EXERCISES FOR THE MINI–COURSE
INTEGRABILITY AND NONHOLONOMIC SYSTEMS WITH SYMMETRY

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1. Part 1

Exercise 1.
- Consider the autonomous ordinary differential equation on $\mathbb{R}^{>}_{+}$ $\dot z = z^2$. Write the conjugate system by the diffeomorphism $\mathcal{C} \in \mathrm{Diff}(\mathbb{R}_+)$ defined by $\mathcal{C}(z) = z^3$.
- Consider the vector field $X = (z_1, z_2)$ on the real plane. Write the conjugated vector field $\tilde X$ by the diffeomorphism $\mathcal{C} \in \mathrm{Diff}(U)$ defined by $\mathcal{C}(z_1, z_2) = (2z_1, z_1 z_2)$, with $U = \{(z_1, z_2) \in \mathbb{R}^2 |\ z_1 \neq 0\}$.

Exercise 2. Consider the vector field on $\mathbb{R}^3$

$$X = yz\frac{\partial}{\partial x} + xz\frac{\partial}{\partial y} - xy\frac{\partial}{\partial z}\,.$$

(1) Prove that the functions

$$f_1(x,y,z) = \frac{1}{4}(x-y)(x+y), \qquad \text{and} \qquad f_2(x,y,z) = \frac{1}{2}(x^2+y^2) + z^2$$

are first integrals of X.

(2) Expect for the compactness and connectedness of the level sets of the first integrals, is X B–integrable?

Exercise 3. Consider the 5–dimensional real Maxwell–Bloch system on $\mathbb{R}^5$[1]

$$X = y_1\frac{\partial}{\partial x_1} + x_1 z\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial x_2} + x_2 z\frac{\partial}{\partial y_2} - (x_1y_1 + x_2y_2)\frac{\partial}{\partial z}\,.$$

(1) Prove that the functions

$$h(x_1,y_1,x_2,y_2,z) = \frac{1}{2}(y_1^2 + y_2^2 + z^2)$$
$$f(x_1,y_1,x_2,y_2,z) = \frac{1}{2}(x_1^2 + x_2^2) + z$$
$$j(x_1,y_1,x_2,y_2,z) = x_2y_1 - x_1y_2$$

are independent first integrals of X.

(2) Is the vector field

$$Y = x_2\frac{\partial}{\partial x_1} + y_2\frac{\partial}{\partial y_1} - x_1\frac{\partial}{\partial x_2} - y_1\frac{\partial}{\partial y_2}$$

a dynamical symmetry of X?

(3) Is the system B–integrable?

2. Part 2

Exercise 4. Consider the nonholonomic particle: $Q = \mathbb{R}^3 \ni q = (x,y,z)$, Lagrangian

$$L(q,\dot q) = \frac{1}{2}|\dot q|^2$$

and nonholonomic constraint

$$\dot z - y\dot x = 0\,.$$

- Compute the constraint distribution $\mathcal{D}$;

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[1] It is well known that this system admits a bi–Hamiltonian structure or a Lagrangian and symplectic realization on $\mathbb{R}^6$, but we with to use it here as an exercise for B–integrability.

- compute the reaction force $R(q, \dot q)$;
- compute the equations of motions;
- compute the reaction annihilator distribution $\mathcal{R}^\circ$.

Exercise 5. Consider now the nonholonomic particle with potential $V = V(x^2+z^2)$, and the same constraint as in Exercise 4. Compare the constraint distributions $\mathcal{D}$, the reaction forces $R(q, \dot q)$ and the reaction–annihilator distributions $\mathcal{R}^\circ$ of the two systems, what do you observe? What do you observe if the constraint change: $\dot z + x\dot y - y\dot x = 0$?

Exercise 6. Consider a nonholonomic particle in $Q = \mathbb{R}^3$ subject to the potential energy $V(q) = z$, with $\in \mathbb{R}$ and to a constraint affine in the velocities

$$\dot z + x\dot y - y\dot x - \kappa = 0\,,$$

with $\kappa \in \mathbb{R} \setminus \{0\}$. Prove that the energy (or Jacobi integral) is not conserved along the flow of the dynamics.

Exercise 7. Consider a vertical disk of mass m that rolls without sliding on a plane and assume if is under the effect of a positional force. The configuration space is $Q \cong \mathbb{R}^2 \times S^1 \times S^1$ with coordinates $q = (x, y, \varphi, \theta)$, where (x, y) are the coordinates of the center of mass of the disk, φ is the angle between the x–axis and the projection of the disk on the plane, θ is the angle between a fixed radius of the disk and the vertical. The Lagrangian of the system is

$$L(q, \dot q) = \frac{1}{2}m(\dot x^2 + \dot y^2) + \frac{1}{2}J\dot\varphi^2 + \frac{1}{2}I\dot\theta^2 - x\sin\varphi - y\cos\varphi$$

where I and J are the pertinent moments of inertia and $V(q) = x\sin\varphi - y\cos\varphi$ is the potential acting on the system. The rolling without slipping nonholonomic constraint is[2]

$$\dot x = \dot\theta\cos\varphi \qquad \text{and} \qquad \dot y = \dot\theta\sin\varphi\,.$$

(1) Write the fibers of the constraint distribution $\mathcal{D}$;
(2) Compute the reaction force $R(q, \cdot q)$;
(3) write the Lagrangian of the system;
(4) Determine the reaction annihilator distribution $\mathcal{R}^\circ$;
(5) the system is invariant with respect to the action of the group $G = S^1 \times S^1$ of translations of the angles. Prove that $T\mathcal{O}_G \cap \mathcal{D} = \emptyset$;
(6) Prove that $T\mathcal{O}_G \cap \mathcal{R}^\circ \neq \emptyset$;
(7) Does the system admit a $\mathcal{R}^\circ$–momentum? If so write its expression.

Exercise 8. Consider a free unit mass particle on the plane. Assuming that the symmetry group is just the group of translations on the plane, prove that the angular momentum along the vertical is a gauge momentum with respect to the group of translations on the plane.

Exercise 9. ('5–dimensional particle') Consider a free particle on $\mathbb{R}^5 \ni q = (q_1, \ldots, q_5)$ subject to the linear nonholonomic constraint given by the non–integrable rank–two distribution with fibers

$$\mathcal{D}_q = \mathrm{span}R\{\partial_{q_1}\,,\ q_1\partial_{q_2} + q_3\partial_{q_3} + q_3\partial_{q_4} + \partial_{q_5}\}\,.$$

The matrix $S(q)$ such that $\mathcal{D}_q = \mathrm{ker}S(q)$ is the 3×5 matrix with block structure

$$S(q) = \begin{pmatrix} 0_3 & \mathbb{I}_3 & -\hat q \end{pmatrix}\,,$$

where $0_3 \in \mathbb{R}^3$ is the zero vector, $\mathbb{I}_3$ is the $(3) \times (3)$ identity matrix, and $\hat q \in \mathbb{R}^3$ is the vector whose components are the first 3 coordinates of q. The Lagrangian and the constraint are invariant with respect to the action of $G = \mathbb{R}^3$ of translations along q_2, q_4, q_5.

- Prove that $\mathcal{D} \cap T\mathcal{O}_G = \emptyset$.
- Write the reaction force $R(q, \dot q)$. (*Hint. Note that the kinetic energy has the identity and there is no potential energy*).
- Write the equations of motions.
- Compute the reaction annihilator distribution $\mathcal{R}^\circ$ and prove that $\mathcal{R}^\circ \cap T\mathcal{O}_G \neq \emptyset$.

[2] The system is analogous to the classical vertical disk with the addiction of an active force of potential energy $V = x\sin\varphi - y\cos\varphi$.

- Prove that the function $J_D = \dot{q}_5 \sqrt{1+q_1^2+q_2^2+q_3^2}$ is a first integral of the system. Prove that J_D is an $\mathcal{R}^\circ$–gauge momentum.

Exercise 10. Does the Chaplygin sleigh admit any $\mathcal{D}$–gauge momenta?

3. Part 3

Exercise 11. Consider a nonholonomic particle in $Q = \mathbb{R} \times S^1 \times \mathbb{R} \ni q = (x, y, z)$ under the effect of a positional force of potential energy $V(q) = \frac{1}{2}(x^2 + z^2)$ and subjected to the linear constraint $z - y\dot{x} = 0$. Is the system B–integrable?

Exercise 12. Consider a nonholonomic particle in $Q = \mathbb{R} \times S^1 \times \mathbb{R} \ni q = (x, y, z)$ under the effect of a positional force of potential energy $V(q) = \frac{1}{2}(x^2 + z^2)$ and subjected to the linear constraint $z - y\dot{x} = 0$. Is the system B–integrable?

Exercise 13. Consider an heavy ball that rolls without sliding inside a vertical cylinder under the action of gravity. Recall that one can reduce in stages the system with respect to $SO(3)$ and then to the S^1 action ending up with a 4–dimensional reduced space $\mathcal{M} \cong \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, parametrized with coordinates $(z, \dot{z}, \dot{\theta}, \vec{n} \cdot \omega)$, where (z, θ) are cylindrical coordinates on $\mathbb{R} \times S^1$, $(\dot{z}, \dot{\theta}) \in \mathbb{R} \times \mathbb{R}$ their velocities and $\vec{n} \cdot \omega$ is the normal component to the cylinder of the angular velocities of the sphere written in the space frame. Moreover, recall that the (full) system admits three $\mathrm{SO}_3(\times)S^1$–invariant, independent first integrals: $J_{1,D} = -\frac{r}{a}(ma^2 + I_C)$, $J_{2,D} = r\,\vec{n} \cdot \omega - \frac{r}{a}\,z\,\dot{\theta}$ and the energy (on D), that go down to $\mathcal{M}$.

Consider the dynamical system defined by the reduced system.

(1) Write the energy $E_{L,D}$ of the full system, and then restrict it to $\mathcal{M}$.
(2) Prove that z oscillates harmonically, provided $\dot{\theta} \neq 0$.[3]
(3) Prove that the three first integrals are independent in an open and dense subset $\bar{\mathcal{M}}$ of $\mathcal{M}$.
(4) What do you need to guarantee that the motions on $\bar{\mathcal{M}}$ are periodic?
(5) Prove that the motion on $\bar{\mathcal{M}}$ are periodic.

[3] If you need some help you can have a look to L.M. Bates, H. Graumann and C. MacDonnell, *Examples of gauge conservation laws in nonholonomic systems*, Rep. Math. Phys., **37** (1996), 295–308.