Introduction to the theory of graded bundles

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We introduce the concept of a *graded bundle* which is a natural generalization of that of a vector bundle. A canonical example is the higher tangent bundle T^nM playing a fundamental role in higher order Lagrangian formalisms. Graded bundles *of degree* n are particular graded manifolds of degree n in the sense that we can choose an atlas whose local coordinates are homogeneous functions of degrees $0, 1, \ldots, n$. Note that graded bundles of degree 1 are just vector bundles. A little more specifically, a vector bundle structure $E \to M$ is encoded by assigning a weight of zero to the base coordinates and one to the linear coordinates on the total space. Thus, there is essentially a one-to-one correspondence between vector bundles and graded bundles for which we can assign the weight zero and one. The condition of the weight to be one for the fibre coordinates is a restatement of linearity. Thus philosophically, a graded bundle should be viewed as a "non-linear or higher vector bundle".

We prove that graded bundles have a convenient equivalent description as *homogeneity structures*, i.e. manifolds with a smooth action of the multiplicative monoid $(\mathbb{R}_{\geq 0}, \cdot)$ of non-negative reals. The main result states that each homogeneity structure admits an atlas whose local coordinates are homogeneous. Considering a natural compatibility condition of homogeneity structures we formulate, in turn, the concept of a *double* (*r-fold*, in general) *graded bundle* which gives a broad generalization of the concept of a *double vector bundle*. Double graded bundles are proven to be locally trivial in the sense that we can find local coordinates which are simultaneously homogeneous with respect to both homogeneity structures.

We then investigate the geometry of graded vector bundles, the *linearization* and "*superization*" functors, and the concepts of *duality* and *tensor products* for graded bundles. Graded bundles equipped with additional compatible structures, such as *graded-linear bundles*, *weighted Lie algebroids* and *weighted Lie groupoids*, together with natural examples, as well as applications to geometrical mechanics with higher order Lagrangians will also be studied.

References

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