Symplectic embedding problems

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This minicourse will survey some recent progress on understanding when one symplectic manifold can be symplectically embedded into another. As the minicourse will illustrate, symplectic embedding problems can be quite subtle, even for seemingly simple examples, and many important questions remain completely open.

Here is a detailed plan of what we will cover:

i) Embedded contact homology capacities and the Fibonacci staircase.

McDuff and Schlenk [1] determined when a four-dimensional symplectic ellipsoid can be symplectically embedded into a four-dimensional ball, and found that if the ellipsoid is close to round, the answer is given by an "infinite staircase" determined by the odd index Fibonacci numbers, while if the ellipsoid is sufficiently stretched, all obstructions vanish except for the volume obstruction. We will review their result, and use this as a jumping off point to introduce the theory of "embedded contact homology" (ECH) capacities [2,3]. ECH capacities give obstructions to embedding any four-dimensional symplectic manifold into another, and McDuff showed that these obstructions are sharp for embeddings of ellipsoids [4]. We will explain why this implies the "Fibonacci staircase" part of McDuff and Schlenk's result [5]. We will briefly discuss some other situations where the obstruction coming from embedded contact homology is sharp [6, 7].

ii) Higher dimensional constructions

Very little is known about higher dimensional symplectic embedding problems, even for ellipsoids. For example, the formula for the ECH capacities of ellipsoids admits a natural analogue in higher dimensions, and one could ask if these numbers still give sharp embedding obstructions. In fact, it turns out that they do not even give obstructions. This follows from a construction due to Guth [8] that I will explain. A related issue is the nonexistence of "intermediate" symplectic capacities, which I will also briefly address [8, 9].

iii) Higher dimensional obstructions

Even though the constructions from above give considerable flexibility, many interesting obstructions still exist in higher dimensions. For example, an analogue of McDuff and Schlenk's Fibonacci staircase holds in any even dimension [10]. I will explain this result and its proof, which involves modifying a technique due to Hind and Kerman [11].

References:

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