INTRODUCTION TO GRADED BUNDLES II – DOUBLE STRUCTURES

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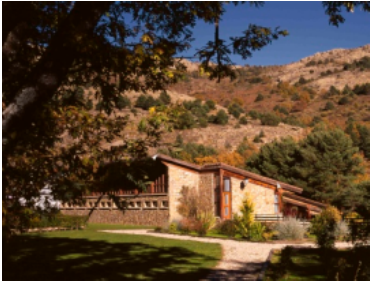

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Double vector bundles

In geometry and applications one often encounters double vector bundles, i.e. manifolds equipped with two vector bundle structures which are compatible in a categorical sense. They were defined by Pradines and studied by Mackenzie, Konieczna (Grabowska), and Urbański as vector bundles in the category of vector bundles. More precisely:

Definition

A double vector bundle $(D; A, B; M)$ is a system of four vector bundle structures

$$\begin{tikzcd} D \arrow[r, "q^D_B"] \arrow[d, "q^D_A"'] & B \arrow[d, "q_B"] \\ A \arrow[r, "q_A"] & M \end{tikzcd}$$

in which D has two vector bundles structures, on bases A and B. The latter are themselves vector bundles on M, such that each of the four structure maps of each vector bundle structure on D (namely the bundle projection, zero section, addition and scalar multiplication) is a morphism of vector bundles with respect to the other structures.

The structure of double vector bundles

- In the above figure, we refer to A and B as the side bundles of D, and to M as the double base.
- In the two side bundles, the addition and scalar multiplication are denoted by the usual symbols $+$ and juxtaposition, respectively.
- We distinguish the two zero-sections, writing $0^A : M \to A$, $m \mapsto 0^A_m$, and $0^B : M \to B$, $m \mapsto 0^B_m$.
- In the vertical bundle structure on D with base A, the vector bundle operations are denoted by $+_A$ and $\cdot_A$, with $\tilde{0}^A : A \to D$, $a \mapsto \tilde{0}^A_a$, for the zero-section.
- Similarly, in the horizontal bundle structure on D with base B we write $+_B$ and $\cdot_B$, with $\tilde{0}^B : B \to D$, $b \mapsto \tilde{0}^B_b$, for the zero-section.
- The two structures on D, namely (D, q^D_B, B) and (D, q^D_A, A) will also be denoted, respectively, by $\tilde{D}_B$ and $\tilde{D}_A$, and called the horizontal bundle structure and the vertical bundle structure.

Double vector bundles - compatibility conditions

The condition that each vector bundle operation in D is a morphism with respect to the other is equivalent to the following conditions, known as the interchange laws:

$$
\begin{aligned}
(d_1 +_B d_2) +_A (d_3 +_B d_4) &= (d_1 +_A d_3) +_B (d_2 +_A d_4),\\
t \cdot_A (d_1 +_B d_2) &= t \cdot_A d_1 +_B t \cdot_A d_2,\\
t \cdot_B (d_1 +_A d_2) &= t \cdot_B d_1 +_A t \cdot_B d_2,\\
t \cdot_A (s \cdot_B d) &= s \cdot_B (t \cdot_A d),\\
\tilde{0}^A_{a_1+a_2} &= \tilde{0}^A_{a_1} +_B \tilde{0}^A_{a_2},\\
\tilde{0}^A_{ta} &= t \cdot_B \tilde{0}^A_{a},\\
\tilde{0}^B_{b_1+b_2} &= \tilde{0}^B_{b_1} +_A \tilde{0}^A_{b_2},\\
\tilde{0}^B_{tb} &= t \cdot_A \tilde{0}^B_{b}.
\end{aligned}
$$

The core

We denote by C the intersection of the two kernels:

$$C = \{c \in D \mid \exists\ m \in M \text{ such that } q^D_B(c) = 0^B_m, \quad q^D_A(c) = 0^A_m\},$$

which is called the core, and together with the map $q_C : c \mapsto m$, (C, q_C, M) is also a vector bundle over M.
Eventually we can write the diagram below to emphasis the core of the relevant double vector bundle.

$$\begin{tikzcd}
& D \arrow[dl, "q^D_A"'] \arrow[dr, "q^D_B"] & \\
A \arrow[dr, "q_A"] & C \arrow[u, hook] \arrow[d, "q_C"] & B \arrow[dl, "q_B"'] \\
& M &
\end{tikzcd}$$

Double vector bundles - reference example

- Let $q_A : A \to M$, $q_B : B \to M$, $q_C : C \to M$ be vector bundles.
- Consider the manifold
$$D = A \times_M B \times_M C\,.$$
- D is a double vector bundle (with side bundles A and B, and the core C) with respect to the obvious projections
$$q^D_A : D \ni (a_m, b_m, c_m) \mapsto a_m \in A\,, \quad q^D_B : D \ni (a_m, b_m, c_m) \mapsto b_m \in B\,,$$
obvious embeddings
$$\tilde{0}^A : A \ni a_m \mapsto (a_m, 0^B_m, 0^C_m) \in D\,, \quad \tilde{0}^B : B \ni b_m \mapsto (0^A_m, b_m, 0^C_m) \in D\,,$$
and obvious vector space structures in fibers:
$$(a_m, b_m, c_m) +_A (a_m, b'_m, c'_m) = (a_m, b_m + b'_m, c_m + c'_m)\,, \text{ etc.}$$
- Actually, every double vector bundle is locally of this form.
- In particular, any Whitney direct sum $A \oplus_M B$, identified with $\simeq A \times_M B$, can be given a double vector bundle structure.

Double vector bundles - canonical example

- Let $\tau : E \to M$ be a vector bundle.
- Consider $D = \mathsf{T}E$. Of course, as every tangent bundle D is canonically a vector bundle over E with respect to the projection $\tau_E : \mathsf{T}E \to E$.
- Applying the tangent functor to all vector bundle structures on E, we get another vector bundle structure on $\mathsf{T}E$, this time with the projection $\mathsf{T}\tau : \mathsf{T}E \to \mathsf{T}M$:

$$\mathsf{T}0_E : \mathsf{T}M \to \mathsf{T}E\,, \quad \mathsf{T}+ : \mathsf{T}E \times_{\mathsf{T}M} \mathsf{T}E \to \mathsf{T}E\,, \quad \mathsf{T}h_t : \mathsf{T}E \to \mathsf{T}E\,.$$

- The proof that these structures are compatible with the ingredients of the vector bundle structure consists of obvious but tiresome calculations.
- But we already know that what matters is only the multiplication by reals, which is in this case the tangent lift $\mathrm{d}_{\mathsf{T}}h$ of the multiplication by reals h in E.

Double Graded Bundles

- We can extend the concept of a double vector bundle of Pradines to double graded bundles.
- However, thanks to our simple description in terms of a homogeneity structure, the 'diagrammatic' definition of Pradines can be substantially simplified.
- As two graded bundle structure on the same manifold are just two homogeneity structures, the obvious concept of compatibility leads to the following:

Definition (Grabowski-Rotkiewicz)

A double graded bundle is a manifold equipped with two homogeneity structures h^1, h^2 which are compatible in the sense that

$$h^1_t \circ h^2_s = h^2_s \circ h^1_t \quad \text{for all } s, t \in \mathbb{R}\,.$$

n-fold Graded Bundles

The above condition can also be formulated as commutation of the corresponding weight vector fields, $[\nabla^1, \nabla^2] = 0$.

For vector bundles this is equivalent to the concept of a double vector bundle in the sense of Pradines and Mackenzie.

Theorem (Grabowski-Rotkiewicz)

The concept of a double vector bundle, understood as a particular double graded bundle in the above sense, coincides with that of Pradines.

All this can be extended to n-fold graded bundles in the obvious way:

Definition

A n-fold graded bundle is a manifold equipped with n homogeneity structures $h^1, \ldots, h^n$ which are compatible in the sense that

$$h^i_t \circ h^j_s = h^j_s \circ h^i_t \quad \text{for all } s, t \in \mathbb{R} \quad \text{and} \quad i, j = 1, \ldots, n \,.$$

Double graded bundles - examples

Proposition

The tangent and phase lifts of graded bundles are compatible with the vector bundle structures of the tangent (resp., cotangent) bundle.

First example: $\mathsf{T}E$.

$$\begin{aligned} \tau : E &\longrightarrow M \\ (x^a, y^i) &\longmapsto (x^a) \end{aligned} \qquad\qquad \begin{aligned} \tau_M : \mathsf{T}M &\longrightarrow M \\ (x^a, \dot{x}^b) &\longmapsto (x^a) \end{aligned}$$

$$\begin{tikzcd}
 & \mathsf{T}E \arrow[dl, "\tau_E"'] \arrow[dr, "\mathsf{T}\tau"] & \\
E \arrow[dr, "\tau"] & E \arrow[u, hook] \arrow[d, "\tau_0"] & \mathsf{T}M \arrow[dl, "\tau_M"] \\
 & M &
\end{tikzcd}$$

$$\nabla^1 = \dot{x}^a \partial_{\dot{x}^a} + \dot{y}^i \partial_{\dot{y}^i}$$

$$\nabla^2 = \mathrm{d}_{\mathsf{T}}(y^i \partial_{y^i}) = y^i \partial_{y^i} + \dot{y}^j \partial_{\dot{y}^j}$$

Double graded bundles - examples

Second example: T^*E^*.

$$\pi_{E^*} : \mathsf{T}^*E^* \longrightarrow E^*$$
$$(x^a, \xi_i, p_b, y^j) \longmapsto (x^a, \xi_i)$$

$$\zeta : \mathsf{T}^*E^* \longrightarrow E$$
$$(x^a, \xi_i, p_b, y^j) \longmapsto (x^a, y^j)$$

$$\begin{tikzcd}
 & \mathsf{T}^*E^* \arrow[dl, "\pi_{E^*}"'] \arrow[dr, "\zeta"] & \\
E^* \arrow[dr, "\pi"] & \mathsf{T}^*M \arrow[u, hook] \arrow[d, "\pi_M"] & E \arrow[dl, "\tau"] \\
 & M &
\end{tikzcd}$$

$$\nabla^1 = p_a\partial_{p_a} + y^i\partial_{y^i}\,, \qquad \nabla^2 = p_a\partial_{p_a} + \xi_i\partial_{\xi_i}\,.$$

Canonical isomorphism

Canonical isomorphism: $\mathsf{T}^*E^* \simeq \mathsf{T}^*E$.

$$
\begin{tikzcd}
 & \mathsf{T}^*E^* \arrow[dl, "\pi_{E^*}"'] \arrow[dr, "\zeta"] & \\
E^* \arrow[dr, "\pi"] & \mathsf{T}^*M \arrow[u, hook] \arrow[d, "\pi_M"] & E \arrow[dl, "\tau"] \\
 & M &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & \mathsf{T}^*E \arrow[dl, "\eta"'] \arrow[dr, "\pi_E"] & \\
E^* \arrow[dr, "\pi"] & \mathsf{T}^*M \arrow[u, hook] \arrow[d, "\pi_M"] & E \arrow[dl, "\tau"] \\
 & M &
\end{tikzcd}
$$

$$(x^a, \xi_i, p_b, y^j) \qquad\qquad (x^a, y^i, p_b, \xi_j)$$

T^*E^* is (symplectically) isomorphic to T^*E. The graph of the canonical d.v.b. anti-symplectic isomorphism $\mathcal{R}$ is the lagrangian submanifold generated in

$$\mathsf{T}^*(E^* \times E) \simeq \mathsf{T}^*E^* \times \mathsf{T}^*E \quad \text{by} \quad E^* \times_M E \ni (\xi, y) \longmapsto \xi(y) \in \mathbb{R}.$$

$$\mathcal{R} : (x^a, y^i, p_b, \xi_j) \longmapsto (x^a, \xi_i, -p_b, y^j)\,.$$

Graded linear bundles

- A double graded bundle whose one structure is linear we will call a graded linear bundle (GrL-bundle). Canonical examples are $\mathsf{T}F$ and T^*F with the lifted and the vector bundle structures.
  Iterated lifts, $\mathsf{T}\mathsf{T}^*F \simeq \mathsf{T}^*\mathsf{T}F$ lead to triple structures of this kind.
- Example. The weight vector field of the lifted graded structure on $\mathsf{T}\mathsf{T}^2M$ with coordinates $(x^a, \dot{x}^b, \ddot{x}^c, \delta x^d, \delta\dot{x}^e, \delta\ddot{x}^f)$ is

$$\nabla^2 = \dot{x}^b\partial_{\dot{x}^b} + \ddot{x}^c\partial_{\ddot{x}^c} + \delta\dot{x}^e\partial_{\delta\dot{x}^e} + \delta\ddot{x}^f\partial_{\delta\ddot{x}^f}\,.$$

  It yields a GrL-bundle with the standard Euler vector field of the tangent bundle structure $\nabla^1 = \delta x^d\partial_{\delta x^d} + \delta\dot{x}^e\partial_{\delta\dot{x}^e} + \delta\ddot{x}^f\partial_{\delta\ddot{x}^f}$.
- Another example: if $\tau: E \to M$ is a vector bundle, then $\wedge^k\mathsf{T}E$ is canonically a GrL-bundle:

$$\begin{tikzcd}
 & \wedge^k\mathsf{T}E \arrow[dl] \arrow[dr] & \\
E \arrow[dr] & & \wedge^k\mathsf{T}M \arrow[dl] \\
 & M &
\end{tikzcd}\,.$$

Linearity

Linearity of different geometrical structures is usually related to some double vector bundle structures.

- A bivector field Π on a vector bundle E is linear if the corresponding map
$$\widetilde{\Pi} : \mathsf{T}^*E \longrightarrow \mathsf{T}E$$
is a morphism of double vector bundles.
- A two-form ω on a vector bundle E is linear if the corresponding map
$$\widetilde{\omega} : \mathsf{T}E \longrightarrow \mathsf{T}^*E$$
is a morphism of double vector bundles.
- A (linear) connection on a vector bundle E is a morphisms of double vector bundles $\Gamma : E \times_M \mathsf{T}M \to \mathsf{T}E$, that acts as the identity on the vector bundles E and $\mathsf{T}M$:
$$(\nabla_X \sigma)^v = \mathsf{T}\sigma(X) - \Gamma(X, \sigma)\,,$$
where $\sigma^v = y^i(x)\partial_{\dot{y}^i}$ is the vertical lift of the section σ:
$$M \ni x \mapsto \sigma(x) = (y^i(x)) \in E\,.$$

Lie algebroids

- $\tau : E \to M$ is a rank-n vector bundle over an m-dimensional manifold M, and $\pi : E^* \to M$ its dual;
- $\mathcal{A}^i(E) = \mathrm{Sec}(\wedge^i E)$, for $i = 0, 1, 2, \ldots$, the module of sections of the bundle $\wedge^i E$.
- $\mathcal{A}(E) = \bigoplus_{i\in\mathbb{Z}} \mathcal{A}^i(E)$ the Grassmann algebra of multisections of E.

We use affine coordinates (x^a, ξ_i) on E^* and the dual coordinates (x^a, y^i) on E.

Definition

A Lie algebroid structure on E is given by a linear Poisson tensor Π on E^*, $[\Pi, \Pi]_{Schouten} = 0$. In local coordinates,

$$\Pi = \frac{1}{2} c^k_{ij}(x)\xi_k \partial_{\xi_i} \wedge \partial_{\xi_j} + \rho^b_i(x)\partial_{\xi_i} \wedge \partial_{x^b}\,,$$

where $c^k_{ij}(x) = -c^k_{ji}(x)$.

Lie algebroids - equivalent definitions

The bivector field Π defines a Poisson bracket $\{\cdot,\cdot\}_\Pi$ on the algebra $C^\infty(E^*)$ of smooth functions on E^* by $\{\phi,\psi\}_\Pi = \langle \Pi, \mathrm{d}\phi\wedge\mathrm{d}\psi\rangle$.

Theorem

A Lie algebroid structure (E,Π) can be equivalently defined as

- *a Lie bracket $[\cdot,\cdot]_\Pi$ on the space $\mathrm{Sec}(E)$, together with a vector bundle morphisms $\rho\colon E\to TM$ (the anchor), such that*

$$[X,fY]_\Pi = \rho(X)(f)Y + f[X,Y]_\Pi, \tag{1}$$

  for all $f\in C^\infty(M)$, $X,Y\in \mathrm{Sec}(E)$,

- *or as a homological derivation d^Π of degree 1 in the Grassmann algebra $\mathcal{A}(E^*)$ (de Rham derivative). The latter is a map $\mathrm{d}^\Pi:\mathcal{A}(E^*)\to\mathcal{A}(E^*)$ such that $\mathrm{d}^\Pi:\mathcal{A}^i(E^*)\to\mathcal{A}^{i+1}(E^*)$, ${\mathrm{d}^\Pi}^2=0$, and that, for $\alpha\in\mathcal{A}^a(E^*)$, $\beta\in\mathcal{A}^b(E^*)$ we have*

$$\mathrm{d}^\Pi(\alpha\wedge\beta) = \mathrm{d}^\Pi\alpha\wedge\beta + (-1)^a\alpha\wedge\mathrm{d}^\Pi\beta\,. \tag{2}$$

Lie algebroids - equivalent definitions

These objects are related to Π according to the formulae

$$\begin{aligned} \iota([X,Y]_\Pi) &= \{\iota(X),\iota(Y)\}_\Pi, \\ \pi^*(\rho(X)(f)) &= \{\iota(X),\pi^* f\}_\Pi\,, \\ (\mathrm{d}^\Pi\mu)^v &= [\Pi,\mu^v]_S\,. \end{aligned}$$

where $\iota(X)(e^*_p)=\langle X(p),e^*_p\rangle$, μ^v is the natural vertical lift of a k-form $\mu\in\mathcal{A}^k(E^*)$ to a vertical k-vector field on E^*, and $[\cdot,\cdot]_S$ is the Schouten bracket of multivector fields. In a local basis of sections $\{e_1,\dots,e_n\}$ of E and the corresponding local coordinates,

$$\begin{aligned} [e_i,e_j]_\Pi(x) &= c^k_{ij}(x)e_k, \\ \rho(e_i)(x) &= \rho^a_i(x)\partial_{x^a}, \\ \mathrm{d}^\Pi f(x) &= \rho^a_i(x)\frac{\partial f}{\partial x^a}(x)e^i, \\ \mathrm{d}^\Pi e^i(x) &= c^i_{lk}(x)e^k\wedge e^l\,. \end{aligned}$$

Lie algebroids - examples

- A Lie algebroid over a single point, with the zero anchor, is a Lie algebra.
- The tangent bundle, $\mathsf{T}M$, of a manifold M, with bracket the Lie bracket of vector fields and with anchor the identity of $\mathsf{T}M$, is a Lie algebroid over M. Any integrable sub-bundle of $\mathsf{T}M$, in particular the tangent bundle along the leaves of a foliation, is also a Lie algebroid.
- If (M, Λ) is a Poisson manifold, then the cotangent bundle T^*M is a Lie algebroid over M. The anchor is the map $\Lambda^\# : \mathsf{T}^*M \to \mathsf{T}M$ The Lie bracket $[,]_\Lambda$ of differential 1-forms satisfies $[\mathrm{d}f, \mathrm{d}g]_\Lambda = \mathrm{d}\{f, g\}_\Lambda$.
- If P is a principal bundle with structure group G, base M and projection p, the G-invariant vector fields on P are the sections of a vector bundle with base M, denoted $E = \mathsf{T}P/G$, and called the Atiyah algebroid of the principal bundle P. This vector bundle is a Lie algebroid, with bracket induced by the Lie bracket of vector fields on P, and with surjective anchor induced by $\mathsf{T}p$.

Lie algebroids - related objects

- For any section $X \in \mathrm{Sec}(E)$, the *Lie derivative* $\mathcal{L}_X$, acting in $\mathcal{A}(E)$ and $\mathcal{A}(E^*)$, is defined in the standard way:

$$
\begin{aligned}
\mathcal{L}_X(f) &= \rho(X)(f)\,, \quad \text{for} \quad f \in C^\infty(M)\,,\\
\mathcal{L}_X(Y_1 \wedge \cdots \wedge Y_l) &= \sum_{i=1}^{l} Y_1 \wedge \cdots \wedge [X, Y_i]_\Pi \wedge \cdots \wedge Y_a\,,\\
\mathcal{L}_X(\alpha) &= i_X \mathrm{d}^\Pi + \mathrm{d}^\Pi i_X\,.
\end{aligned}
$$

- We know that the linear bivector field Π on E^* induces a morphism of double vector bundles $\Pi^\# : \mathsf{T}^*E^* \to \mathsf{T}E^*$, covering the identity on E^*. Composing it with the canonical isomorphism $\mathcal{R} : \mathsf{T}^*E \to \mathsf{T}^*E^*$, we get a morphism of double vector bundles

$$
\varepsilon_\Pi : \mathsf{T}^*E \to \mathsf{T}E^*\,,
$$

covering the identity on E^*.

General algebroids

- A general algebroid is a double vector bundle morphism covering the identity on E^*:

$$
\begin{tikzcd}
 & \mathsf{T}^*E \arrow[rrr, "\varepsilon"] \arrow[dr, "\pi_E"] \arrow[ddl, "\mathsf{T}^*\tau"'] & & & \mathsf{T}E^* \arrow[dr, "\mathsf{T}\pi"] \arrow[ddl, "\tau_{E^*}"'] & \\
 & & E \arrow[rrr, "\rho"] \arrow[ddl, "\tau"'] & & & \mathsf{T}M \arrow[ddl, "\tau_M"'] \\
E^* \arrow[rrr, "id"] \arrow[dr, "\pi"] & & & E^* \arrow[dr, "\pi"] & & \\
 & M \arrow[rrr, "id"] & & & M &
\end{tikzcd}
$$

In local coordinates,

$$\varepsilon(x^a, y^i, p_b, \xi_j) = (x^a, \xi_i, \rho^b_k(x)y^k, c^k_{ij}(x)y^i\xi_k + \sigma^a_j(x)p_a)\,.$$

Algebroids

Any such morphism is associated with a linear tensor field on E^*,

$$\Pi_\varepsilon = c_{ij}^k(x)\xi_k\partial_{\xi_i}\otimes\partial_{\xi_j} + \rho_i^b(x)\partial_{\xi_i}\otimes\partial_{x^b} - \sigma_j^a(x)\partial_{x^a}\otimes\partial_{\xi_j}.$$

We speak about a skew algebroid (resp. Lie algebroid) if the tensor Π_ε is skew-symmetric (resp., Poisson tensor).

Theorem

An algebroid structure (E,ε) can be equivalently defined as a bilinear bracket $[\cdot,\cdot]_\varepsilon$ on sections of $\tau\colon E\to M$, together with vector bundle morphisms a_l^ε, $a_r^\varepsilon\colon E\to \mathsf{T}M$ (left and right anchors), such that

$$[fX,gY]_\varepsilon = f(a_l^\varepsilon\circ X)(g)Y - g(a_r^\varepsilon\circ Y)(f)X + fg[X,Y]_\varepsilon$$

for $f,g\in\mathcal{C}^\infty(M)$, $X,Y\in\mathrm{Sec}(E)$.
For skew-algebroids the bracket is skew-symmetric, thus $a_l^\varepsilon = a_r^\varepsilon = \rho^\varepsilon$, and for Lie algebroids it satisfies the Jacobi identity,

$$[[X,Y]_\varepsilon,Z]_\varepsilon = [X,[Y,Z]_\varepsilon]_\varepsilon - [Y,[X,Z]_\varepsilon]_\varepsilon\,.$$

Non-holonomic reduction

Let ε be a Lie algebroid structure on a vector bundle E over M associated with the tensor Π_ε.
For a linear subbundle D in E, supported on the whole M, consider a decomposition

$$E = D \oplus_M D^\perp \tag{3}$$

and the associated projection $P : E \to D$. With such a decomposition we can associate a skew-algebroid structure on D.
The projection P induces a map on sections: $P : \mathrm{Sec}(E) \to \mathrm{Sec}(D)$ and thus a bracket

$$[X, Y]_{\varepsilon_P} = P[X, Y]_\varepsilon \tag{4}$$

on sections of D – the nonholonomic restriction of $[\cdot, \cdot]$ along P.
This is a skew algebroid bracket with the original anchor.
A particular case of this construction can be applied to a vector subbundle D of $\mathsf{T}M$, for M equipped with a Riemannian structure.

Homework

- Problem 1. Prove that the tangent and cotangent bundles of a double graded bundle are canonically triple graded bundles.
- Problem 2. On the space of curves $\gamma : \mathbb{R} \to M$ in a manifold M, consider the $(\mathbb{R}, \cdot)$-action $\hat{h}_t(\gamma)(s) = \gamma(ts)$.
  Prove that this action induces the canonical homogeneity structure on the space $\mathsf{T}^2 M$ of second jets of these curves.
- Problem 3. Show that the second tangent lift of a homogeneity structure h on F, defined by $(\mathsf{T}^2 h)_t = \mathsf{T}^2(h_t)$, is a homogeneity structure on $\mathsf{T}^2 F$. Here $\mathsf{T}^2\phi : \mathsf{T}^2 M \to \mathsf{T}^2 N$ denotes the obvious second-jet prolongation of $\phi : M \to N$ to the second tangent bundles.
- Problem 4. Prove that the lifted homogeneity structure $\mathsf{T}^2 h$ from the previous problem is compatible with the canonical homogeneity structure on the second tangent bundle $\mathsf{T}^2 F$.
- Problem 5. Show that the anchor map induces, for any Lie algebroid E, a homomorphism of the Lie algebroid bracket into the Lie bracket of vector fields:
  $$\rho([X, Y]_\varepsilon) = [\rho(X), \rho(Y)]_{vf} .$$