)$, i.e. at a k-fold vector bundle. This functor from $\mathsf{GrB}[k]$ to $\mathsf{VB}[k]$ we call a total linearization. Its image consists of k-fold vector bundles equipped with an action of the symmetry group S_k permuting the order of vector bundle structures (symmetric k-fold vector bundles).

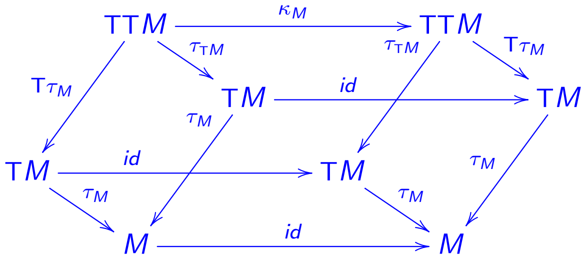
Theorem (Bruce-Grabowski-Rotkiewicz)

There is a canonical functor $\mathsf{L}[k] : \mathsf{GrB}[k] \to \mathsf{VB}[k]$ from the category of graded bundles of degree k into the category of k-fold vector bundles. It gives an equivalence of $\mathsf{GrB}[k]$ with the subcategory (not full) SymVB of symmetric k-fold vector bundles. There is a canonical graded embedding $\iota[k] : F_k \hookrightarrow \mathsf{L}(F_k)$ of F_k as a subbundle of symmetric (holonomic) vectors.

Example. We have $\mathsf{L}(\mathsf{T}^k M) \simeq \mathsf{T}^{(k)} M$, where $\mathsf{T}^{(k)} M = \mathsf{T}\mathsf{T} \cdots \mathsf{T} M$ is the iterated tangent bundle. The action of S_k comes from iterations of the canonical "flips" $\kappa : \mathsf{TT}M \to \mathsf{TT}M$ (see the homework).

Homework

- Problem 1. As tangent vectors are 'infinitesimal curves', elements of the iterated tangent bundle $\mathsf{T}\mathsf{T}M$ are represented by homotopies $f:\mathbb{R}^2 \ni (s,t) \to f(s,t) \in M$. Show that the transposition $(\kappa f)(s,t) = f(t,s)$ induces an automorphism of the double vector bundle $\mathsf{T}\mathsf{T}M$:



- Problem 2. Prove that holonomic vectors in $\mathsf{T}\mathsf{T}M$ are described as those $v \in \mathsf{T}\mathsf{T}M$ which are invariant with respect to κ, $\kappa(v) = v$.