POSTER SESSION. THURSDAY, JULY 2, 11:00-11:30.

The motion of a Chaplygin sleigh coupled with a skate

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We consider the motion of a Chaplygin sleigh that is pulling a skate. The skate is assumed to be attached to the Chaplygin sleigh via a massless rigid rod that connects the contact points of the respective blades with the plane. Both the Chaplygin sleigh and the skate are subjected to the nonholonomic constraint that prohibits motion in the direction perpendicular to their respective blade. Our setup is a toy model for the motion of a trailer with two cars that is driven by its own inertia.

We observe that although the nonholonomic constraints define a smooth constraint distribution on the configuration space, the growth vector of this distribution is not constant. Indeed, its values drop at those configurations for which the blades of the Chaplygin sleigh and the skate are perpendicular.

Using the symmetries of the problem, we are able to formulate the reduced equations of motion on a 3-dimensional reduced space that is a rank 2 vector bundle over the circle. If the center of mass of the leading Chaplygin sleigh coincides with its contact point, the reduced dynamics is integrable and is either periodic or asymptotic depending on the initial angular velocity of the Chaplygin sleigh. If this is not the case, the dynamics appears to be non-integrable and takes place on two-dimensional invariant tori that arise as the level sets of the preserved energy of the system. We present numerical simulations for the reduced dynamics in both cases.

Reduction of optimal control problems for left-invariant systems on Lie groups

Leonardo Colombo-University of Michigan (joint work with Anthony Bloch, Rohit Gupta and Tomoki Ohsawa)

We will discuss some new developments in the study of reduction methods for optimal control problems of left-invariant systems on Lie groups. We will study Euler-Poincaré reduction and Lie-Poisson reduction.

Interesting applications as, for instance, the motion planning of a Unicycle with obstacles, Linear Quadratic Regulator type problems on SO(3) and the optimal control of the Brockett integrator will be shown.

Numerical Methods and Periodic Orbits in the Kepler-Heisenberg Problem

Corey Shanbrom (California State University, Sacramento, USA)

and Victor Dods (Leap Motion, Inc. USA)

The Kepler problem is among the oldest and most fundamental problems in mechanics. It has been studied in curved spaces, such as the sphere and hyperbolic plane. Here, we formulate the problem on the Heisenberg group, the simplest sub-Riemannian manifold. We take the sub-Riemannian Hamiltonian as our kinetic energy, and our potential is the fundamental solution to the Heisenberg sub-Laplacian. The resulting dynamical system is known to contain a fundamental integrable subsystem. We discuss the use of variational methods in proving the existence of periodic orbits with k-fold rotational symmetry for any odd integer k greater than one, and show approximations for k=3. Numerical methods which take advantage of the variational formulation are used to find approximate solutions having the sought-after symmetries. The sub-Riemannian structure on the Heisenberg group allows us to parameterize the optimization problem in terms of a single complex-valued curve, the Fourier decomposition of which lends itself to a particularly simple expression of the symmetry conditions.

The two-dimensional Taub–NUT system

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Abstract

In this contribution we study the Hamiltonian system defined by

$$\mathcal{H}_\eta = \frac{|\mathbf{q}|}{2(\eta + |\mathbf{q}|)}\,\mathbf{p}^2,$$

which describes the geodesic motion on the two-dimensional Taub–NUT space. This is a maximally superintegrable Hamiltonian system on a space of nonconstant curvature, that can be connected with the Kepler–Coulomb problem on the Euclidean space through a suitable Stäckel transform [1, 2]. Several interesting geometric and algebraic features of this exactly solvable system are analysed, and the quantization problem for negative values of η is also sketched.

References

[1] A. Ballesteros, A. Enciso, F.J. Herranz, O. Ragnisco, D. Riglioni: *Superintegrable oscillator and Kepler systems on spaces of nonconstant curvature via the Stäckel transform*, SIGMA **7**, 048 (2011).

[2] A. Ballesteros, A. Enciso, F.J. Herranz, O. Ragnisco, D. Riglioni: *An exactly solvable deformation of the Coulomb problem associated with the Taub–NUT metric*, Annals of Physics **351**, 540 (2014).

Hamilton-Jacobi theory in multisymplectic classical field theories

Pere Daniel Prieto (UPC, Spain)

The geometric framework for the Hamilton-Jacobi theory developed in previous works is extended for multisymplectic first-order classical field theories. The Hamilton-Jacobi problem is stated for the Hamiltonian formalism of these theories as a particular case of a more general problem, and the classical Hamilton-Jacobi equation for field theories is recovered from this geometrical setting. The use of distributions in jet bundles that represent the solutions to the field equations is the fundamental tool in this formulation.

This work has been carried out in collaboration with Manuel de León, Narciso Román-Roy and Silvia Vilariño.

'Moving' energy and integrability of a ball on a turning cup

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Abstrat

Energy is in general not conserved for mechanical nonholonomic systems with affine constraints. In this poster I give necessary a sufficient conditions for conservation of energy in nonholonomic systems with affine constraints and I point out that, if it is not the case, there might exist a modification of the energy—that may be interpreted as the energy of the system relative to a different reference frame in which the constraints are linear and for this reason will be called *moving energy*—that is conserved. After giving sufficient conditions for this to happen, we point out the role of symmetry in this mechanism. Then I characterize the conditions for the conservation of the components of the momentum maps of lifted actions, and of their 'gauge-like' generalizations, in mechanical nonholonomic systems with affine constraints. Lastly, we apply these ideas to prove that the motions of a heavy homogeneous solid sphere that rolls inside a convex surface of revolution in uniform rotation about its vertical figure axis, are (at least for certain parameter values and in open regions of the phase space) quasi-periodic on tori of dimension up to three.

Lie Hamilton systems on the plane: properties and applications

Cristina Sardón (Universidad de Salamanca, Spain)

We study Lie-Hamilton systems on the plane, i.e. systems of first-order differential equations describing the integral curves of a *t*-dependent vector field taking values in a finite-dimensional real Lie algebra of planar Hamiltonian vector fields with respect to a Poisson structure. We start with the local classification of finite-dimensional real Lie algebras of vector fields on the plane obtained in [A. González-López, N. Kamran and P.J. Olver, Proc. London Math. Soc. 64, 339 (1992)] and we interpret their results as a local classification of Lie systems. Moreover, by determining which of these real Lie algebras consist of Hamiltonian vector fields with respect to a Poisson structure, we provide the complete local classification of Lie-Hamilton systems on the plane. We present and study through our results new Lie-Hamilton systems of interest which are used to investigate relevant non-autonomous differential equations, e.g. we get explicit local diffeomorphisms between such systems. In particular, the Milne-Pinney, second-order Kummer-Schwarz, complex Riccati and Buchdahl equations as well as some Lotka-Volterra and nonlinear biomathematical models are analysed from this Lie-Hamilton approach.

N-Point Vortices in SU(3) and the non-torus manifold

Amna Shaddad (University of Manchester, UK)

The geometry of a momentum map is shown via the moment polytope, and the momentum map is a tool for exploring, amongst other things, the dynamics of the Hamiltonian function of a mechanical system. The torus action and the resulting convex moment polytope have been studied in detail but little has been done concerning non-torus action. This poster will introduce current work on an example of 'weighted' non-torus action resulting from research into vortices in $SU(3)$.

COVARIANT HAMILTONIAN FIELD THEORIES ON MANIFOLDS WITH BOUNDARY: YANG-MILLS THEORIES

A. IBORT AND A. SPIVAK

ABSTRACT. The multisymplectic formalism for first order covariant Hamiltonian field theories on manifolds with boundary is described and a general geometric formalism for the theory of boundary conditions based on the preservation of the conservation laws along the boundary is presented. This approach provides a natural geometrical realization of Fock's description of field theories as used for instance in recent work by Cattaneo, Mnev and Reshetikhin [Ca14].

It is also shown that the natural way to interpret Euler-Lagrange equations as an evolution system near the boundary is as a presymplectic system. The consistency conditions at the boundary are analyzed and the reduced phase space of the system is determined to be a symplectic manifold with a distinguished isotropic submanifold corresponding to the boundary data of the solutions of Euler-Lagrange equations. This setting makes it possible to define well-posed boundary conditions, and provides the adequate setting for the canonical quantization of the system.

The notions of the theory will be tested against three significant examples: scalar fields, Poisson σ-model and Yang-Mills theories.

HAMILTON-JACOBI THEORY, SYMMETRIES AND COISOTROPIC REDUCTION

Miguel Vaquero (ICMAT, Spain)

Reduction theory has played a major role in the study of Hamiltonian systems. On the other hand the Hamilton-Jacobi theory is one of the main tools to integrate the dynamics of certain Hamiltonian problems and a topic of research on its own. The natural question that we answer in this poster is how these two topics fit together and how to obtain a reduction and reconstruction procedure for the Hamilton-Jacobi equation, even in a generalized sense to be clarified in the poster.

EXISTENCE OF KNOTS AND LINKS IN STATIONARY FLUID FLOWS

FRANCISCO J. TORRES DE LIZAUR

A stationary ideal fluid flow on a three dimensional Riemannian manifold (M,g) is described by a velocity field $u(x)$ which satisfies the stationary Euler equations

$$i_u i_\omega \mu = dB\,, \qquad di_u \mu = 0$$

where μ is the volume form on (M, g), $B(x)$ is the so-called Bernoulli function and ω :=curl u is the *vorticity field*.

A major topic in topological hydrodynamics, going back to Lord Kelvin [5], concerns the existence of knotted and linked structures in stationary fluid flows, specially of knotted and linked *stream and vortex lines* (integral curves of the velocity and vorticity fields, respectively) and *vortex tubes* (bounded domains whose boundaries are invariant tori of the vorticity field).

In light of his structure theorem for the stationary Euler equations [1], V. Arnold suggested that this rich topological behaviour was to be sought within a particular class of solutions called *Beltrami fields*. A Beltrami field is an eigenfield of the curl operator, $\operatorname{curl} u = \lambda u$, or equivalently, a stationary flow whose velocity and vorticity are proportional. If they are everywhere non zero, they can also be seen as the Reeb field of a contact form.

A. Enciso and D. Peralta-Salas showed [2, 3] that knots and links of any type arise as vortex lines and vortex tubes of Beltrami fields in $\mathbb{R}^3$. Their techniques, relying on a Runge-type approximation theorem, cannot be applied to compact manifolds. In this talk we will sketch a proof of the fact that on the round sphere $\mathbb{S}^3$ and on the flat torus $\mathbb{T}^3$ high-energy Beltrami fields can indeed exhibit arbitrarily knotted and linked vortex lines and tubes. More precisely:

Theorem: Let $\mathcal{S}$ be a finite union of (pairwise disjoint, but possibly knotted and linked) closed curves and tubes in $\mathbb{S}^3$ or $\mathbb{T}^3$. In the case of the torus, we assume that $\mathcal{S}$ is contained in a contractible subset of $\mathbb{T}^3$. Then for infinitely many sufficiently large eigenvalues λ of the curl operator there exists a Beltrami field u with energy λ and a diffeomorphism Φ of $\mathbb{S}^3$ or $\mathbb{T}^3$ such that $\Phi(\mathcal{S})$ is a union of vortex lines and vortex tubes of u. Furthermore, this set is structurally stable.

One key to the proof is the growing degeneracy of the spectra of the curl operator on the round sphere and on the flat torus: our strategy does not work in general Riemannian 3-manifolds. Another feature of the result is that it gives a very precise understanding of the nature of the diffeomorphism Φ : its effect is to rescale a contractible subset containing $\mathcal{S}$ to have diameter of order $\frac{1}{\lambda}$.

This is a joint work with Alberto Enciso and Daniel Peralta-Salas[4].

References

[1] V. I. Arnold, Sur la topologie des ecoulements stationnaires des fluides parfaits, C. R. Acad. Sci. Paris 261 (1965) 1720.

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