Geometry and Dynamics of Nonholonomic Systems

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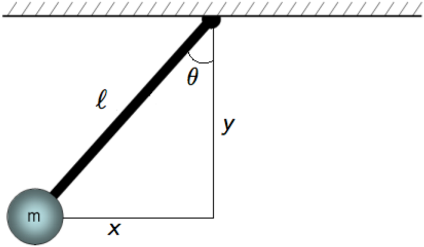
Holonomic Constraints

Pendulum

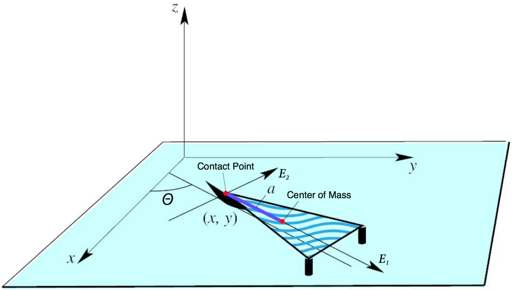


Constraint: $x^2 + y^2 = \ell^2$

Geometric Constraint. Restriction in coordinate values.
Configuration space is not $\mathbb{R}^2$ but S^1.

Nonholonomic Constraints

Chaplygin Sleigh

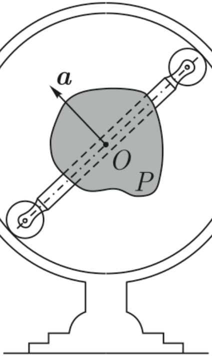


Kinematic Constraint:

$$\dot{y}\cos\phi = \dot{x}\sin\phi$$

No constraint on the configurations.

Suslov Problem

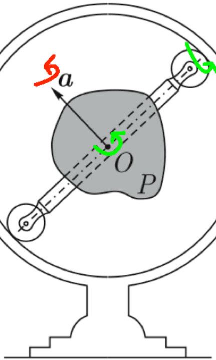


Kinematic Constraint:

$$\langle a, \Omega \rangle = 0$$

No constraint on the configurations. $Q = SO(3)$.

Suslov Problem



Kinematic Constraint:

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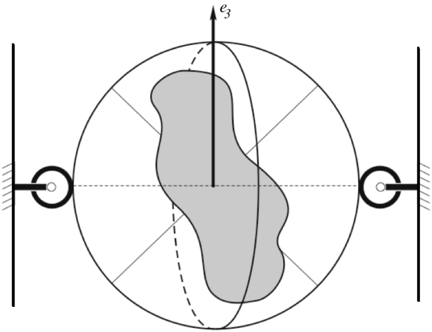
Veselova Problem

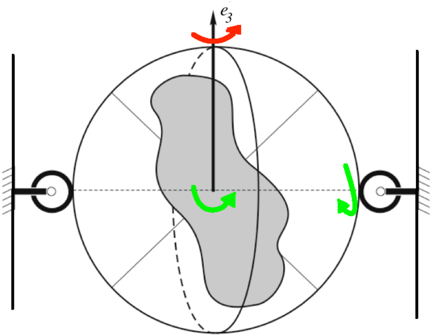


Kinematic Constraint:

$$\langle e_3, \omega \rangle = 0$$

No constraint on the configurations. $Q = SO(3)$.

Veselova Problem

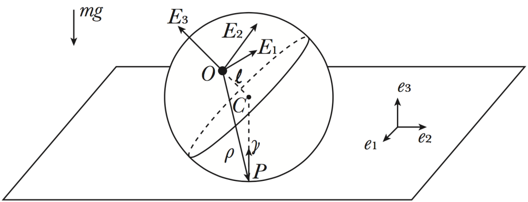


Kinematic Constraint:

$$\langle e_3, \omega \rangle = 0$$

No constraint on the configurations. $Q = SO(3)$.

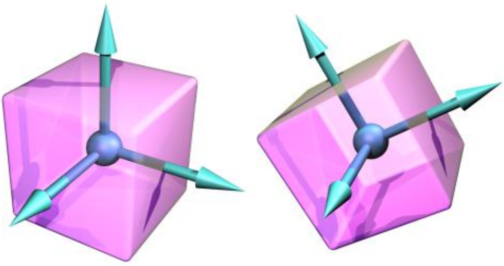
Chaplygin Top



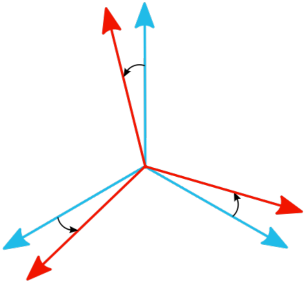
Kinematic Constraint: Rolling without slipping.
No constraint on the configurations. $Q = SO(3) \times \mathbb{R}^2$.

Rigid body dynamics

Body frame. Not inertial. Attached to the body.



Space and body frames



Change of basis matrix is $B \in SO(3)$.

$$q = BQ$$

Angular velocity vector

In body coordinates:

$$B^{-1}\dot{B} = \hat{\Omega} = \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix}$$

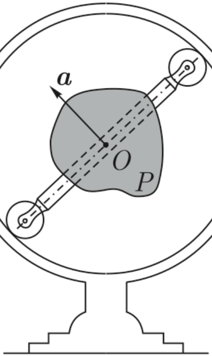
Left invariant

In space coordinates:

$$\dot{B}B^{-1} = \hat{\omega} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

Right invariant

Suslov Problem



Constraint:

$$\langle a, \Omega \rangle = 0$$

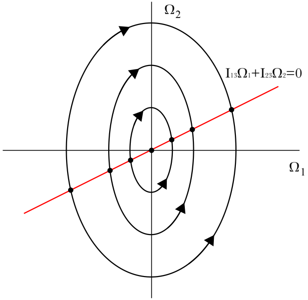
Lagrangian:

$$L = \langle \mathbb{I}\Omega, \Omega \rangle$$

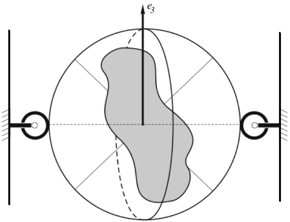
Suslov problem

$$\dot{\Omega}_1 = -\frac{1}{I_{11}}\left((I_{13}\Omega_1 + I_{23}\Omega_2)\Omega_2\right),$$
$$\dot{\Omega}_2 = \frac{1}{I_{22}}\left((I_{13}\Omega_1 + I_{23}\Omega_2)\Omega_1\right).$$

$$E = \frac{1}{2}(I_{11}\Omega_1^2 + I_{22}\Omega_2^2)$$



Veselova problem



$$\langle e_3, \omega \rangle = \langle \gamma, \Omega \rangle = 0$$

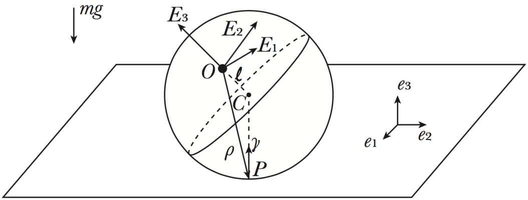
$$\mathbb{I}\dot{\Omega} = (\mathbb{I}\Omega) \times \Omega + \lambda\gamma$$

$$\lambda = -\frac{\langle (\mathbb{I}\Omega) \times \Omega, \mathbb{I}^{-1}\gamma \rangle}{\langle \gamma, \mathbb{I}^{-1}\gamma \rangle}$$

$$\dot{\gamma} = \gamma \times \Omega$$

Equations on TS^2

Chaplygin Top

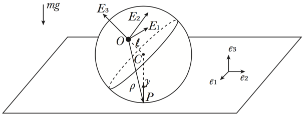


$$\dot{u} = B(\rho \times \Omega)$$

$$m\ddot{u} = -mge_3 + R_1, \qquad \mathbb{I}\dot{\Omega} = (\mathbb{I}\Omega) \times \Omega + R_2$$

$$R_2 = m\rho \times (\Omega \times (\rho \times \Omega)) + m\rho \times (\dot{\rho} \times \Omega) + m\rho \times (\rho \times \dot{\Omega}) + mg\rho \times \gamma$$

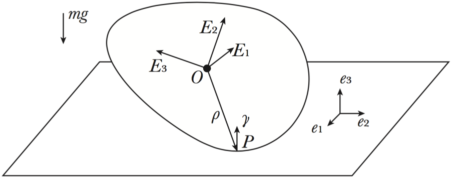
$$\mathbb{I}\dot{\Omega} = (\mathbb{I}\Omega) \times \Omega + R_2$$



$$\rho = -R\gamma - \ell E_3 \qquad \dot{\gamma} = \gamma \times \Omega$$

Equations on $S^2 \times \mathbb{R}^3$

Smooth convex body rolling on the plane



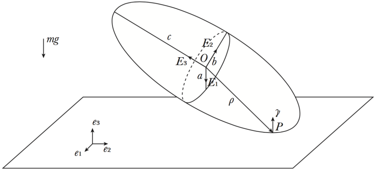
$$\begin{aligned} R_2 =& m\rho \times (\Omega \times (\rho \times \Omega)) + m\rho \times (\dot{\rho} \times \Omega) \\ &+ m\rho \times (\rho \times \dot{\Omega}) + mg\rho \times \gamma \end{aligned}$$

$$\mathbb{I}\dot{\Omega} = (\mathbb{I}\Omega) \times \Omega + R_2$$

If surface is given by $f(\rho) = 0$ with $f : \mathbb{R}^3 \to \mathbb{R}$

$$\gamma = -\frac{\nabla f(\rho)}{||\nabla f(\rho)||}, \qquad \dot{\gamma} = \gamma \times \Omega$$

Ellipsoid rolling on the plane



$$
\begin{aligned}
R_2 = & m\rho \times (\Omega \times (\rho \times \Omega)) + m\rho \times (\dot{\rho} \times \Omega) \\
& + m\rho \times (\rho \times \dot{\Omega}) + mg\rho \times \gamma
\end{aligned}
$$

$$\mathbb{I}\dot{\Omega} = (\mathbb{I}\Omega) \times \Omega + R_2$$

$$\rho = \frac{-A\gamma}{\sqrt{\langle A\gamma, \gamma \rangle}}, \qquad A = diag(a^2, b^2, c^2)$$

$$\dot{\gamma} = \gamma \times \Omega$$

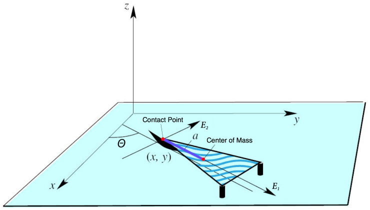
Nonholonomic equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = R_i, \qquad \beta(q)\dot{q} = 0$$

Equivalent to:

$$\frac{d}{dt}\left(\frac{\partial L_c}{\partial v^\alpha}\right) = \rho^i_\alpha(q)\frac{\partial L_c}{\partial q^i} - C^\gamma_{\alpha\beta}(q) v^\beta \frac{\partial L_c}{\partial v^\gamma}$$
$$\dot{q}^i = \rho^i_\alpha(q) v^\alpha$$

Example: Chaplygin sleigh

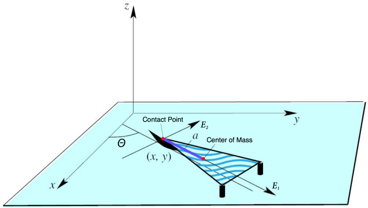


$$\dot{x}\sin\theta - \dot{y}\cos\theta = 0$$

$$L = \frac{1}{2}((J + ma^2)\dot{\theta}^2 + m(\dot{x}^2 + \dot{y}^2) + 2ma\dot{\theta}(\dot{y}\cos\theta - \dot{x}\sin\theta))$$

$$\mathcal{D}^\perp = \operatorname{span}\left\{ Y = \frac{ma}{J + ma^2}\frac{\partial}{\partial\theta} + \sin\theta\frac{\partial}{\partial x} - \cos\theta\frac{\partial}{\partial y} \right\}$$

Example: Chaplygin sleigh



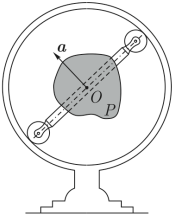
$$C_{12}^1 = 0, \qquad C_{12}^2 = -\frac{ma}{J + ma^2}$$

$$L_c(u, \omega, x, y, \theta) = \frac{1}{2}((J + ma^2)\omega^2 + mu^2)$$

$$m\dot{u} = ma\omega^2 \qquad (J + ma^2)\dot{\omega} = -mau\omega$$

Example: Suslov problem

$$\frac{d}{dt}\left(\frac{\partial L_c}{\partial v^\alpha}\right) = \rho^i_\alpha(q)\frac{\partial L_c}{\partial q^i} - C^\gamma_{\alpha\beta}(q) v^\beta \frac{\partial L_c}{\partial v^\gamma}, \qquad \dot{q}^i = \rho^i_\alpha(q) v^\alpha$$



$$a = E_3, \qquad \Omega_3 = 0, \qquad L_c(B, \Omega_1, \Omega_2) = \frac{1}{2}(I_{11}\Omega_1^2 + I_{22}\Omega_2^2)$$

$$I_{11}\dot{\Omega}_1 = -\left((I_{13}\Omega_1 + I_{23}\Omega_2)\Omega_2\right), \quad I_{22}\dot{\Omega}_2 = \left((I_{13}\Omega_1 + I_{23}\Omega_2)\Omega_1\right).$$

$$\dot{B} = B\hat{\Omega}.$$

Hamiltonian formulation

$$\frac{d}{dt}\left(\frac{\partial L_c}{\partial v^\alpha}\right) = \rho^i_\alpha(q)\frac{\partial L_c}{\partial q^i} - C^\gamma_{\alpha\beta}(q)v^\beta\frac{\partial L_c}{\partial v^\gamma}$$
$$\dot{q}^i = \rho^i_\alpha(q)v^\alpha$$

Equivalent to:

$$\dot{p}_\alpha = -\rho^k_\alpha(q)\frac{\partial H}{\partial q^k} - C^\gamma_{\alpha\beta}(q)p_\gamma\frac{\partial H}{\partial p_\beta}$$
$$\dot{q}^i = \rho^i_\alpha(q)\frac{\partial H}{\partial p_\alpha}$$

X_{nh} is tangent to $\mathcal{F}$

Properties of $\mathcal{F}$

1. Integrable if and only if constraints are holonomic.
2. $\mathcal{F}$ is symplectic
3. X_{nh} characterized by

$$i_{X_{nh}}\Omega|_{\mathcal{F}} = dH|_{\mathcal{F}}$$

Distributional Hamiltonian approach Bates, Cushman, etc

4. Ibort, de León, Marrero, Martín de Diego:

$$\{f, g\}_{nh}(m) = \Omega_m(R_m(X_{\tilde{f}}(m)), R_m(X_{\tilde{g}}(m)))$$

$\Longrightarrow$ The span of the "Hamiltonian" vector fields X_f^{nh} with $f \in C^\infty(D^*)$ is the distribution $\mathcal{F}$.

Conclusion: The bracket $\{\cdot, \cdot\}_{nh}$ satisfies the Jacobi identity if and only if the constraints are holonomic.

Example. Chaplygin sleigh

$$L_c = \frac{1}{2}\left((J+ma^2)\omega^2 + mu^2\right).$$

and

$$C_{12}^1 = 0, \qquad C_{12}^2 = -\frac{ma}{J+ma^2}.$$

We have

$$p_u = \frac{\partial L_c}{\partial u} = mu, \qquad p_\omega = \frac{\partial L_c}{\partial \omega} = (J+ma^2)\omega.$$

The Hamiltonian is

$$H = \frac{1}{2}\left(\frac{p_\omega^2}{J+ma^2} + \frac{p_u^2}{m}\right).$$

The equations of motion:

$$\dot{x} = \frac{p_u}{m}\cos\theta, \qquad \dot{y} = \frac{p_u}{m}\sin\theta, \qquad \dot{\theta} = \frac{p_\omega}{J+ma^2}$$

$$\dot{p}_u = \frac{map_\omega^2}{(J+ma^2)^2}, \qquad \dot{p}_\omega = -\frac{ap_u p_\omega}{J+ma^2},$$

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ \theta \\ p_u \\ p_\theta \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & \sin\theta & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -\cos\theta & -\sin\theta & 0 & 0 & \frac{ma}{J+ma^2} p_\omega \\ 0 & 0 & -1 & -\frac{ma}{J+ma^2} p_\omega & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H_c}{\partial x} \\ \frac{\partial H_c}{\partial y} \\ \frac{\partial H_c}{\partial \theta} \\ \frac{\partial H_c}{\partial p_u} \\ \frac{\partial H_c}{\partial p_\omega} \end{pmatrix}.$$

Rank 4 matrix. Null space

$$(-\sin\theta, \cos\theta, 0, 0, 0)$$

Constraint one-form

$$-\sin\theta\, dx + \cos\theta\, dy.$$

$$\mathcal{F} = \left\{ \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial p_u}, \frac{\partial}{\partial p_\omega} \right\}$$

Measure preservation of homogeneous systems in vector spaces

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n.$$

f homogeneous of degree $k \in \mathbb{N}$: i.e. $f(\lambda x) = \lambda^k f(x)$.

Kozlov '88: f preserves a smooth measure $\mu(x)\, dx$ if and only if it preserves the euclidean measure dx and $\mu(x)$ is a conserved quantity.

$$
\begin{aligned}
\dot{p}_\alpha &= -\rho^k_\alpha(q)\frac{\partial H}{\partial q^k} - C^\gamma_{\alpha\beta}(q)p_\gamma\frac{\partial H}{\partial p_\beta} \\
\dot{q}^i &= \rho^i_\alpha(q)\frac{\partial H}{\partial p_\alpha}
\end{aligned}
$$

Suppose there is no potential energy

$$
H(q^i, p_\alpha) = \frac{1}{2}p^T(A_D(q))^{-1}p
$$

Kozlov: Invariant measure must be basic $\mu(q)\,dq \wedge dp$.

Example: Suslov problem

$$\dot{\Omega}_1 = -\frac{1}{I_{11}}\left((I_{13}\Omega_1 + I_{23}\Omega_2)\Omega_2\right),$$
$$\dot{\Omega}_2 = \frac{1}{I_{22}}\left((I_{13}\Omega_1 + I_{23}\Omega_2)\Omega_1\right).$$

Invariant measure if and only if $I_{13} = I_{23}$ axis of forbidden rotations is a principal axis of inertia.

Symmetry reduction of nonholonomic systems

Nonholonomic system: Q, L, $\mathcal{D}$.
Free and proper action of Lie group G on Q

$$\Psi : G \times Q \to Q$$

Tangent lift

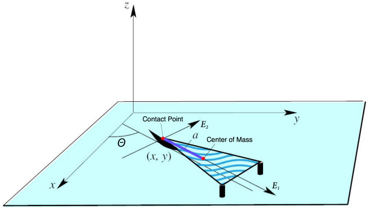
$$\hat{\Psi} : G \times TQ \to TQ$$

Suppose that $\hat{\Psi}$ preserves L and $\mathcal{D}$. Action on D

$$\Phi : G \times D \to D, \qquad \Phi_g = \hat{\Psi}_g \Big|_D$$

Vector field X_{nh} is equivariant. Reduced dynamics on D/G=vector bundle over $S = Q/G$.

Example: Chaplygin sleigh

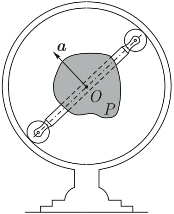


$$Q = \mathbb{R}^2 \times S^1$$

$$D/G = \mathbb{R}^2$$

$$m\dot{u} = ma\omega^2 \qquad (J + ma^2)\dot{\omega} = -mau\omega$$

Example: Suslov problem



$$Q = SO(3)$$

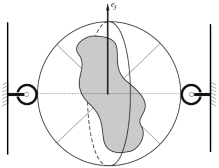
$$L = \frac{1}{2}\langle \mathbb{I}\Omega, \Omega\rangle, \qquad \langle a, \Omega\rangle = 0$$

$$D/G = \mathbb{R}^2$$

$$\dot{\Omega}_1 = -\frac{1}{I_{11}}\left((I_{13}\Omega_1 + I_{23}\Omega_2)\Omega_2\right),$$

$$\dot{\Omega}_2 = \frac{1}{I_{22}}\left((I_{13}\Omega_1 + I_{23}\Omega_2)\Omega_1\right).$$

Example: Veselova problem



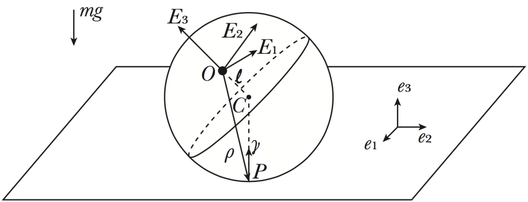
$$Q = SO(3)$$

$$L = \frac{1}{2}\langle \mathbb{I}\Omega, \Omega\rangle, \qquad \langle \gamma, \Omega\rangle = \omega_3 = 0$$

$$D/G = TS^2$$

$$||\gamma|| = 1, \qquad \langle \gamma, \Omega\rangle = 0$$

Example: Chaplygin top



$$Q = SO(3) \times \mathbb{R}^2$$

$$D/G = \text{ Rank 3 vector bundle over } S^2$$

$$\Omega, \qquad ||\gamma|| = 1$$

Reduced equations of motion

Work on Hamiltonian formulation.

$$\Phi : G \times D^* \to D^*$$

$$\pi : D^* \to D^*/G := \mathcal{R}$$

Hamiltonian is invariant

$$H = h \circ \pi, \qquad h : \mathcal{R} \to \mathbb{R}$$

Nonholonomic bracket is invariant

$$\{f_1 \circ \Phi_g, f_2 \circ \Phi_g\}_{nh} = \{f_1, f_2\}_{nh} \circ \Phi_g$$

Bracket on $\mathcal{R}$:

$$\{F_1, F_2\}_{\mathcal{R}} = \{F_1 \circ \pi, F_2 \circ \pi\}_{nh}$$

Coordinates on $\mathcal{R} = D^*/G$

$\mathcal{R} = D^*/G$ is a vector bundle over $S = Q/G$

If the basis $\{X_\alpha\}$ is equivariant then p_α is invariant.
Coordinates for $\mathcal{R} = D^*/G$ are (s^i, p_α)

$$\{p_\alpha, p_\beta\}_\mathcal{R} = -C^\gamma_{\alpha\beta}(s)p_\gamma$$
$$\{s^i, s^j\}_\mathcal{R} = 0,$$
$$\{s^i, p_\alpha\}_\mathcal{R} = (\pi_* X_\alpha)[s^i]$$

Reduced equations of motion

$$\dot{p}_A = -\rho^i_A(s)\frac{\partial h}{\partial s^i} - C^\beta_{A\alpha} p_\beta \frac{\partial h}{\partial p_\alpha}$$

$$\dot{p}_a = \cancel{-\rho^i_a(s)\frac{\partial h}{\partial s^i}} - C^\gamma_{a\beta}(s) p_\gamma \frac{\partial h}{\partial p_\beta}$$

$$\dot{s}^i = \rho^i_A(s)\frac{\partial h}{\partial p_A} + \cancel{\rho^i_a(s)\frac{\partial h}{\partial p_a}}$$

Reduced equations of motion

$$\dot{p}_A = -\rho^i_A(s)\frac{\partial h}{\partial s^i} - C^\beta_{A\alpha} p_\beta \frac{\partial h}{\partial p_\alpha}$$

$$\dot{p}_a = -C^\gamma_{a\beta}(s) p_\gamma \frac{\partial h}{\partial p_\beta}$$

$$\dot{s}^i = \rho^i_A(s)\frac{\partial h}{\partial p_A}$$

Look for measure $e^{\sigma(s)}\, ds \wedge dp_A \wedge dp_a$:

$$\frac{\partial}{\partial s^i}\left(e^{\sigma(s)}\rho^i_A(s)\frac{\partial h}{\partial p_A}\right) + e^{\sigma(s)}\frac{\partial}{\partial p_A}\left(-\rho^i_A(s)\frac{\partial h}{\partial s^i} - C^\beta_{A\alpha}(s) p_\beta \frac{\partial h}{\partial p_\alpha}\right)$$
$$+ e^{\sigma(s)}\frac{\partial}{\partial p_a}\left(-C^\gamma_{a\beta}(s) p_\gamma \frac{\partial h}{\partial p_\beta}\right) = 0$$

Necessary conditions for invariant measure $e^{\sigma(s)}\, ds \wedge dp_A \wedge dp_a$

$$\rho^i_A(s)\frac{\partial\sigma}{\partial s^i} + \frac{\partial\rho^i_A}{\partial s^i} + C^\alpha_{A\alpha} = 0 \qquad \text{for all } A$$

$$C^\alpha_{a\alpha} = 0 \qquad \text{for all } a$$

Equivalent form given by Bloch, Zenkov 2003.

Example: Chaplygin sleigh

$$\mathcal{D}_q \subset T_q \mathrm{Orb}_G(q)$$

$$Z_1 = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}$$

$$Z_2 = \frac{\partial}{\partial \theta}$$

$$C_{12}^1 = 0 \qquad C_{12}^2 = -\frac{ma}{J + ma^2}$$

Invariant measure only if $a = 0$

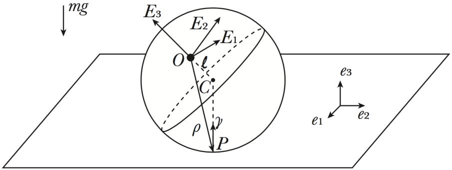
$$m\dot{u} = ma\omega^2$$

$$(J + ma^2)\dot{\omega} = -mau\omega$$

$$E = \frac{1}{2}\left((J + ma^2)\omega^2 + mu^2\right).$$

m

Example: Chaplygin top



$$\dim(\mathcal{D}_q \cap T_q \mathrm{Orb}_G(q)) = 1$$

One necessary condition

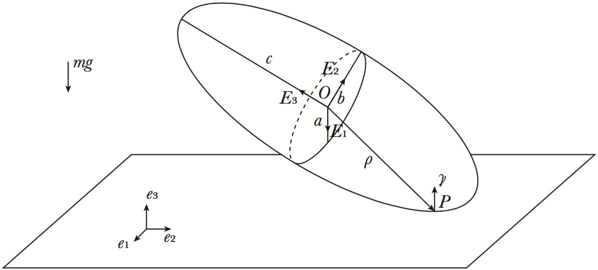
$$C_{1\alpha}^{\alpha}(s) = 0$$

s are coordinates on S^2

Fedorov, GN, Marrero 2015: (In the absence of gravity) there exists an invariant measure if and only if

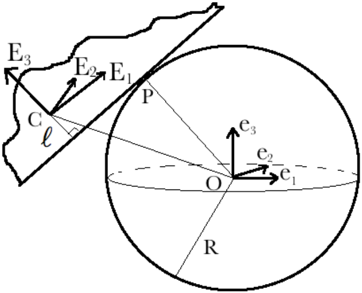
$\ell = 0$	Chaplygin sphere 1903	or	$\mathbb{I} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_1 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$	Routh sphere 1884

Homogeneous ellipsoid rolling on the plane



Invariant measure if and only if two of the semi-axes are equal.

Rigid body with planar section that rolls over a sphere



$$C^{\alpha}_{\alpha,1} = \frac{m}{R^3 \det(T)}\big(I_{23}(I_{11} + m\ell^2)s_1 \\ - I_{13}(I_{22} + m\ell^2)s_2 + m\ell(I_{11} - I_{22})s_1 s_2\big).$$

Necessary conditions for existence of invariant measure:

$$I_{12} = I_{23} = I_{13} = 0 \qquad\qquad (I_{11} - I_{22})\ell = 0.$$

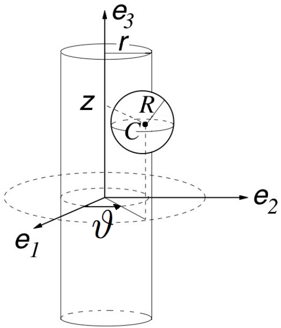
Invariant measures for Chaplgyin systems

$$T_q = \mathcal{D}_q \oplus T_q\mathrm{Orb}_G(q)$$

$$\rho^i_A(s)\frac{\partial\sigma}{\partial s^i} + \frac{\partial\rho^i_A}{\partial s^i}(s) + C^\alpha_{A\alpha}(s) = 0 \qquad \text{for all } A$$

$\xcancel{C^\alpha_{a\alpha} = 0}$

Inhomogeneous sphere rolling on a circular cylinder



Existence of invariant measure if and only if:
Sphere is homogeneous.

The modular vector field of an (almost) Poisson structure

Grabowski (2012), Marrero, GN, Fedorov (2015)

Consider bracket of functions in $\mathbb{R}^n$:

$$\{F, G\}(x) = (\nabla F(x))^T \pi(x) \nabla G(x)$$

Skew-symmetry: $\pi_{\alpha\beta} = -\pi_{\beta\alpha}$

Jacobi identity: $\pi_{\delta\alpha}\frac{\partial \pi_{\beta\gamma}}{\partial x_\delta} + \pi_{\delta\gamma}\frac{\partial \pi_{\alpha\beta}}{\partial x_\delta} + \pi_{\delta\beta}\frac{\partial \pi_{\gamma\alpha}}{\partial x_\delta} = 0$

Hamiltonian vector fields:

$$\dot{x} = \pi(x)\nabla H(x) := X_H(x); \qquad \dot{x}_\alpha = \pi_{\alpha\beta}(x)\frac{\partial H}{\partial x_\beta}(x)$$

Taking (euclidean) divergence

$$\begin{aligned}\operatorname{div}(X_H(x)) &= \frac{\partial \pi_{\alpha\beta}}{\partial x_\alpha}(x)\frac{\partial H}{\partial x_\beta}(x) + \overbrace{\pi_{\alpha\beta}(x)\frac{\partial^2 H}{\partial x_\alpha \partial x_\beta}(x)}^{0} \\ &= \mathcal{M}(x) \cdot \nabla H(x). \qquad \textit{modular vector field}\end{aligned}$$

In general:

$$\mathrm{div}(e^{\sigma(x)}X_H(x)) = e^{\sigma(x)}\left(\overbrace{\nabla\sigma(x)\cdot X_H(x)}^{\{\sigma,H\}} + \mathrm{div}(X_H(x))\right)$$
$$= e^{\sigma(x)}\left(\mathcal{M}(x) - X_\sigma(x)\right)\cdot\nabla H(x)$$

Definition: If $\mathcal{M}(x)$ is Hamiltonian $\implies$ π is *unimodular*

Unimodularity: Sufficient condition for the existence of an invariant measure.

With some extra conditions (related to homogeneity of the Hamiltonian vector fields) unimodularity is also a necessary condition for the existence of an invariant measure.

Note: The definition of unimodularity and the above conclusions only depend on the skew-symmetry of π.

The modular class of a Poisson manifold

If the Jacobi identity holds then the entries of $\mathcal{M}$ satisfy

$$\mathcal{M}_\gamma \frac{\partial \pi_{\alpha\beta}}{\partial x_\gamma} + \pi_{\gamma\alpha} \frac{\partial \mathcal{M}_\beta}{\partial x_\gamma} - \pi_{\gamma\beta} \frac{\partial \mathcal{M}_\alpha}{\partial x_\gamma} = 0, \qquad \mathcal{L}_{\mathcal{M}} \pi = 0.$$

$\mathcal{M}$ is a Poisson vector field.

$$\begin{array}{c}\text{First Poisson}\\ \text{cohomology group}\end{array} = \frac{\{\text{Vector fields that preserve } \pi \}}{\{\text{Hamiltonian vector fields}\}}$$

Representative of $\mathcal{M}$ is the *modular class* of π (Weinstein '96).

Important objects in the study (topology, classification) of Poisson manifolds (Weinstein, Xu, Dufour, Grabowski, Lu, Evens,...)

Unimodularity $\iff$ modular class is zero.

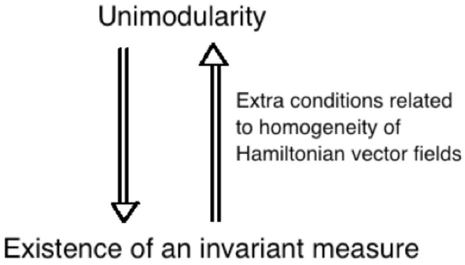
Summary: Unimodularity and invariant measures for (almost) Poisson Hamiltonian systems

Unimodularity

$$\Downarrow$$

Existence of an invariant measure

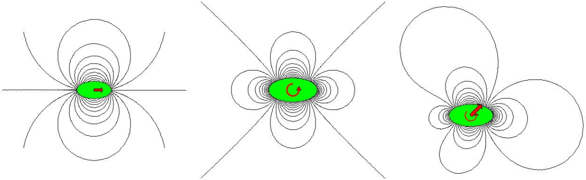
Summary: Unimodularity and invariant measures for (almost) Poisson Hamiltonian systems



Note: The discussion can be generalized to *orientable* (almost) Poisson manifolds. The unimodularity is a global and intrinsic concept.

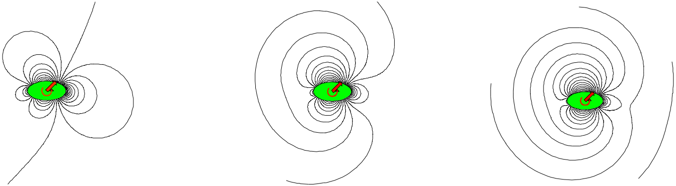
Example: Hydrodynamic Chaplygin sleigh (zero circulation)

This system asymptotically approaches periodic orbits. No invariant measures.



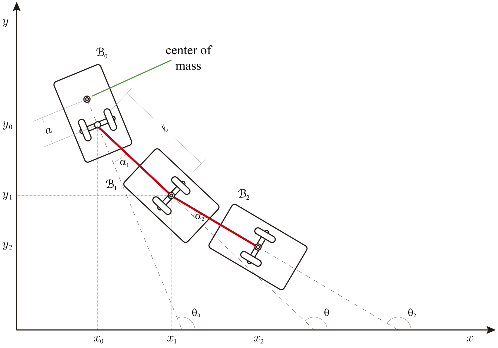
Fedorov, GN, (2010)

Example: Hydrodynamic Chaplygin sleigh with circulation



For low energies the system is driven by the circulation and the behavior is Hamiltonian-like.
For large energies the system asymptotically approaches periodic orbits. No invariant measures.
Fedorov, GN, Vankerschaver (2013).

n-Trailer vehicle



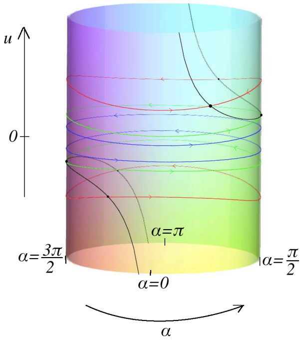
Bravo, GN (2015)

Case $a = 0$ $n = 1$

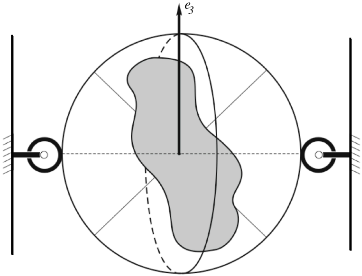
$$\dot{u} = \frac{(m\ell^2 - J)u\cos\alpha\sin\alpha(\ell\omega - u\sin\alpha)}{\ell((M + m\cos^2\alpha)\ell^2 + J\sin^2\alpha)},$$

$$\dot{\omega} = 0,$$

$$\dot{\alpha} = \omega - \frac{u\sin\alpha}{\ell}.$$



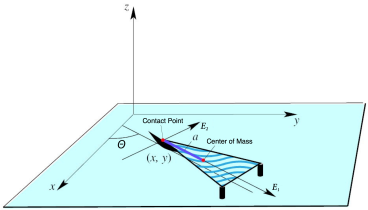
LR systems: Veselova problem



$$Q = SO(3)$$

$$L = \frac{1}{2}\langle \mathbb{I}\Omega, \Omega \rangle \qquad \omega_3 = 0$$

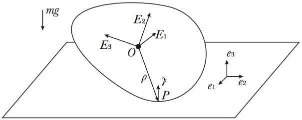
Example: Chaplygin sleigh



$$\dot{x}\sin\theta - \dot{y}\cos\theta = 0$$

$$L = \frac{1}{2}((J + ma^2)\dot{\theta}^2 + m(\dot{x}^2 + \dot{y}^2) + 2ma\dot{\theta}(\dot{y}\cos\theta - \dot{x}\sin\theta))$$

Convex rolling body rolling on the plane



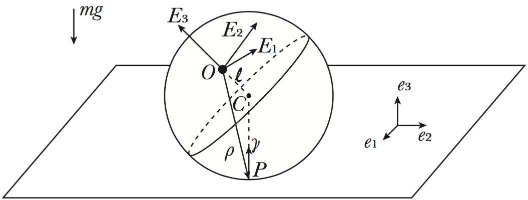
Equations of motion

$$\dot{K} = K \times \Omega + m\dot{\rho} \times (\Omega \times \rho) + mg\rho \times \gamma$$
$$\dot{\gamma} = \gamma \times \Omega$$

$$K = \mathbb{I}\Omega + m\rho \times (\Omega \times \rho) \qquad \rho = \rho(\gamma)$$

Chaplygin top

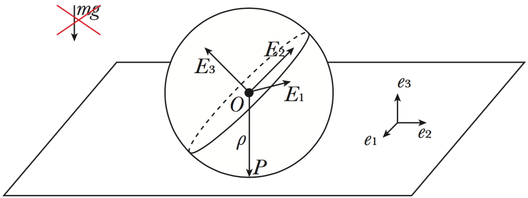
$$\rho(\gamma) = -R\gamma - \ell E_3 \qquad \mathbb{I} = \begin{pmatrix} I_{11} & 0 & I_{13} \\ 0 & I_{22} & I_{23} \\ I_{13} & I_{23} & I_{33} \end{pmatrix}$$



$$\dot{K} = K \times \Omega + m\dot{\rho} \times (\Omega \times \rho) + mg\rho \times \gamma$$
$$\dot{\gamma} = \gamma \times \Omega$$

Chaplygin sphere

$$\rho(\gamma) = -R\gamma \qquad \mathbb{I} = diag(I_1, I_2, I_3)$$



$$\dot{K} = K \times \Omega + \cancel{m\dot{\rho} \times (\Omega \times \rho)} + \cancel{mg\rho \times \gamma}$$
$$\dot{\gamma} = \gamma \times \Omega$$

Linear first integral

$$F = \langle K, \gamma \rangle$$

Liouville Integrability of Hamiltonian systems

(M, Ω) symplectic manifold, $\dim(M) = 2n$.

n =number of degrees of freedom.

$H \in C^\infty(M)$ Hamiltonian function.

X_H Hamiltonian vector field on M defined by

$$\mathbf{i}_{X_H}\Omega = dH.$$

Poisson bracket $F, G \in C^\infty(M)$:

$$\{F, G\} := \Omega(X_F, X_G)$$

Closeness of $\Omega \iff$ Jacobi identity.

$$d\Omega = 0 \iff \{F_1, \{F_2, F_3\}\} + \{F_3, \{F_1, F_2\}\} + \{F_2, \{F_3, F_1\}\} = 0.$$

Theorem (Liouville, Complete Integrability)

Suppose that the smooth functions $H = F_1, F_2, \ldots, F_n$ are (pairwise) in involution $\{F_i, F_j\} = 0$.
Consider a level set of the functions F_i:

$$M_{\mathbf{f}} = \{x \in M \,:\, F_i(x) = f_i, \qquad i = 1, \ldots, n\}.$$

Assume that the n functions F_i are independent on $M_{\mathbf{f}}$ and that $M_{\mathbf{f}}$ is compact and connected. Then

Theorem (Liouville, Complete Integrability)

- *$M_{\mathbf{f}}$ is a smooth manifold, invariant under the flow of X_H.*
- *Every connected component of $M_{\mathbf{f}}$ is diffeomorphic to $\mathbb{T}^k \times \mathbb{R}^{n-k}$.*
- *There are coordinates $\varphi_1, \ldots, \varphi_k \mod 2\pi$, $y_1, \ldots, y_{n-k}$ on $\mathbb{T}^k \times \mathbb{R}^{n-k}$ in which Hamilton's equations on $M_{\mathbf{f}}$ take the form*

$$\dot{\varphi}_m = \omega_m, \qquad \dot{y}_s = c_s \qquad (\omega, c = const).$$

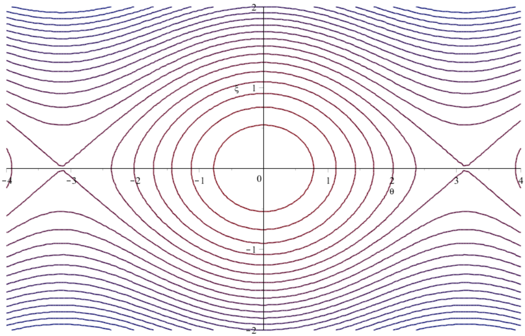
Action - Angle Variables

- If $M_{\mathbf{f}}$ is compact and connected then it is diffeomorphic to $\mathbb{T}^n$.
- There exist *action* - *angle* coordinates $J_1, \ldots, J_n,\ \ \varphi_1, \ldots, \varphi_n$ mod 2π in a neighborhood of $M_{\mathbf{f}}$:
  - They are symplectic
  $$\Omega = dJ \wedge d\varphi$$
  - The functions F_i depend only on J.
  - In particular $H = H(J)$. Hamilton's equations:

$$\dot{J}_k = -\frac{\partial H}{\partial \varphi_k} = 0, \qquad \dot{\varphi}_k = \frac{\partial H}{\partial J_k} = \omega_k(J).$$

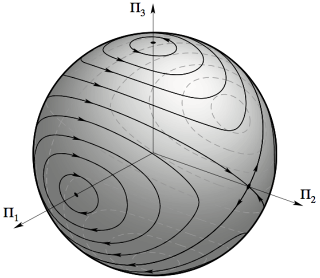
Example: Pendulum

$$M = T^* S^1$$

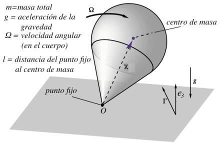


Example: Symplectic Reduction of Euler Top

$$M = S^2$$



The Heavy Top



$$
\begin{aligned}
\mathbb{I}\dot{\Omega} &= (\mathbb{I}\Omega) \times \Omega + mg\ell\gamma \times \chi \\
\dot{\gamma} &= \gamma \times \Omega.
\end{aligned}
$$

- Hamiltonian system. Lie Poisson equations on $\mathfrak{se}(3)^* = \{(M = \mathbb{I}\Omega, \gamma) \in \mathbb{R}^3 \times \mathbb{R}^3\}$.
- Casimirs of the bracket

$$\langle \mathbb{I}\Omega, \gamma \rangle \qquad \text{and} \qquad ||\gamma||.$$

- 4 dimensional symplectic leaves. Liouville integrability requires extra first integral independent of the Hamiltonian.

Known cases of integrability of the Heavy Top

- Euler-Poinsot case: $\ell = 0$ (Free rigid body). Extra integral:

$$F = \langle \mathbb{I}\Omega, \mathbb{I}\Omega \rangle$$

- Lagrange Top: $I_1 = I_2$, $\chi_1 = \chi_2 = 0$. Extra integral:

$$F = \Omega_3$$

- Kovalevskaya Top: $I_1 = I_2 = 2I_3$ and $\chi_2 = \chi_3 = 0$. Extra integral:

$$F = [(\Omega_1 + i\Omega_2)^2 + \chi_1(\gamma_1 + i\gamma_2)][(\Omega_1 - i\Omega_2)^2 + \chi_1(\gamma_1 - i\gamma_2)].$$

Key ingredient in the proof of Liouville's Theorem

- The Hamiltonian vector fields X_{F_i}, $i = 1, \ldots, n$
  - tangent to $M_{\mathbf{f}}$, (skew-symmetry of Poisson bracket)
  - are linearly independent on $M_{\mathbf{f}}$ (non-degeneracy of Ω)
  - *commute*

$$[X_{F_i}, X_{F_j}] = -X_{\{F_i, F_j\}}$$

Consequence of the Jacobi identity!

Theorem (Jacobi's last Multiplier)

Consider the system

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n$$

and suppose that it preserves a smooth invariant measure. If the system has $n-2$ first integrals $F_1, F_2, \ldots, F_{n-2}$ that are independent on the invariant set $E_c = \{x \in \mathbb{R}^n : F_s(x) = c_s, 1 \leq s \leq n-2\}$ then

1. *the solutions that belong to E_c can be found by quadratures. If E_c is compact and connected, and $f \neq 0$ on E_c then*
2. *E_c is a smooth surface diffeomorphic to a two-dimensional torus,*
3. *it is possible to choose angle variables $\varphi_1, \varphi_2 \bmod 2\pi$ on E_c so that,*

$$\dot{\varphi}_1 = \frac{\lambda}{\Phi(\varphi_1, \varphi_2)}, \quad \dot{\varphi}_2 = \frac{\mu}{\Phi(\varphi_1, \varphi_2)}$$

where $\lambda, \mu =$const, $|\lambda| + |\mu| \neq 0$ and Φ is a smooth positive function that is 2π-periodic in φ_1 and φ_2.

Important ingredients in integrability of nonholonomic systems

- Existence of first integrals.
- Existence of an invariant measure.
- Reduction
- *Hamiltonization.* When do the reduced equations possess a Hamiltonian structure?

Example: Chaplygin sleigh

Reduced equations of motion

$$\dot{p}_u = \frac{map_\omega^2}{(J+ma^2)^2}, \qquad \dot{p}_\omega = -\frac{ap_u p_\omega}{J+ma^2},$$

The Hamiltonian is

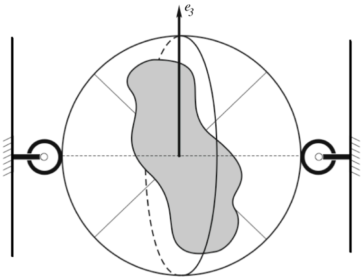
$$H = \frac{1}{2}\left(\frac{p_\omega^2}{J+ma^2} + \frac{p_u^2}{m}\right).$$

$$\dot{p}_u = \{p_u, H\}, \qquad \dot{p}_\omega = \{p_\omega, H\}$$

Where

$$\{F, G\} = -\frac{map_\omega}{J+ma^2}\left(\frac{\partial F}{\partial p_u}\frac{\partial G}{\partial p_\omega} - \frac{\partial F}{\partial p_\omega}\frac{\partial G}{\partial p_u}\right)$$

Veselova problem

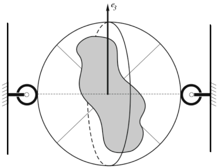


$$Q = SO(3)$$

$$L = \frac{1}{2}\langle \mathbb{I}\Omega, \Omega\rangle \qquad \omega_3 = 0$$

Example of an LR system!
Exceptional class of nonholonomic systems that always have an invariant measure (Veselov, Veselova 1988).

Veselova problem



Phase space TS^2: $||\gamma|| = 1, \quad \langle \gamma, \Omega \rangle = 0$

Existence of invariant measure.

$$\sqrt{\langle \gamma, \mathbb{I}^{-1}\gamma \rangle}\; d\Omega \; d\gamma.$$

First integrals

$$H = \frac{1}{2}\langle \mathbb{I}\Omega, \Omega \rangle, \qquad F = \frac{1}{2}\langle \mathbb{I}\Omega, \mathbb{I}\Omega \rangle - \frac{1}{2}\langle \mathbb{I}\Omega, \gamma \rangle^2$$

Integrable by Jacobi's theorem.

Hamiltonization of Veselova system

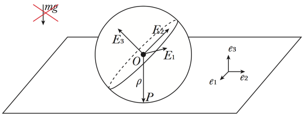
Theorem (Chaplygin's Reducing Multiplier Theorem)

If a Chaplygin system with $dim(S) = 2$ possesses an invariant measure, then it is Hamiltonizable.

> After a *time reparametrization* the Veselova system is a Liouville-integrable Hamiltonian system on TS^2.

Fedorov, Jovanović (2004) have found integrability and Hamiltonization of multi-dimensional versions of the Veselova problem.

Chaplygin sphere



$$\dot{K} = K \times \Omega, \qquad \dot{\gamma} = \gamma \times \Omega$$
$$K = \mathbb{I}\Omega + mr^2 \gamma \times (\Omega \times \gamma)$$

Phase space $\mathbb{R}^3 \times S^2, \quad (K, \gamma)$

First integrals

$$H = \frac{1}{2}\langle K, \Omega \rangle, \qquad F_1 = \langle K, \gamma \rangle, \qquad F_2 = \langle K, K \rangle$$

Invariant measure

$$\left(\frac{1}{mR^2} - \langle \gamma, (\mathbb{I} + mR^2)^{-1} \gamma \rangle \right)^{-\frac{1}{2}} \, dK \, d\gamma.$$

Integrable by Jacobi's theorem: Chaplygin 1903.

Hamiltonization of Chaplygin sphere

- Duistermaat [2000] *Although the system is integrable in every sense of the word, it neither arises as a Hamiltonian system, nor is the integrability an immediate consequence of the symmetries*.
- Borisov and Mamaev [2002] *Chaplygin's Ball Rolling Problem is Hamiltonian*. After a *time reparametrization* write the reduced equations of motion with respect to a nonlinear (yet mechanical!) bracket of functions that satisfies the Jacobi identity.
- Ehlers, Koiller, Montgomery, Rios [2004] failed to obtain the Hamiltonian structure of the reduced equations by their (geometric) methods.
- GN[2007] Understand the geometry of B.& M. bracket and tie it with the general theory of almost Poisson brackets for nonholonomic systems.

Families of Almost Poisson Brackets for a nonholonomic system (GN [2010])

- Idea: Equations of motion

$$\mathbf{i}_{X_{\mathrm{nh}}}\Omega_Q = dH + \sum_{i=1}^{k} \lambda_i \tau^* \beta^i, \qquad X_{nh}(m) \in \mathcal{F}_m$$

  can also be written as:

$$\mathbf{i}_{X_{\mathrm{nh}}}(\Omega_Q + B) = dH + \sum_{i=1}^{k} \lambda_i \tau^* \beta^i,$$

  for a 2-form B satisfying $\mathbf{i}_{X_{\mathrm{nh}}} B = 0$.
- Construct bracket using the non-canonical form $\tilde{\Omega}_Q := \Omega_Q + B$.

Gauged Almost Symplectic Structures

Definition

A nontrivial two-form B on T^*Q defines a Gauged (almost) Symplectic Structure, $\tilde{\Omega}_Q := \Omega_Q + B$, for our nonholonomic system if the following conditions hold:

- $\mathbf{i}_{X_H} B = 0$.
- The form B is linear semi-basic.

$$B = B_{ij}^k(q)\, p_k\, dq^i \wedge dq^j .$$

Gauged Almost Poisson Brackets

Theorem

- *A Gauged Almost Symplectic Structure* $\tilde{\Omega}_Q$ *is non-degenerate (not necesarilly closed!)*
- *The equations of motion can be written as*

$$\mathbf{i}_{X_{\mathrm{nh}}} \tilde{\Omega}_Q = d\mathcal{H} + \sum_{i=1}^{k} \lambda_i \tau^* \beta^i.$$

- *Constraints remain the same:* $X_{\mathrm{nh}}(m) \in \mathcal{F}_m \quad \forall m \in D^*$.
- *For all* $m \in \mathcal{M}$ *we have the symplectic decomposition*

$$T_m(T^*Q) = \mathcal{F}_m \oplus \mathcal{F}_m^{\tilde{\Omega}_Q}.$$

- *Same relevant properties as* Ω_Q*!*

Definition of B-Gauged Brackets

- Let $\tilde{\mathcal{P}}_m : T_m(T^*Q) \to \mathcal{F}_m$ be the projector associated to the decomposition $T_m(T^*Q) = \mathcal{F}_m \oplus \mathcal{F}_m^{\tilde{\Omega}_Q}$.
- $X_{\mathrm{nh}}(m) = \tilde{\mathcal{P}}_m X_H(m)$.
- For $f_1, f_2 \in C^\infty(D^*)$ define the B-gauged bracket:

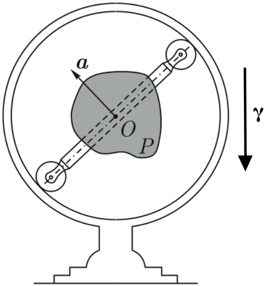
$$\{f_1, f_2\}_{\mathrm{nh}}^B(m) = \tilde{\Omega}_Q(\tilde{\mathcal{P}}_m \tilde{X}_{f_1}(m), \tilde{\mathcal{P}}_m \tilde{X}_{f_2}(m)),$$

  with $\tilde{X}_{f_j}$ defined by $\mathbf{i}_{\tilde{X}_{f_j}} \tilde{\Omega}_Q = df_j, \;\; j = 1, 2.$
- Equations of motion can be written with respect to the new bracket

$$X_{\mathrm{nh}}(f)(m) = \{f, H\}_{\mathrm{nh}}^B(m),$$

- Both brackets have the same characteristic distribution $\mathcal{F}$.
- In general $\{f_1, f_2\}_{\mathrm{nh}}^B \neq \{f_1, f_2\}_{\mathrm{nh}}$. Different way of encoding the constraint forces!

Suslov problem with potential



- The system has a smooth preserved measure $\iff$ a is an eigenvector of $\mathbb{I}$
  Restrict to this case. Assume $a = E_3$.

First integrals

$$
\begin{aligned}
\mathbb{I}\dot{\Omega} &= (\mathbb{I}\Omega) \times \Omega + \gamma \times \frac{\partial U}{\partial \gamma} + \lambda a, \\
\dot{\gamma} &= \gamma \times \Omega.
\end{aligned}
$$

- Constraint $\langle a, \Omega \rangle = 0$.
- Geometric integral $||\gamma|| = 1$.
- Energy $H = \frac{1}{2}\langle \mathbb{I}\Omega, \Omega \rangle + U(\gamma)$.

Integrability? (In the sense of Jacobi's last multiplier Theorem)
Depends on the existence of one additional independent integral.
(Reminiscent of Heavy top)

Known cases of integrability of the Suslov Problem

- Lagrange Top: $I_1 = I_2$, $U(\gamma) = \chi_3\gamma_3$. Extra integral:

  $$F = \langle \mathbb{I}\Omega, \gamma \rangle$$

  (Constraint is a preserved quantity. The system is Hamiltonian.)

- Generalized Kharlamova and Klebsh-Tisserand cases:

  $$I_1 \neq I_2 \qquad U(\gamma) = U_1(\gamma_1, \gamma_2^2 + \gamma_3^2) + U_2(\gamma_2, \gamma_1^2 + \gamma_3^2).$$

  Extra integral:

  $$K = \frac{1}{2}\langle \mathbb{I}\Omega, \mathbb{I}\Omega \rangle + I_2 U_1(\gamma_1, \gamma_2^2 + \gamma_3^2) + I_1 U_2(\gamma_2, \gamma_1^2 + \gamma_3^2).$$

Okuneva's Work

- Okuneva (1986,1987) studied the particular integrable case:

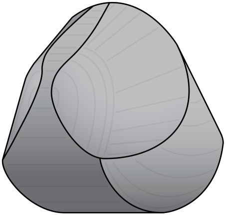
$$U(\gamma) = U_1(\gamma_1) + U_2(\gamma_2)$$

Striking result:

Two dimensional invariant manifolds may have genus from zero to five!
Very different from integrable Hamiltonian Systems.
Very different from integrable, Hamiltonizable nonholonomic systems (Chaplygin sphere, Veselova problem) where the system describes non-uniform rectilinear motion on tori.

Open problems in nonholonomic systems

Rolling of bodies with non-smooth surfaces



- Gömböc

Open problems in nonholonomic systems

- Perturbations of systems with an invariant measure.
- Perturbations of integrable nonholonomic systems. Nonholonomic KAM theory?
- Validity of Lagrange-D'Alembert principle and better understanding of friction-related phenomena.
- Discrete nonholonomic mechanics. *Nonholonomic standard map?*

And finally...

Thank you!