On the relationship between the energy shaping and the Lyapunov constraint based methods

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Energy shaping method

Consider a function $H \in C^\infty(T^*Q)$, a codistribution $W \subset T^*Q$ and a critical point $\alpha^\bullet \in T^*Q$ of X_H. Note that $(H, \mathrm{vlift}(W))$ defines an underactuated Hamiltonian system.

Problem P. Find a control signal $Y \subset \mathrm{vlift}(W)$ such that the closed-loop system defined by $X := X_H + Y$ is stable at $\alpha^\bullet$.

ESM. Find $V \in C^\infty(T^*Q)$ and two vertical fields $Z_g, Z_d \in \mathfrak{X}(T^*Q)$ such that:

1. V is positive-definite w.r.t. $\alpha^\bullet$,
2. $\langle dV(\alpha), Z_g(\alpha)\rangle = 0$, i.e. Z_g is a *gyroscopic* force,
3. $\langle dV(\alpha), Z_d(\alpha)\rangle \leq 0$, i.e. Z_d is a *dissipative* force,
4. $X_V + Z_g - X_H, Z_d \subset \mathrm{vlift}(W)$,
5. and $\alpha^\bullet$ is critical for $X := X_V + Z_g + Z_d$.

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$$X = X_H + \underbrace{(X_V + Z_g - X_H) + Z_d}_{Y} = X_H + Y$$

and $Y \subset \mathrm{vlift}(W)$. So, Y solves the problem **P**. This stabilization method has its roots 35 years ago. Its present form has more than 15 years.

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It is based on the idea of **feedback equivalence**. Usually, H and V are assumed to be simple, i.e. of the form

$$F(q,p) = \frac{1}{2} p_i \, \mathbb{F}^{ij}(q) \, p_j + f(q), \quad \text{with} \quad \mathbb{F} > 0.$$

In a recent version (Chang, 2010), where particular Z_g's are chosen, it was shown that V, Z_g and Z_d (i.e Y) exist if and only if

$$\left(\frac{\partial \mathbb{V}^{ij}(q)}{\partial q^a} \, \mathbb{H}^{ak}(q) - \frac{\partial \mathbb{H}^{ij}(q)}{\partial q^a} \, \mathbb{V}^{ak}(q) \right) \, p_i p_j p_k = 0$$

(**kinetic matching conditions**) and

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(**potential matching conditions**) hold, for all $p \in \mathbb{V}^{-1}\left(W^0\right)$. The control laws Y are explicitly given by the method.

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Lyapunov constraint based method

Given $X \in \mathfrak{X}(M)$ and $\alpha^{\bullet} \in M$ critical for X, the existence of a related Lyapunov function $V \in C^{\infty}(M)$ implies the *constraint*

$$\frac{d}{dt} V(\Gamma(t)) = \langle dV(\Gamma(t)), \Gamma'(t) \rangle = -\mu(\Gamma(t)),$$

with $\mu(\alpha) := -\langle dV(\alpha), X(\alpha) \rangle \geq 0$ and Γ a trajectory of X.

Fix $H \in C^{\infty}(T^*Q)$, a codistribution $W \subset T^*Q$, and $\alpha^{\bullet} \in T^*Q$ critical for X_H, and consider the problem **P**. If we impose the constraint above for some positive-definite function V w.r.t. $\alpha^{\bullet}$ and a non-negative function μ, any constraint force Y would satisfy

$$\langle dV(\alpha), X_H(\alpha) + Y(\alpha) \rangle = -\mu(\alpha) \leq 0.$$

Consequently, if $Y \subset \mathrm{vlift}(W)$, then Y would solve **P**. This gives rise to the following stabilization method (Grillo, 2009).

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Any control law Y that solves **P**, and such that its related closed-loop system has a Lyapunov function (in particular, those given by the **ESM**), can be constructed with this method. Assuming that H and V are simple, such Y exists if and only if (Grillo, Marsden, Nair, 2011)

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Suppose that $dimQ = 2$ and choose canonical coordinates (x, y, p_x, p_y) such that $W = \langle dy \rangle$ and $\alpha^\bullet = (0, 0, 0, 0)$. Suppose also that

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By using the ESM, Chang showed that the condition:

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Suppose that $dimQ = 2$ and choose canonical coordinates (x, y, p_x, p_y) such that $W = \langle dy \rangle$ and $\alpha^\bullet = (0, 0, 0, 0)$. Suppose also that

$$H(x, y, p_x, p_y) = \frac{1}{2} \begin{pmatrix} p_x \\ p_y \end{pmatrix}^t \begin{bmatrix} a(x, y) & b(x, y) \\ b(x, y) & c(x, y) \end{bmatrix} \begin{pmatrix} p_x \\ p_y \end{pmatrix} + h(x, y).$$

By using the ESM, Chang showed that the condition:

$$\left(b \frac{\partial^2 h}{\partial^2 x} + c \frac{\partial^2 h}{\partial x \partial y} \right)(0, 0) \neq 0 \quad \text{or} \quad \frac{\partial^2 h}{\partial^2 x}(0, 0) > 0,$$

is sufficient to ensure the stabilizability, by means of the existence of a Lyapunov function, for $(H, \mathrm{vlift}(W))$.

With the LCBM, we showed that (Grillo, Salomone, Zuccalli, 2016-b)

$$\left(b \, \frac{\partial^2 h}{\partial^2 x} + c \, \frac{\partial^2 h}{\partial x \partial y} \right) (0,0) \neq 0 \quad \text{or} \quad \frac{\partial^2 h}{\partial^2 x} (0,0) > 0,$$

- it is also a necessary condition to ensure the stabilizability by means of the existence of a Lyapunov function;
- it is sufficient not only to ensure stabilizatility, but also to ensure asymptotic stabilizatility.

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With the LCBM, we showed that (Grillo, Salomone, Zuccalli, 2016-b)

$$\left(b\,\frac{\partial^2 h}{\partial^2 x} + c\,\frac{\partial^2 h}{\partial x \partial y} \right)(0,0) \neq 0 \quad \text{or} \quad \frac{\partial^2 h}{\partial^2 x}(0,0) > 0,$$

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THANK YOU!