Universal model for locally conformally symplectic manifolds

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- $S^3 \times S^1$ has canonical complex structure

$$(\mathbb{C}^2 \backslash \{0\}, J_{\mathrm{st}})/\sim \tfrac{1}{2}\mathrm{Id} := (H_{\frac{1}{2}\mathrm{Id}}, J_{\frac{1}{2}\mathrm{Id}})$$

- $(S^3 \times S^1, J_{\frac{1}{2}\mathrm{Id}})$ is not Kahler! $(\pi_1(S^3 \times S^1) = \mathbb{Z})$
  - $g_{\mathrm{st}} \rightsquigarrow \frac{1}{2}\mathrm{Id}^* g_{\mathrm{st}} = \frac{1}{4} g_{\mathrm{st}}$
  - $F = (z\bar{z})^{-1} > 0 \rightsquigarrow \frac{1}{2}\mathrm{Id}^* F = 4F$

$\rightsquigarrow Fg_{\mathrm{st}}$ descends to $S^3 \times S^1$.

- $(S^3 \times S^1, J_{\frac{1}{2}\mathrm{Id}}, g)$ is a locally conformally Kahler (l.c.K). manifold

Definition

A l.c.K. structure in (X, J) is given by a Hermitian metric g locally conformal to a Kahler one. The associated 2-form $\omega_g := g(\cdot, J\cdot)$ is l.c. to a Kahler form.

Example (Linear Hopf manifolds (H_A, J_A, g_A))

$(\mathbb{C}^n \backslash \{0\}, J_{\mathrm{st}})/\sim A$, A has eigenvalues of norm < 1.

▷ L.c.K. subject of much recent research (Ornea, Verbitsky, Kamishima, Belgun, Gauduchon...)

- Deciding which complex surfaces admit l.c.K. structures.
- Structure results; relations with Sasakian geometry.
- Group of automorphisms.
- Reduction.
- Kodaira type holomorphic embeddings in (H_A, J_A).

$$\text{KAHLER} \overset{\text{forget } J}{\subset} \text{SYMPLECTIC}$$
$$\text{L. C. KAHLER} \overset{\text{forget } J}{\subset} \text{L. C. SYMPLECTIC}$$

Definition

A l.c.s structure on M is given by $\omega \in \Omega^2(M)$

- max. non degenerate,
- l.c. to a symplectic form ($d\omega = \tau \wedge \omega$, $d\tau = 0$)

▷ In U, $\mathrm{e}^{-f_U}\omega$ symplectic, f_U unique up to constant.

$\rightsquigarrow$ τ Lee form

- L.c. symplectic = Transitive (even dim.) Jacobi manifold

$\rightsquigarrow$ Good setting for Hamiltonian mechanics (Vaisman).

- L.c. symplectic $\rightsquigarrow$ Lie algebroid structure on $\mathcal{J}^1 M$.
  - τ exact iff $(\mathcal{J}^1 M, [\cdot,\cdot], \rho)$ unimodular.

$\rightsquigarrow$ Recent activity on l.c.s geometry (Banyaga, Haller, Rybicky, Kotschick, Bande)

- Group of l.c.s. transformations.
- Reduction.
- Moser stability.

- Some results at the level of the conformal class $\langle\omega\rangle$

- Kahler (Kodaira): $(X, J, \Omega_g) \overset{holomorphic}{\hookrightarrow} (\mathbb{CP}^N, J_{\mathrm{st}}, \Omega_{FS})$
- Symplectic (Tischler): $(M, \Omega) \overset{symplectic}{\hookrightarrow} (\mathbb{CP}^N, \Omega_{\mathrm{FS}})$
- Kodaira $\overset{+Moser}{\rightsquigarrow}$ Tischler
- L.c.K. (Ornea, Verbitsky): $(X, J, \omega_g) \overset{holomorphic}{\hookrightarrow} (H_A, J_A, \omega_{g_A})$

Question 1

Are there (compact) universal models for l.c.s. structures?

$$(M, \omega) \overset{l.c.s.}{\hookrightarrow} ?$$

Question 2

If so and $? = (H_A, \omega_{g_A})$, do we have l.c.K + Moser = l.c.s?

- The compact model:

$$(S^{2n-1} \times S^1, \omega_{\mathrm{st}} = d\eta_{\mathrm{st}} - d\theta \wedge \eta_{\mathrm{st}})$$

$$\eta_{\mathrm{st}} = \sum_{j=1}^{n} x_j dy_j - y_j dx_j$$

Theorem (Marrero, M. T., Padrón)

Let (M^{2n}, ω) be a compact, exact, integral l.c.s. manifold. Then

$$(M^{2n}, \omega) \overset{l.c.s}{\hookrightarrow} (S^{8n+5} \times S^1, c\omega_{\mathrm{st}}),\ c > 0$$

Proposition (Marrero, M. T., Padrón)

$$(H^{2n}_A, J_A) \overset{\phi}{\cong} S^{2n-1} \times S^1$$

$$\phi_* \omega_{g_A} = f \omega_{\mathrm{st}},\ f > 0$$

Theorem (Marrero, M. T., Padrón)

For a compact L.c.K (+condition), the holomorphic embedding followed by a Moser type correction gives a l.c.s. embedding up to conformal factor.

- Symplectic (Gotay, Tuynman):

$$(M,\Omega) \cong (C_{\mathrm{red}},\Omega_{\mathrm{red}})\ , C \hookrightarrow (\mathbb{R}^{2N},\Omega_{\mathrm{st}})$$

- Reduction exist for l.c.s. manifolds (Haller, Rybicki)

Theorem (Marrero, M. T., Padrón)

Let (M,ω) be a l.c.s. manifold of the first kind with finite rank k and basis μ. Then

$$(M,\omega) \cong (C_{\mathrm{red}},\omega_{\mathrm{red}}), C \hookrightarrow (\mathbb{R}\times\mathcal{J}^1(\mathbb{T}^k\times\mathbb{R}^N),\omega_\mu)$$