Polysymplectic manifolds and Marsden-Weinstein reduction

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Outline

1. Polysymplectic manifolds
2. Marsden-Weinstein reduction for polysymplectic manifolds
3. Kirillov-Kostant-Souriau theorem for polysymplectic manifolds
4. References

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1. Polysymplectic manifolds
2. Marsden-Weinstein reduction for polysymplectic manifolds
3. Kirillov-Kostant-Souriau theorem for polysymplectic manifolds
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Polysymplectic structure

Definition

Let M be a differentiable manifold. A polysymplectic structure on M is a family $(\omega_1, \ldots, \omega_k)$ where:

- $\omega_A \in \Lambda^2(M)$ are closed 2-forms and
- $\bigcap_{A=1}^{k} \ker \omega_A = \{0\}$.

$(M, \omega_1, \ldots, \omega_k)$ is called a polysymplectic manifold.

The tangent bundle of k^1-covelocities

- $(T^1_k)^*Q = T^*Q \oplus \overset{k}{\ldots} \oplus T^*Q$

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- Projection map

$$\begin{array}{rccc} \pi^k_Q: & T^*Q \oplus \overset{k}{\cdots} \oplus T^*Q & \to & Q \\ & (\alpha_{1_q}, \ldots, \alpha_{k_q}) & \mapsto & q \end{array}$$

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- Canonical polysymplectic structure on $(T^1_k)^*Q$.

$$\omega_A := (\pi^{k,A}_Q)^*\omega\,, \quad 1 \leq A \leq k$$

where

- $\pi^{k,A}_Q : (T^1_k)^*Q \to T^*Q\,, \quad \pi^{k,A}_Q(\alpha_{1_q}, \ldots, \alpha_{k_q}) = \alpha_{A_q}$
- ω is the canonical symplectic structure of T^*Q

1 Polysymplectic manifolds

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Polysymplectic Hamiltonian space

Definition

A polysymplectic Hamiltonian space is a family $(M, \omega_1, \ldots, \omega_k; \Phi, J)$ formed by:

- A polysymplectic manifold $(M, \omega_1, \ldots, \omega_k)$,
- A polysymplectic action $\Phi\colon G \times M \to M$
- A $coad^k$-equivariant momentum map for Φ,

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$$\Phi_g^* \omega_A = \omega_A$$

  for all $g \in G$ and all $A = 1, \ldots, k$.
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Momentum map

Definition

Let $(M, \omega_1, \ldots, \omega_k)$ be a polysymplectic manifold and $\Phi\colon G \times M \to M$ be a polysymplectic action. A map

$$J \equiv (J^1, \ldots, J^k) : M \to \mathfrak{g}^* \times \overset{k}{\ldots} \times \mathfrak{g}^*$$

is said to be a momentum map for the action Φ if

$$i_{\xi_M}\omega_A = d\hat{J}^A_\xi \,, \text{ for all } \xi \in \mathfrak{g} \text{ and } A = 1, \ldots, k$$

where $\hat{J}^A_\xi : M \to \mathbb{R}$ is the map defined by

$$\hat{J}^A_\xi(x) = J^A(x)(\xi) \quad x \in M \,.$$

Momentum map

Definition

A momentum map $J \equiv (J^1, \ldots, J^k) : M \to \mathfrak{g}^* \times \overset{k}{\ldots} \times \mathfrak{g}^*$ is said to be $coad^k$-equivariant if

$$J(\Phi_g(x)) = coad_g^k J(x) \ , \ \forall g \in G, x \in M.$$

$$\begin{array}{ccc} M & \xrightarrow{J} & \mathfrak{g}^* \times \overset{k}{\ldots} \times \mathfrak{g}^* \\ \Phi_g \downarrow & & \downarrow coad_g^k \\ M & \xrightarrow{J} & \mathfrak{g}^* \times \overset{k}{\ldots} \times \mathfrak{g}^* \end{array}$$

$$coad_g^k(\mu) = coad_g^k(\mu_1, \ldots, \mu_k) = (coad_g(\mu_1), \ldots, coad_g(\mu_k)) \, ,$$

Theorem (Reduction theorem)

Let $(M, \omega_1, \ldots, \omega_k; \Phi, J)$ be a Hamiltonian polysymplectic space and $\mu = (\mu_1, \ldots, \mu_k) \in \mathfrak{g}^* \times \overset{k}{\ldots} \times \mathfrak{g}^*$ be a regular value of J. We denote by $i \colon S = J^{-1}(\mu) \to M$ the canonical inclusion.

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Let us suppose that:

- *The distribution $\bigcap_{A=1}^{k} \ker(i^*\omega_A)$ has constant rank*
- *$S/\mathcal{F}_S$ is a manifold and the canonical projection $\pi : S \to S/\mathcal{F}_S$ is a submersion.*

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Then there exist a polysymplectic structure

$$(\widetilde{\omega}_1^S, \dots, \widetilde{\omega}_k^S)$$

on $S/\mathcal{F}_S$ such that for all $A = 1, \dots, k$

$$\pi^* \widetilde{\omega}_A^S = i^* \omega_A$$

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Lemma

If $x \in J^{-1}(\mu) = J^{-1}(\mu_1, \ldots, \mu_k)$ *then*

$$T_x(G_\mu \cdot x) \subseteq \bigcap_{A=1}^{k} \ker i^*\omega_A(x)$$

Can we assure that the distribution $\bigcap_{A=1}^{k} \ker(i^*\omega_A)$ has constant rank?

When can we assure that $T_x(G_\mu \cdot x) = \bigcap_{A=1}^{k} \ker i^*\omega_A(x)$ for each $x \in J^{-1}(\mu)$?

Step 1

For each $A = 1, \dots, k$ we shall prove that

$$\frac{\left(\dfrac{\ker (J^A)_*(x)}{\ker \omega_A(x)}\right)}{\{[\xi_M(x)] \,/\, \xi \in \mathfrak{g}_{\mu_A}\}} \quad , x \in J^{-1}(\mu)$$

is a symplectic vector space.

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Step 2

We shall define a linear map

$$\widetilde{\pi}_x^A : \frac{T_x(J^{-1}(\mu))}{T_x(G_\mu \cdot x)} \equiv T_{[x]}(J^{-1}(\mu)/G_\mu) \longrightarrow \frac{\left(\dfrac{\ker(J^A)_*(x)}{\ker\,\omega_A(x)}\right)}{\{[\xi_M(x)]\,/\,\xi \in \mathfrak{g}_{\mu_A}\}}$$

such that:

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such that: if $\widetilde{\pi}_x^A$ is a linear epimorphism and $\bigcap_{A=1}^k \ker \widetilde{\pi}_x^A = \{0\}$ holds then

$$T_x(G_\mu \cdot x) = \bigcap_{A=1}^{k} \ker i^* \omega_A(x) \qquad x \in J^{-1}(\mu)$$

Lemma(Step 1)

Let $(M, \omega_1, \ldots, \omega_k; \Phi, J)$ be a Hamiltonian polysymplectic space and $\mu = (\mu_1, \ldots, \mu_k) \in \mathfrak{g}^* \times \overset{k}{\ldots} \times \mathfrak{g}^*$ be a regular value of J. Then for each $A = 1, \ldots, k$

$$\frac{\left(\dfrac{\ker (J^A)_*(x)}{\ker \omega_A(x)}\right)}{\{[\xi_M(x)] \,/\, \xi \in \mathfrak{g}_{\mu_A}\}} \quad , x \in J^{-1}(\mu)$$

is a symplectic vector space.

Step 1: Sketch of the proof

$$\begin{array}{ccc}
\ker\left(J^A\right)_*(x) & \xrightarrow{i_x^A} & T_xM \\
\Big\downarrow{\scriptstyle pr^{J^A}} & & \Big\downarrow{\scriptstyle pr_A^M} \\
\dfrac{\ker\left(J^A\right)_*(x)}{\ker\omega_A(x)} & \xrightarrow{\widetilde{i_x^A}} & \dfrac{T_xM}{\ker\omega_A(x)} \\
\Big\downarrow{\scriptstyle \widetilde{pr_A}} & & \\
\dfrac{\left(\dfrac{\ker\left(J^A\right)_*(x)}{\ker\omega_A(x)}\right)}{\left\{[\xi_M(x)]\,/\,\xi\in\mathfrak{g}_{\mu_A}\right\}} & &
\end{array}$$

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$$\begin{array}{ccc}
\ker\left(J^A\right)_*(x) & \xrightarrow{i_x^A} & T_xM\ ,\ \omega_A(x) \\
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\end{array}$$

Step 2

We shall define a linear map

$$\widetilde{\pi}_x^A : \frac{T_x(J^{-1}(\mu))}{T_x(G_\mu \cdot x)} \equiv T_{[x]}(J^{-1}(\mu)/G_\mu) \longrightarrow \frac{\left(\dfrac{\ker(J^A)_*(x)}{\ker \omega_A(x)}\right)}{[\xi_M(x)] \,/\, \xi \in \mathfrak{g}_{\mu_A}\}}$$

such that:
If $\widetilde{\pi}_x^A$ *is a linear epimorphism and* $\bigcap_{A=1}^k \ker \widetilde{\pi}_x^A = \{0\}$ *holds then*

$$T_x(G_\mu \cdot x) = \bigcap_{A=1}^{k} \ker i^* \omega_A(x) \qquad x \in J^{-1}(\mu)$$

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Lemma

Let $(M, \omega_1, \ldots, \omega_k; \Phi, J)$ be a Hamiltonian polysymplectic space and $\mu = (\mu_1, \ldots, \mu_k) \in \mathfrak{g}^* \times \overset{k}{\ldots} \times \mathfrak{g}^*$ be a regular value of J.

If G_μ acts freely and properly on $J^{-1}(\mu)$, then:

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Lemma

Let $(M, \omega_1, \ldots, \omega_k; \Phi, J)$ be a Hamiltonian polysymplectic space and $\mu = (\mu_1, \ldots, \mu_k) \in \mathfrak{g}^* \times \overset{k}{\ldots} \times \mathfrak{g}^*$ be a regular value of J.

If G_μ acts freely and properly on $J^{-1}(\mu)$, then:

- *The orbit space $J^{-1}(\mu)/G_\mu$ is a manifold with the canonical projection $\pi_\mu \colon J^{-1}(\mu) \to J^{-1}(\mu)/G_\mu$ a submersion.*

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If G_μ acts freely and properly on $J^{-1}(\mu)$, then:

- *The orbit space $J^{-1}(\mu)/G_\mu$ is a manifold with the canonical projection $\pi_\mu : J^{-1}(\mu) \to J^{-1}(\mu)/G_\mu$ a submersion.*
- $T_{\pi_\mu(x)}(J^{-1}(\mu)/G_\mu) \cong \dfrac{T_x(J^{-1}(\mu))}{T_x(G_\mu \cdot x)} \cong \dfrac{T_x(J^{-1}(\mu))}{\{\xi_{J^{-1}(\mu)}(x) \,/\, \xi \in \mathfrak{g}_\mu\}}$
- $T_x(J^{-1}(\mu)) = \bigcap_{A=1}^{k} \ker (J^A)_*(x)$

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Lemma

Let $(M, \omega_1, \ldots, \omega_k; \Phi, J)$ be a Hamiltonian polysymplectic space and $\mu = (\mu_1, \ldots, \mu_k) \in \mathfrak{g}^* \times \overset{k}{\ldots} \times \mathfrak{g}^*$ be a regular value of J.

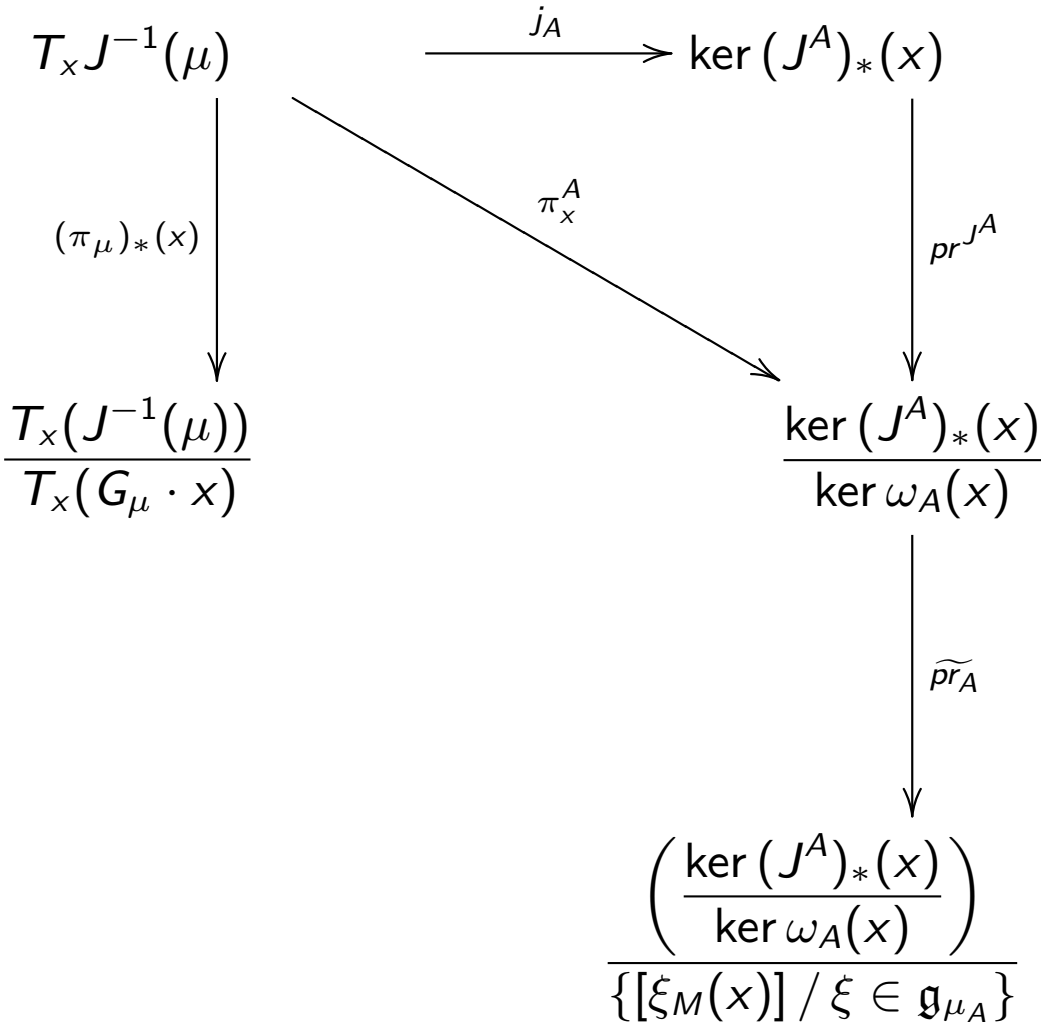
If G_μ acts freely and properly on $J^{-1}(\mu)$, then one can define a linear map

$$\widetilde{\pi}_x^A : \frac{T_x(J^{-1}(\mu))}{T_x(G_\mu \cdot x)} \equiv T_{[x]}(J^{-1}(\mu)/G_\mu) \longrightarrow \frac{\left(\dfrac{\ker(J^A)_*(x)}{\ker \omega_A(x)}\right)}{[\xi_M(x)] \,/\, \xi \in \mathfrak{g}_{\mu_A}\}}$$

Step 2: Sketch of the proof

$$
\begin{array}{ccc}
T_x J^{-1}(\mu) & \xrightarrow{j_A} & \ker (J^A)_*(x) \\
\Big\downarrow{\scriptstyle (\pi_\mu)_*(x)} & & \Big\downarrow{\scriptstyle pr^{J^A}} \\
\dfrac{T_x(J^{-1}(\mu))}{T_x(G_\mu \cdot x)} & & \dfrac{\ker (J^A)_*(x)}{\ker \omega_A(x)} \\
 & & \Big\downarrow{\scriptstyle \widetilde{pr_A}} \\
 & & \dfrac{\left(\dfrac{\ker (J^A)_*(x)}{\ker \omega_A(x)}\right)}{\{[\xi_M(x)] \,/\, \xi \in \mathfrak{g}_{\mu_A}\}}
\end{array}
$$

Step 2: Sketch of the proof



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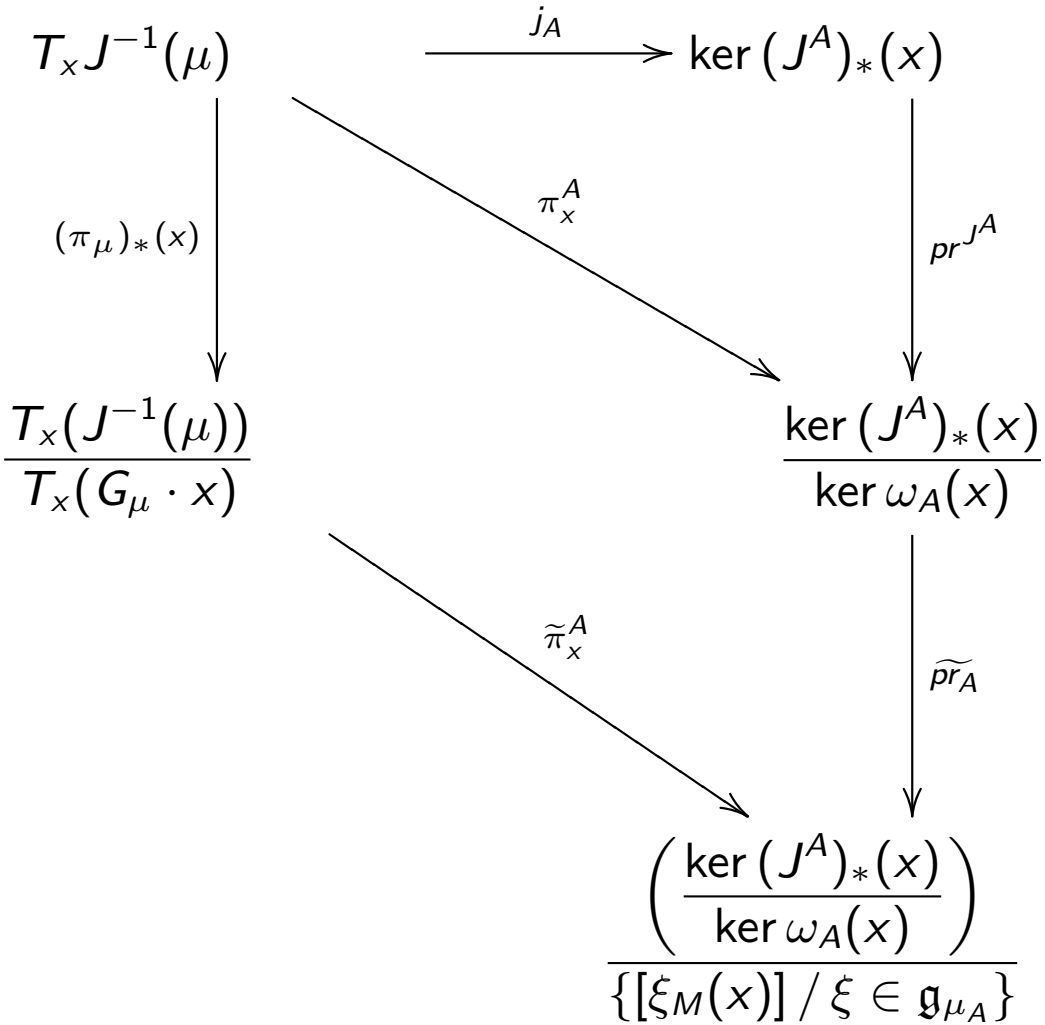


Step 2

Theorem

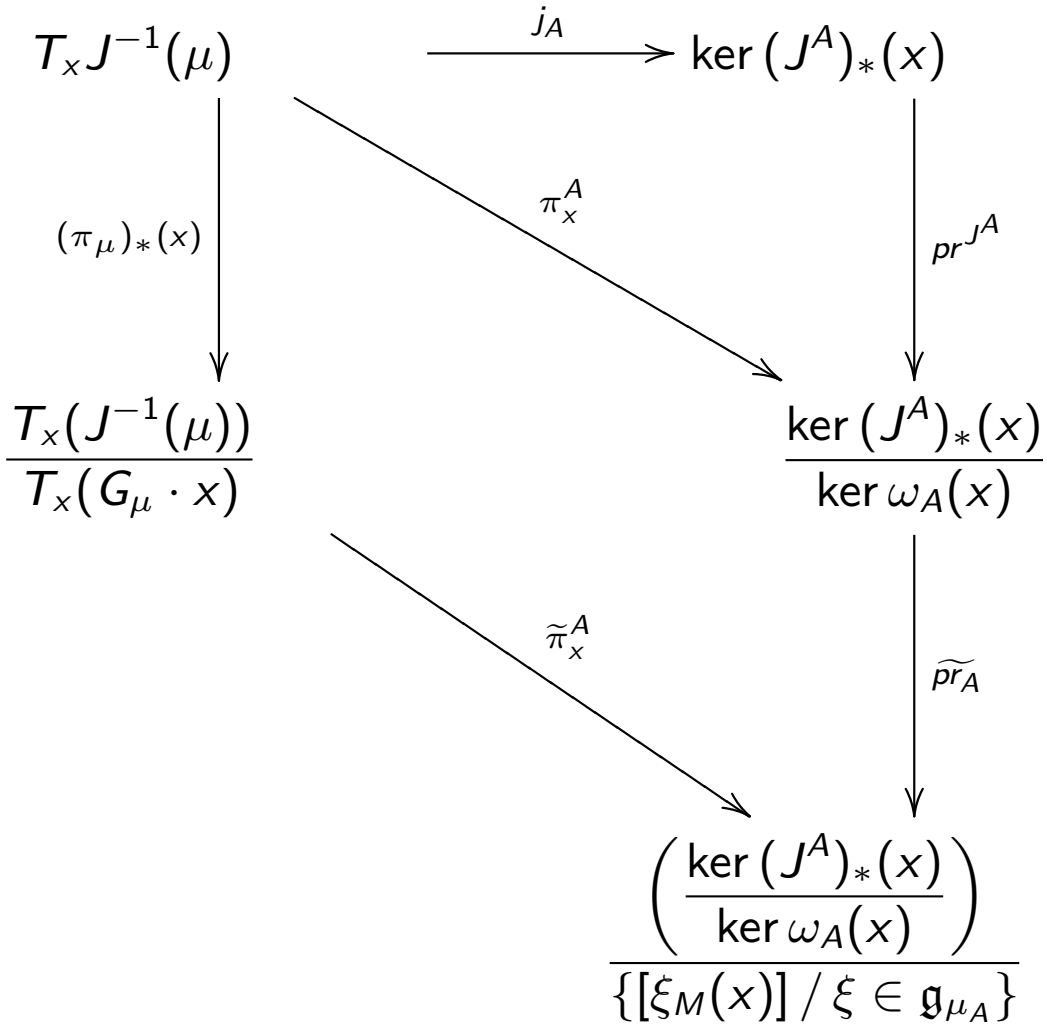
If $\widetilde{\pi_x^A}$ *is an epimorphism and* $\bigcap_{A=1}^{k} \ker \widetilde{\pi_x^A} = 0$ *then*

1. $\bigcap_{A=1}^{k} \ker \omega_A^S(x) = T_x(G_\mu \cdot x)$
2. $\bigcap_{A=1}^{k} \ker \omega_A^\mu(x) = \{0\}$

and then $(\omega_1^\mu, \ldots, \omega_k^\mu)$ *is a polysymplectic structure on* $J^{-1}(\mu)/G_\mu$.

Step 2

Sketch of the proof



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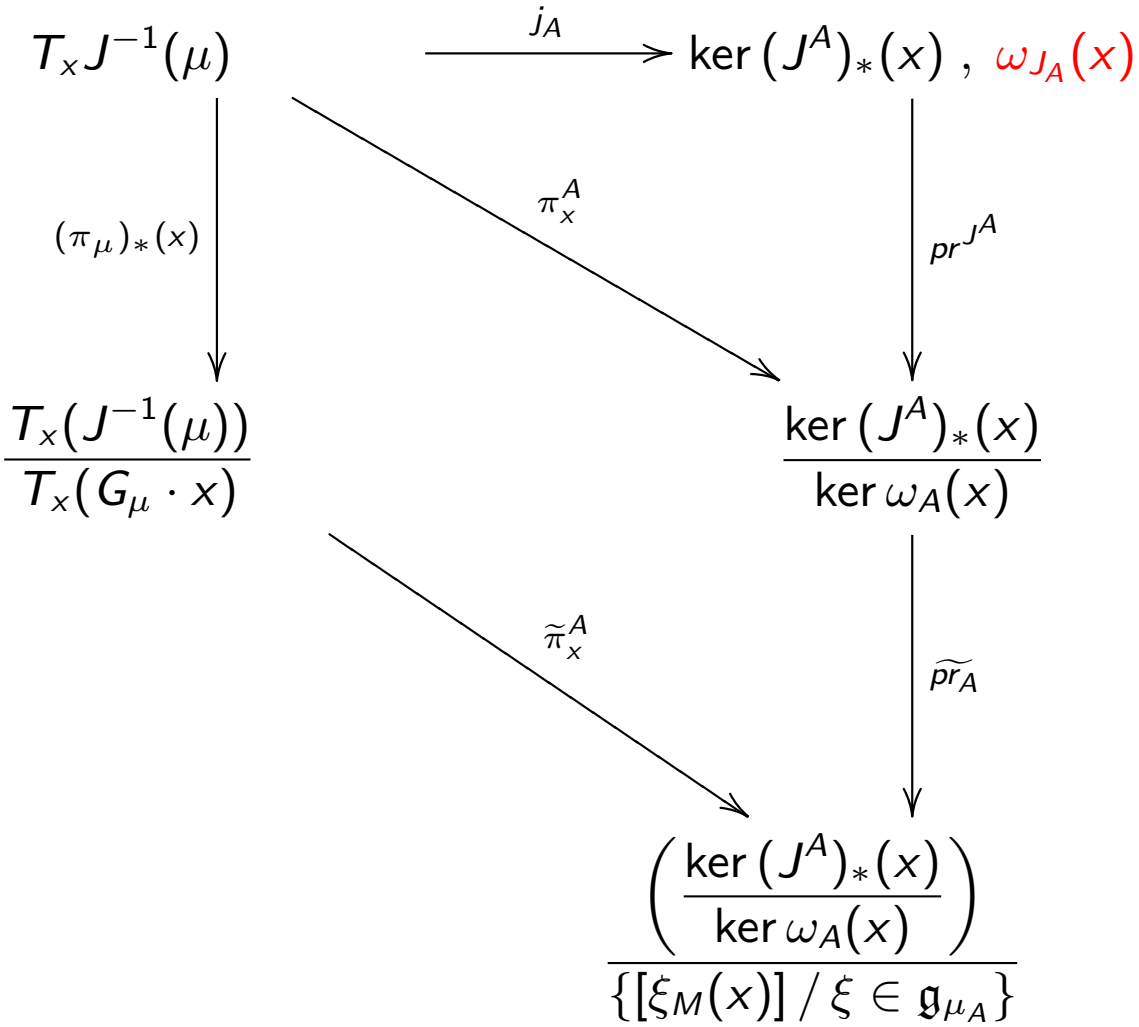

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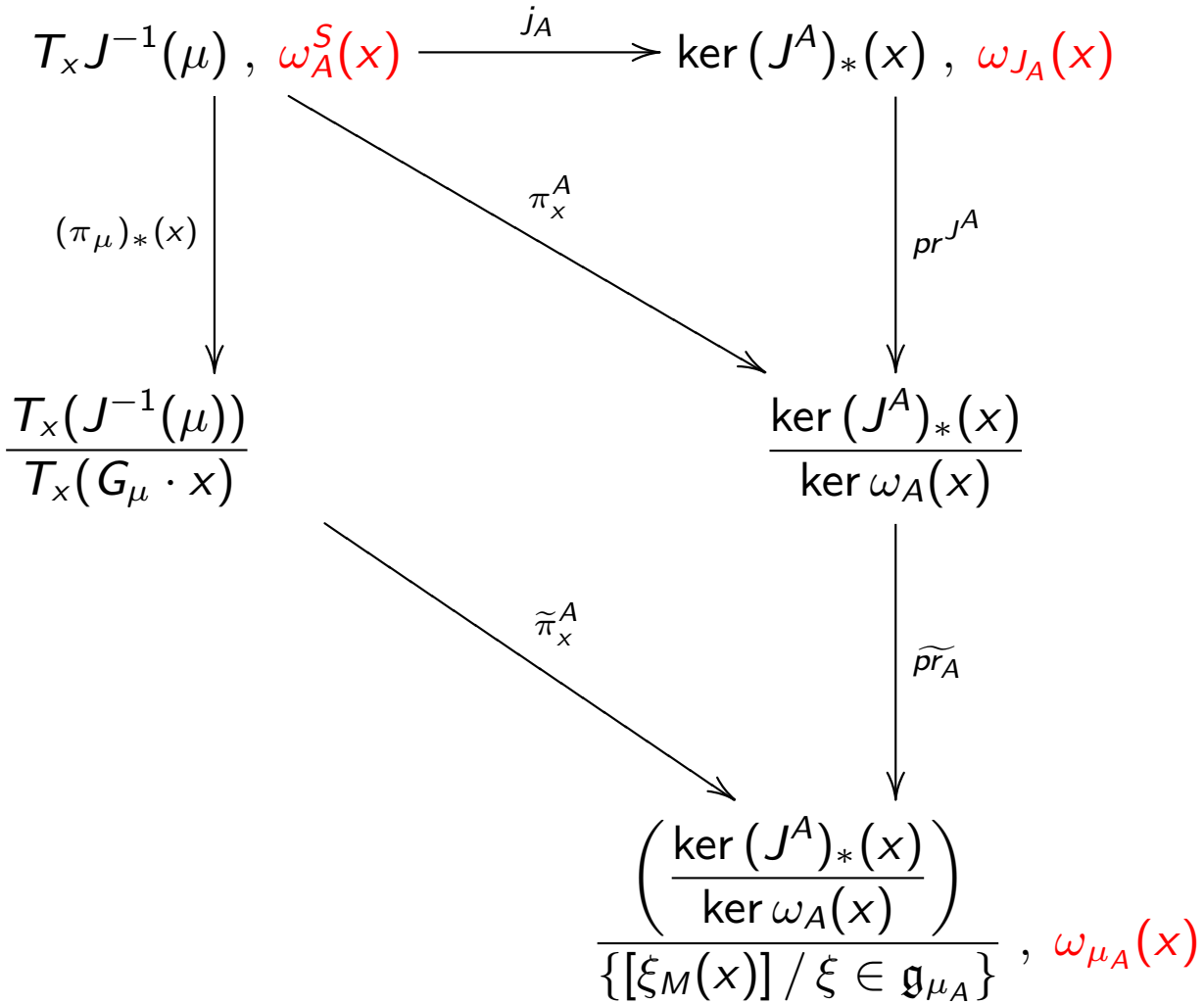

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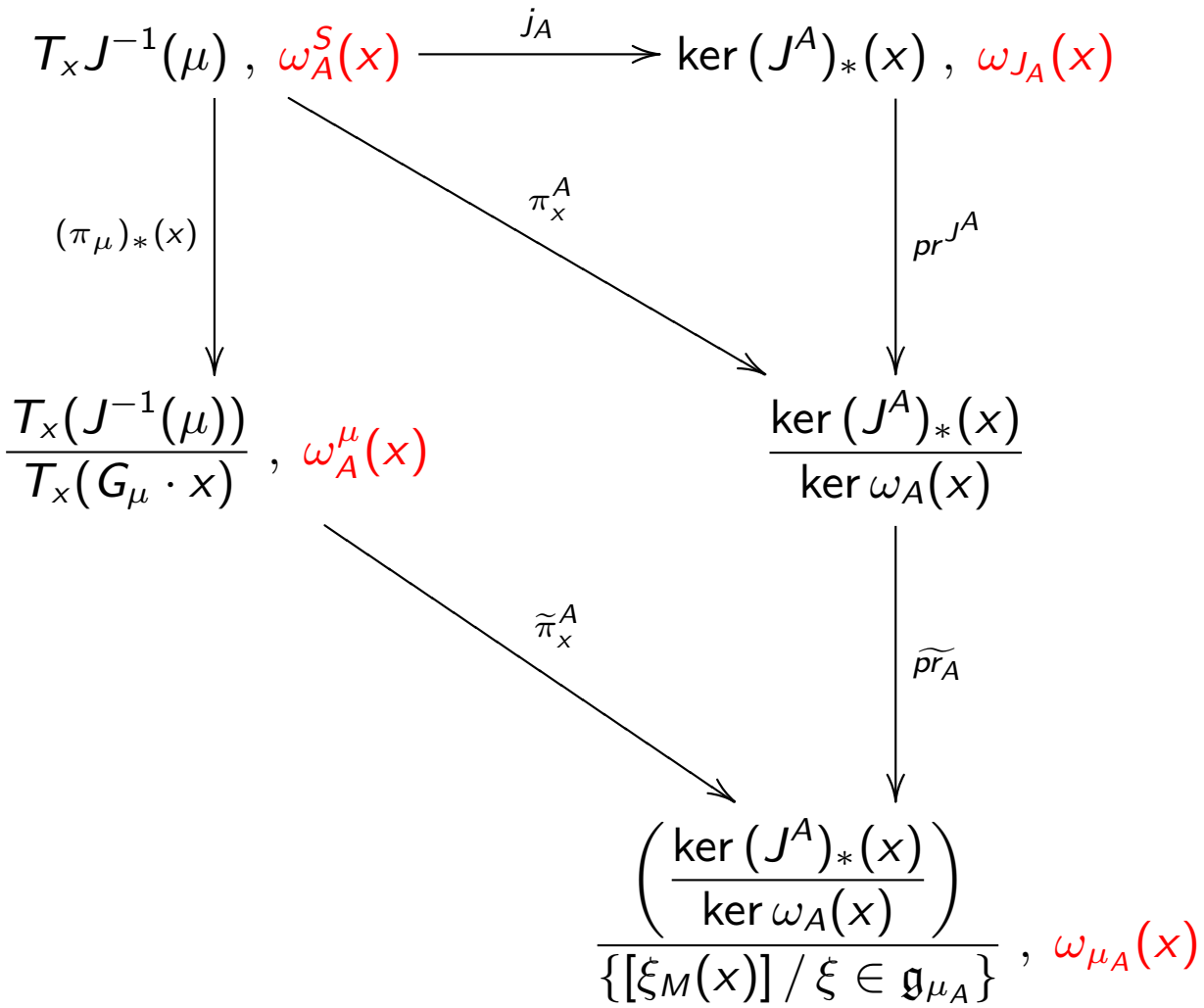

Lemma

If

$$(T_xJ^1, \dots, T_xJ^{A-1}, T_xJ^{A+1}, \dots, T_xJ^k) :$$

$$\ker \omega_A(x) \to T_{J^1(x)}\mathfrak{g}^* \times \dots \times T_{J^{A-1}(x)}\mathfrak{g}^* \times T_{J^{A+1}(x)}\mathfrak{g}^* \times \dots \times T_{J^k(x)}\mathfrak{g}^*$$

is a linear epimorphism then

$$\pi_x^A : T_x(J^{-1}(\mu)) \to \frac{\ker(J^A)_*(x)}{\ker \omega_A(x)}$$

is epimorphism and induces the following epimorphism,

$$\widetilde{\pi_x^A} : \frac{T_x(J^{-1}(\mu))}{T_x(G_\mu \cdot x)} \longrightarrow \frac{\left(\frac{\ker(J^A)_*(x)}{\ker \omega_A(x)}\right)}{\{[\xi_{J^{-1}(\mu)}(x)] \,/\, \xi \in \mathfrak{g}_{\mu_A}\}}$$

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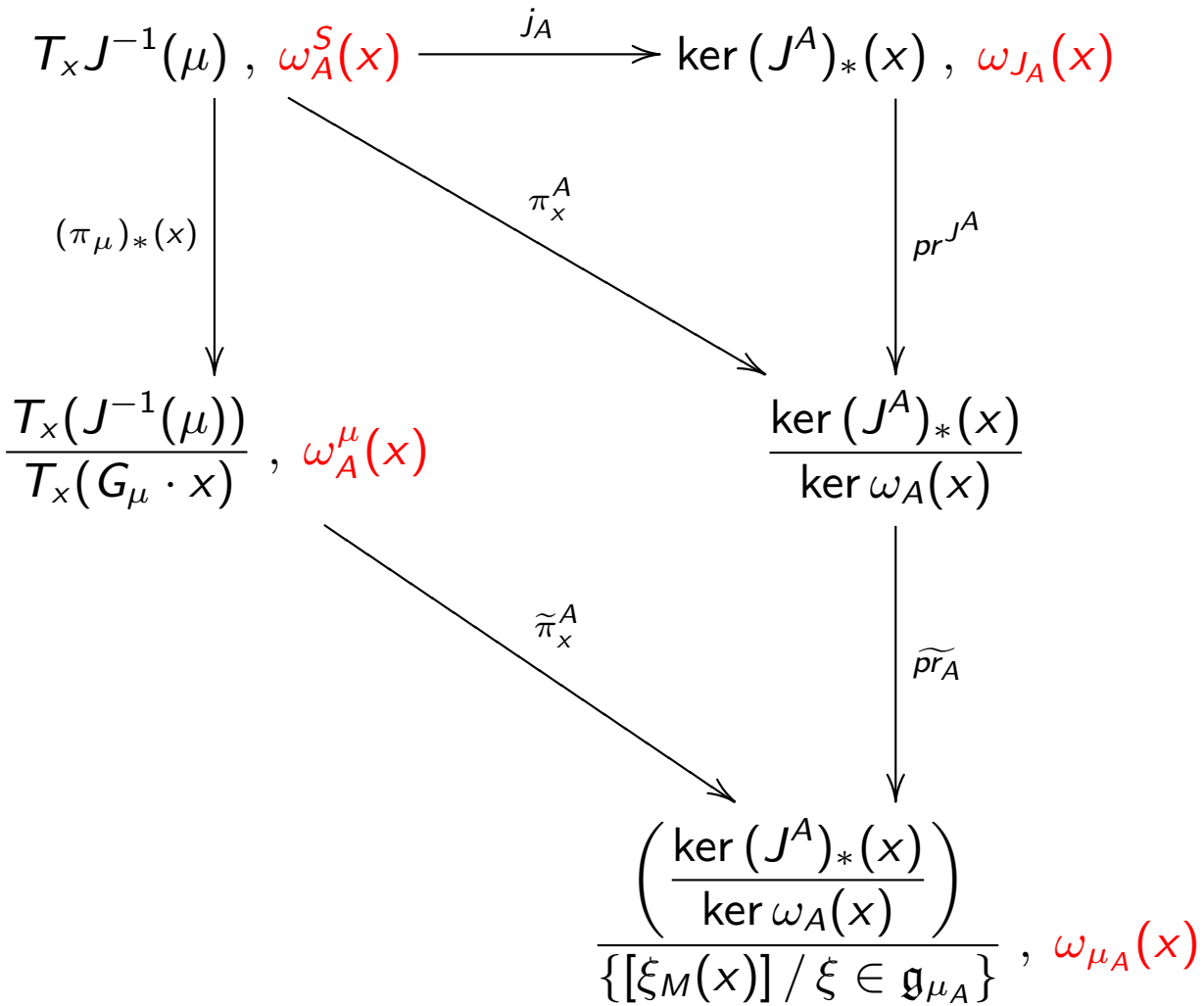

Lemma

If there exists $A_0 \in \{1, \dots, k\}$ *such that*

$$T_x(G_\mu \cdot x) \cap (\ker\, \omega_{A_0}(x) + \ker\, \omega_A(x)) = \{0\} \quad \forall A = 1, \dots, k$$

and the map

$$\mathfrak{g}_{\mu_{A_0}} + \mathfrak{g}_{\mu_A} \to T_x M, \quad \xi \to \xi_M(x)$$

is one to one for each $A = 1, \dots, k$ *then*

$$\bigcap_{A=1}^{k} \ker\, \widetilde{\pi_x^A} = \{0\}\ .$$

Marsden-Weinstein reduction

Theorem

Let $(M, \omega_1, \ldots, \omega_k; \Phi, J)$ be a Hamiltonian polysymplectic space. Assume

- *$\mu = (\mu_1, \ldots, \mu_k) \in \mathfrak{g}^* \times \overset{k}{\ldots} \times \mathfrak{g}^*$ is a regular value of J and that*
- *G_μ acts freely and properly on $J^{-1}(\mu)$.*

If for each $x \in J^{-1}(\mu)$:

1. *$(T_x J^1, \ldots, T_x J^{A-1}, T_x J^{A+1}, \ldots, T_x J^k) : \ker \omega_A(x) \to T_{J^1(x)}\mathfrak{g}^* \times \ldots \times T_{J^{A-1}(x)}\mathfrak{g}^* \times T_{J^{A+1}(x)}\mathfrak{g}^* \times \ldots \times T_{J^k(x)}\mathfrak{g}^*$ is a linear epimorphism and,*
2. *there exists $A_0 \in \{1, \ldots, k\}$ such that*

$$T_x(G_\mu \cdot x) \cap (\ker \omega_{A_0}(x) + \ker \omega_A(x)) = \{0\} \quad \forall A = 1, \ldots, k$$

and the map

$$\mathfrak{g}_{\mu_{A_0}} + \mathfrak{g}_{\mu_A} \to T_x M, \quad \xi \to \xi_M(x)$$

is one to one for each $A = 1, \ldots, k$

Marsden-Weinstein reduction

Theorem

Then the orbit space $J^{-1}(\mu)/G_\mu$ has a only polysymplectic structure $(\omega_1^\mu, \dots, \omega_k^\mu)$ with the property

$$\pi_\mu^* \omega_A^\mu = i^* \omega_A \,, \quad \forall A = 1, \dots, k \,,$$

where $\pi_\mu \colon J^{-1}(\mu) \to J^{-1}(\mu)/G_\mu$ is the canonical projection and $i \colon J^{-1}(\mu) \to M$ is the inclusion.

1 Polysymplectic manifolds

2 Marsden-Weinstein reduction for polysymplectic manifolds

3 Kirillov-Kostant-Souriau theorem for polysymplectic manifolds

4 References

Kirillov-Kostant-Souriau theorem

- Let G be a Lie group.
- Let Φ be the action of G on itself by left traslations.
- Consider the induced action Φ^{T^*} on $(T^1_K)^*G = T^*G \oplus \ldots \oplus T^*G$

$$\begin{array}{rccc} \Phi^{T^*}: & G \times (T^1_K)^*G & \to & (T^1_K)^*G \\ & (g', (\alpha_{1_g}, \ldots, \alpha_{k_g})) & \mapsto & ((T^{k*}L_{g'^{-1}})(\alpha_{1_g}, \ldots, \alpha_{k_g})) \end{array}$$

Momemtum mapping of the action Φ^{T^*} is

$$\begin{array}{rccl} J=(J^1,\ldots,J^k): & (T^1_K)^*G & \to & \mathfrak{g}^*\times \overset{k}{\ldots} \times\mathfrak{g}^* \\ & (\alpha_{1g},\ldots,\alpha_{kg}) & \to & (\alpha_{1g}\circ(R_g)_*(e),\ldots,\alpha_{kg}\circ(R_g)_*(e)) \end{array}$$

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Using the identification

$$\begin{array}{rcl} T^*G\oplus \overset{k}{\ldots} \oplus T^*G & \cong & G\times(g^*\times \overset{k}{\ldots} \times\mathfrak{g}^*) \\ (\alpha_{1_g},\ldots,\alpha_{k_g}) & \equiv & (g,\alpha_{1_g}\circ T_eL_g,\ldots,\alpha_{k_g}\circ T_eL_g) \end{array}$$

Momemtum mapping of the action Φ^{T^*} is

$$\begin{array}{rccl} J=(J^1,\ldots,J^k): & (T^1_K)^*G & \to & \mathfrak{g}^*\times \overset{k}{\ldots} \times\mathfrak{g}^* \\ & (\alpha_{1_g},\ldots,\alpha_{k_g}) & \to & (\alpha_{1_g}\circ(R_g)_*(e),\ldots,\alpha_{k_g}\circ(R_g)_*(e)) \end{array}$$

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the momemtum map can be written as follows:

$$\begin{array}{rccl} J: & G\times(\mathfrak{g}^*\times \overset{k}{\ldots} \times\mathfrak{g}^*) & \to & \mathfrak{g}^*\times \overset{k}{\ldots} \times\mathfrak{g}^* \\ & (g,(\mu'_1,\ldots,\mu'_k)) & \to & coad^k_g(\mu'_1,\ldots,\mu'_k) \end{array}$$

that is

$$J^A(g,(\mu'_1,\ldots,\mu'_k))=coad_g(\mu'_A)$$

$$J^{-1}(\mu_1, \ldots, \mu_k)$$

$$= \{(g, (\alpha_1, \ldots, \alpha_k)) \in G \times (\mathfrak{g}^* \times \overset{k}{\ldots} \times \mathfrak{g}^*) \,|\, Coad(g, \alpha_A) = \mu_A \,,\, 1 \leq A \leq k\}$$

$$J^{-1}(\mu_1, \ldots, \mu_k)$$

$$= \{(g, (\alpha_1, \ldots, \alpha_k)) \in G \times (\mathfrak{g}^* \times \overset{k}{\ldots} \times \mathfrak{g}^*) \mid Coad(g, \alpha_A) = \mu_A \,,\, 1 \leq A \leq k\}$$

Then $J^{-1}(\mu_1, \ldots, \mu_k)$ is diffeomorphism to G

$$\begin{array}{ccc} G & \to & J^{-1}(\mu_1, \ldots, \mu_k) \\ g & \to & (g, (coad_{g^{-1}}(\mu_1), \ldots, coad_{g^{-1}}(\mu_k)) \end{array}$$

$$J^{-1}(\mu_1,\ldots,\mu_k)$$
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Then $J^{-1}(\mu_1,\ldots,\mu_k)$ is diffeomorphism to G

$$\begin{array}{ccc} G & \to & J^{-1}(\mu_1,\ldots,\mu_k) \\ g & \to & (g,(coad_{g^{-1}}(\mu_1),\ldots,coad_{g^{-1}}(\mu_k)) \end{array}$$

Hence, the reduced phase space is naturally identifiable with the orbit $\mathcal{O}_{(\mu_1,\ldots,\mu_k)}$

$$\frac{J^{-1}(\mu_1,\ldots,\mu_k)}{G_{(\mu_1,\ldots,\mu_k)}} \cong \frac{G}{G_{(\mu_1,\ldots,\mu_k)}} \cong \mathcal{O}_{(\mu_1,\ldots,\mu_k)} \subseteq \mathfrak{g}^* \times \overset{k}{\ldots} \times \mathfrak{g}^*$$

1 Polysymplectic manifolds

2 Marsden-Weinstein reduction for polysymplectic manifolds

3 Kirillov-Kostant-Souriau theorem for polysymplectic manifolds

4 References


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Thank you for your attention!!!

Outline

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4. References