Geometric nonlinear control
- an introduction -

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- Lecture I -

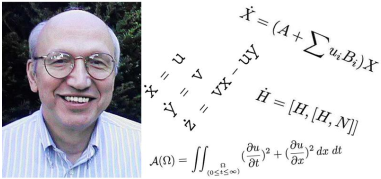
Basics notions of nonlinear control systems on manifolds and controllability issues

- Control theory is a theory that deals with influencing the behavior (controlling) of dynamical systems.
- Many processes in industries like robotics and aerospace industry have strong nonlinear dynamics.
- The configuration spaces of many mechanical systems are smooth manifolds (Lie groups, symmetric spaces,...). Techniques from differential and Riemannian geometry are fundamental in modern control theory.

The evolution ...

- Beginning - 60's

Roger Brockett
- the father -



- Adultness - mid 70's

Agrachev, Bloch, Crouch, Nijmeijer, Jurdjevic, Krener, Sachkov, Sontag, Sussmann, Van der Schaft, ...

- The steam of publications has grown sharply in recent years and gives every indication of continuing to grow...

What is a control system?

Vector fields and flows

Let M be a n-dimensional smooth ($\mathcal{C}^\infty$) manifold and $x \in M$.

- A (smooth) vector field on M is a (smooth) mapping

$$x \in M \mapsto f(x) \in T_x M.$$

- A dynamical system on M is an equation of the form

$$\dot{x} = f(x), \quad x \in M.$$

γ is a solution of $\dot{x} = f(x) \Leftrightarrow \gamma$ is an integral curve of f

A dynamical system is a vector field

Vector fields and flows (cont.)

Assume that the vector field f is complete, i.e., for all $x_0 \in M$ the solution $x(t, x_0)$ of the Cauchy problem

$$\dot{x} = f(x), \quad x(0, x_0) = x_0$$

is defined for all $t \in \mathbb{R}$.

Flow generated by the vector field f:

$$t \;\to\; \exp(t\,f) : \begin{array}{ccc} M & \to & M \\ x_0 & \mapsto & x(t, x_0) \end{array} \qquad , \; t \in \mathbb{R}$$

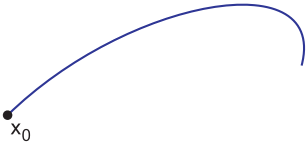
If $\dot{x} = f(x)$ describes the dynamics of a moving fluid in M, then $\exp(t\,f)$ takes any particle of the fluid from a position x_0 and moves it for a time $t \in \mathbb{R}$ to the position $\exp(t\,f)(x_0) = x(t, x_0)$.

(If f not complete, flow is local)

Dynamical system

A dynamical system evolving on a smooth manifold M is a vector field $f(.)$ on M

$$\dot{x}(t) = f(x(t)), \quad x(t) \in M.$$



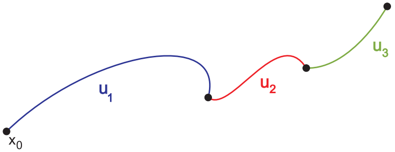
The dynamics of this system is determined by the flow of one vector field only. The future $x(t, x_0)$ is completely determined by the present state $x_0 = x(0, x_0)$.

In order to affect (control) dynamics we consider a family of vector fields.

Control system

A control system evolving on M is a family of vector fields $f(., u)$ on M, parameterized by the controls u.

$$\dot{x}(t) = f(x(t), u(t)), \quad x(t) \in M, \quad u(t) \in U \subset \mathbb{R}^m$$



x is the state of the system, M is the state space

u is the input or control of the system

In control theory we can change the dynamics of the control system at any moment of time by changing the control u.

Control system (cont.)

$$\dot{x}(t) = f(x(t), u(t)), \quad x(t) \in M, \quad u(t) \in U \subset \mathbb{R}^m$$

The controls belong to a class $\mathcal{U}$ of admissible controls. Its choice depends on what the control system is modeling.

Our assumptions:

- On the set of admissible controls:

$\mathcal{U}$ contains all the piecewise constant functions with values in U, which are piecewise continuous from the right.

- On the vector fields:

For each $x_0 \in M$ and each $u \in \mathcal{U}$, the differential equation

$$\dot{x}(t) = f(x(t), u(t)), \qquad x(0) = x_0$$

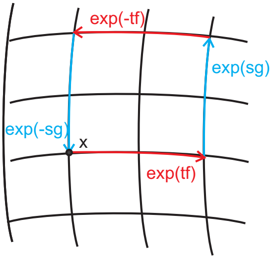
has a solution for all $t \in [0, \infty[$.

Control system / family of vector fields

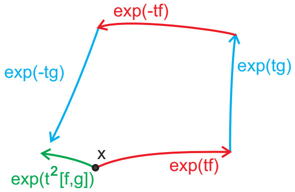
• The set of all smooth vector fields on M forms a Lie algebra. We can perform linear combinations and Lie brackets.

• The Lie bracket of 2 vector fields measures non-commutativity of the corresponding flows:

$$[f, g] \equiv 0 \ \Leftrightarrow \ \exp(t f) \circ \exp(sg) = \exp(sg) \circ \exp(t f), \ \forall s, t \in \mathbb{R}.$$



f and *g* commute



f and *g* don't commute

Control system / family of vector fields (cont.)

The Lie bracket allows studying interconnections between different dynamical systems.

From the control theory point of view it is particularly useful that $[f, g] \notin \text{span}\{f, g\}$.

Example

$$f(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0 \\ x_1 \end{bmatrix},$$

$$[f, g](x) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

At the origin, $[f, g](0)$ adds a new direction to $\text{span}\{f(0), g(0)\}$.

Orbit of a control system

Let $f_u(.) := f(., u)$

Consider the family of vector fields on M : $\mathcal{F} = \{f_u\}_{u \in \mathcal{U}}$

$Lie(\mathcal{F})$ denotes the Lie algebra generated by $\mathcal{F}$

Orbit of $\mathcal{F}$ through a point $x_0 \in M$:

$$\mathcal{O}(x_0) = \{exp(t_k f_{u_k}) \cdots exp(t_1 f_{u_1})(x_0) \mid k \in \mathbb{N},\ f_{u_i} \in \mathcal{F},\ t_i \in \mathbb{R}\}$$

Points in the orbit of x_0 are those points which can be reached by starting at x_0 and traveling along the integral curves of vector fields in $\mathcal{F}$, *either forward or backward* in time in all possible sequences.

Properties of orbits

Orbit theorem (Nagano-Sussmann)

Each orbit $\mathcal{O}(x)$ is an immersed submanifold of M and, for every $x \in M$,

$$Lie_x(\mathcal{F}) \subset T_x\mathcal{O}(x).$$

Corollary (Rashevsky-Chow theorem)

If M is connected and $Lie_x(\mathcal{F}) = T_xM$, for every $x \in M$, then

$$\mathcal{O}(x) = M, \quad \forall x \in M.$$

$$\mathcal{O}(x_0) = \{exp(t_k f_{u_k}) \cdots exp(t_1 f_{u_1})(x_0) \mid k \in \mathbb{N},\ f_{u_i} \in \mathcal{F},\ t_i \in \mathbb{R}\}$$

Reachable set of $\mathcal{F}$ from a point $x_0 \in M$:

$$\mathcal{R}(x_0) = \{exp(t_k f_{u_k}) \cdots exp(t_1 f_{u_1})(x_0) \mid k \in \mathbb{N},\ f_{u_i} \in \mathcal{F},\ t_i \geq 0\}$$

Points in the reachable set are those points which can be reached by traveling along the integral curves of vector fields in $\mathcal{F}$, going *only forward* in time.

$$\mathcal{R}(x_0) \subset \mathcal{O}(x_0)$$

If $\mathcal{F}$ is symmetric (i.e., $f \in \mathcal{F}$ if and only if $-f \in \mathcal{F}$), then orbits and reachable sets coincide.

Difference between reachable sets and orbits:

Geometric nonlinear control

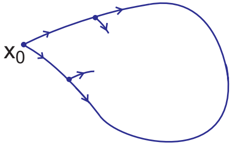
What is a control system?
Dynamical systems / Control systems
Typical problems of control theory
Examples

Accessibility / Controllability
Reachable sets
Accessibility and controllability
Distributions and integrability

Control systems on Lie groups
Main theorems on controllability
Examples

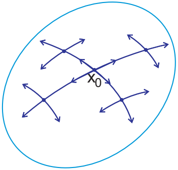
Representation of trajectories
Reduction to the "simple" case

Only forward-in-time motions are allowed



Reachable set

Both forward and backward motions are allowed



Orbit

Typical problems of control theory

Controllability

Controllability is the ability to steer a system from a given initial state to any final state, in finite time, using the available controls.

Controllability doesn't care about the quality of the trajectory between two states, neither the amount of control effort!

Optimal control

What is the optimal way to control the system? We may require smooth trajectories, minimizing costs, ...

Control affine systems

$$\dot{x} = \underbrace{g_0(x) + \sum_{i=1}^{m} u_i g_i(x)}_{f(x,u)}$$

- g_0 is the drift vector field - specifies the dynamics in the absence of controls.
- $g_i,\ i = 1, \cdots m$, are called the control vector fields

Control affine systems (cont.)

$$\dot{x} = g_0(x) + \sum_{i=1}^{m} u_i g_i(x)$$

Assumptions:

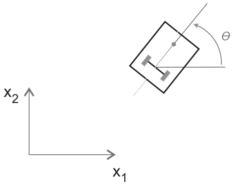
- **On the set of admissible controls:**

$\mathcal{U}$ consists of all the piecewise constant functions with values in U, which are piecewise continuous from the right.

- **On the vector fields:**

$g_0, g_1, \cdots, g_m$ are smooth (of class $\mathcal{C}^\infty$). $m \leq n = \dim(M)$.

The car



The state of the car:

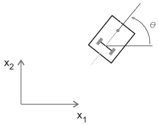
- position of its center of mass $(x_1, x_2) \in \mathbb{R}^2$
- orientation angle $\theta \in S^1$ (relative to the positive direction of the axis x_1)

The state space:

$$M = \{x = (x_1, x_2, \theta) | x_1, x_2 \in \mathbb{R}, \theta \in S^1\} = \mathbb{R}^2 \times S^1.$$

The car (cont.)

Possible kinds of motion:



Linear motion: drive the car forward and backward with some fixed linear velocity $u_1 = \sqrt{\dot{x}_1^2 + \dot{x}_2^2}$

$$\begin{cases} \dot{x}_1 &= u_1 \cos\theta \\ \dot{x}_2 &= u_1 \sin\theta \\ \dot{\theta} &= 0 \end{cases}$$

(dynamical system for linear motion)

Rotational motion: turn the car around its center of mass with with some fixed angular velocity $u_2 = \dot{\theta}$

$$\begin{cases} \dot{x}_1 &= 0 \\ \dot{x}_2 &= 0 \\ \dot{\theta} &= u_2 \end{cases}$$

(dynamical system for rotational motion)

The car (cont.)

In vector form:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \theta \end{bmatrix}, \quad g_1(x) = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}, \quad g_2(x) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{aligned} \dot{x} &= u_1 g_1(x) && \text{(linear motion)} \\ \dot{x} &= u_2 g_2(x) && \text{(rotational motion)} \end{aligned}$$

Combining both kinds of motion in an admissible way:

$$\dot{x} = \underbrace{u_1 g_1(x) + u_2 g_2(x)}_{f(x,u)} \qquad \text{(affine control system)}$$

The control $u = (u_1, u_2)$ can take any value in $U \subset \mathbb{R}^2$

Linear control systems

Linear systems are particular cases of affine control systems.

$$\dot{x} = Ax + Bu, \qquad \text{(Linear)}$$

$x \in \mathbb{R}^n$, $u = \begin{pmatrix} u_1, \cdots, u_m \end{pmatrix}^\top \in \mathbb{R}^m$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{m \times n}$.

$$\dot{x} = \underbrace{Ax}_{g_0(x)} + \sum_{i=1}^{m} u_i \underbrace{Be_i}_{g_i(x)}, \qquad \text{(Affine)}$$

$M = \mathbb{R}^n$, $U = \mathbb{R}^m$.

Controllability of linear control systems

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \; u \in \mathbb{R}^m \qquad (L)$$

The linear control system (L) is controllable if and only if

$$\operatorname{rank} \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} = n.$$

Kalman rank condition for controllability

Linearization

Nonlinear system:

$$\dot{x} = g_0(x) + \sum_{i=1}^{m} u_i g_i(x), \qquad g_0(x_0) = 0.$$

Its linearization in a neighborhood of x_0 and $u = 0$:

$$\dot{z} = \underbrace{\frac{\partial g_0}{\partial x}(x_0)}_{A}\, z + \underbrace{\sum_{i=1}^{m} \tilde{u}_i g_i(x_0)}_{B\tilde{u}}, \quad z \in \mathbb{R}^n,\ \tilde{u} \in \mathbb{R}^m,$$

where $Be_i = g_i(x_0)$.

If the linearization is controllable,
then the nonlinear system is locally controllable.

Linearization of the model of a car

$$\dot{x} = u_1 g_1(x) + u_2 g_2(x) = u_1 \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} + u_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Linearization in a neighborhood of $(x_0, 0)$:

$$A = \frac{\partial g_0}{\partial x}(x_0) = 0, \quad B = \begin{bmatrix} \cos\theta_0 & 0 \\ \sin\theta_0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\operatorname{rank}\begin{bmatrix} B & AB & A^2B \end{bmatrix} = \operatorname{rank}[B] = 2 \neq 3.$$

The linearization is not controllable!

But the nonlinear system is (globally) controllable, in accordance to our experience.

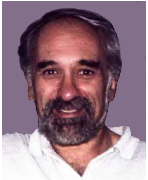
Linear control systems (cont.)

The theory of linear control systems is well developed.

Important contributions with a geometric flavor:


Brockett


Kailath


Sontag


Willems


Wonham

Some local questions of nonlinear systems may be answered using linear theory.

But one may loose much of the structure!

Control system / family of vector fields (cont.)

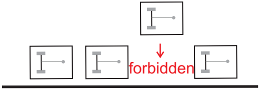
Back to the car model

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \theta \end{bmatrix} \in \mathbb{R}^2 \times S^1, \quad g_1(x) = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}, \quad g_2(x) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

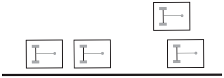
$$[g_1, g_2](x) = -\begin{bmatrix} 0 & 0 & -\sin\theta \\ 0 & 0 & \cos\theta \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sin\theta \\ -\cos\theta \\ 0 \end{bmatrix}.$$

- The vector field g_1 generates the forward/backward motion.
- The vector field g_2 generates the clockwise/counterclockwise rotation.
- The vector field $[g_1, g_2]$ generates the motion in the direction perpendicular to the orientation of the car.

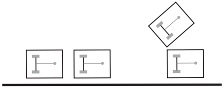
Typical maneuver in parking a car



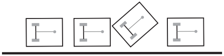
Four motions with the same amplitude perform forbidden motion:



1. motion forward



2. rotation counterclockwise



3. motion backward

4. rotation clockwise

Reachable sets

Characterize the states that can be reached from a given initial state $x_0 \in M$ in finite time, by choosing various controls and switching from one to another from time to time.

Reachable sets (cont.)

Characterize the states that can be reached from a given initial state $x_0 \in M$ in finite time, by choosing various controls and switching from one to another from time to time.

Reachable set from x_0 at time $t \geq 0$:

$$\mathcal{R}_{x_0}(t) = \{x_u(t) |\, u \in \mathcal{U}\}\,.$$

Reachable set from x_0 up to time t:

$$\mathcal{R}^t_{x_0} = \bigcup_{0 \leq \tau \leq t} \mathcal{R}_{x_0}(\tau).$$

Reachable set from x_0:

$$\mathcal{R}(x_0) = \bigcup_{0 \leq \tau < \infty} \mathcal{R}^{\tau}_{x_0}.$$

Reachable sets (cont.)

Example: $M = \mathbb{R}^3$

$$\dot{x} = u_1 \begin{bmatrix} x_2 \\ -x_1 \\ 0 \end{bmatrix} + u_2 \begin{bmatrix} 0 \\ x_3 \\ -x_2 \end{bmatrix} \quad \Leftrightarrow \quad \left\{ \begin{array}{rcl} \dot{x}_1 & = & u_1 x_2 \\ \dot{x}_2 & = & -u_1 x_1 + u_2 x_3 \\ \dot{x}_3 & = & -u_2 x_2 \end{array} \right.$$

Multiplying the first equation by x_1, the second by x_2, the third by x_3 and adding, we obtain

$$2x(t)^\top \dot{x}(t) = \frac{d}{dt}\left[x(t)^\top x(t)\right] = 0.$$

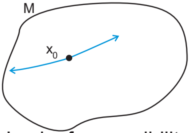
So, the solutions of the control systems must satisfy $x(t)^\top x(t) = x(0)^\top x(0)$. Consequently,

$$\mathcal{R}(x) \subseteq \{z \in \mathbb{R}^3 \,|\, z^\top z = x^\top x\}.$$

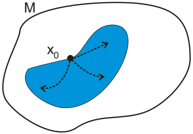
The reachable sets are constrained to lie in a submanifold of dimension ≤ 2.

Definitions of accessibility and controllability

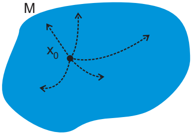
- The system has the accessibility property from a state $x \in M$ if $\mathcal{R}(x)$ has nonempty interior.
- The system has the accessibility property if it has the accessibility property from every state $x \in M$.
- The system is controllable from $x \in M$ if $\mathcal{R}(x) = M$.
- The system is controllable if $\mathcal{R}(x) = M, \ \forall x \in M$.



Lack of accessibility



Accessible
not controllable



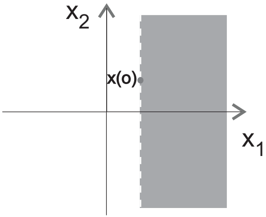
Controllable

Example

Control system on $\mathbb{R}^2$

$$\left\{ \begin{array}{lcl} \dot{x}_1 & = & x_2^2 \\ \dot{x}_2 & = & u \end{array} \right. \Leftrightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} x_2^2 \\ 0 \end{bmatrix}}_{g_0(x)} + u \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{g_1(x)}$$

$x_2^2 \geq 0$, so the x_1-coordinate is always non-decreasing.



Reachable set from $x(0)$

Accessibility property

Distributions and integrability

• A distribution Δ on the manifold M is a map which assigns to each point in M a subspace of the tangent space at this point:

$$M \ni x \mapsto \Delta(x) \subset T_x M.$$

$\dim(\Delta) = k$ if $\dim(\Delta(x)) = k$, $\forall x \in M$.

Example: $M = \mathbb{R}^3 \backslash \{0\}$, $\qquad \Delta(x) = \{v \in \mathbb{R}^3 \,|\, v^\top x = 0\}$

$\Delta(x)$ is the tangent space at x to the sphere centered at 0 passing through x.

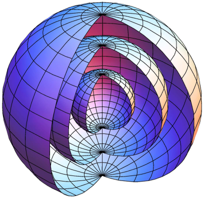
This distribution Δ has a special property: for every $x \in M$, there exists a smooth 2-dim submanifold N_x of M (the sphere centered at 0 passing through x) which is everywhere tangent to Δ.

This property is called integrability

• An integral manifold of a distribution Δ is a submanifold N of M satisfying

$$T_xN = \Delta(x), \quad \forall x \in N.$$

• A distribution Δ is integrable if, for every $x \in M$, there exists a (maximal) integral manifold, $N(x)$, of Δ through every point $x \in M$, or equivalently, there exists a (integral) foliation on M whose tangent bundle is Δ.



An integral foliation on $M = \mathbb{R}^3 \backslash \{0\}$

Involutive distribution

The integrability of a distribution depends on its involutivity.

- A distribution Δ on M is involutive if $\forall x \in M$

$$f(x), g(x) \in \Delta(x) \Rightarrow [f, g](x) \in \Delta(x).$$

Frobenius theorem

Suppose a distribution Δ has constant dimension. Then,

$$\Delta \text{ is integrable if and only if } \Delta \text{ is involutive.}$$

Control systems without drift

$$\dot{x} = \sum_{i=1}^{m} u_i g_i(x), \quad x \in M, \qquad \text{(unconstrained inputs).}$$

Control distribution

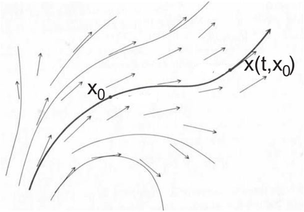
$$\Delta(x) = \mathrm{span}\{g_1(x), \cdots, g_m(x)\} \subset T_x M.$$

Control distribution (cont.)

Control distribution: $\Delta(x) = \mathrm{span}\{g_1(x), \cdots, g_m(x)\}$

Example: $M = \mathbb{R}^2$, $m = 1$, $g_1(x) \neq 0$, for all x.

The control distribution is 1-dimensional. Through each point $x_0 \in \mathbb{R}^2$ passes a curve $\gamma(x_0) = x(t, x_0)$ which is everywhere tangent to Δ. Δ is integrable.



What is the reachable set from x_0?

$$\boxed{\mathcal{R}(x_0) = \gamma(x_0) \neq \mathbb{R}^2}$$

The system is not controllable!

Control distribution (example)

$$\dot{x} = u_1 \begin{bmatrix} x_2 \\ -x_1 \\ 0 \end{bmatrix} + u_2 \begin{bmatrix} 0 \\ x_3 \\ -x_2 \end{bmatrix} + u_3 \begin{bmatrix} x_3 \\ 0 \\ -x_1 \end{bmatrix}, \quad x \in \mathbb{R}^3 \backslash \{0\}.$$

The control distribution Δ is 2-dimensional.

$$[g_1, g_2] = -g_3 \quad [g_2, g_3] = -g_1 \quad [g_3, g_1] = -g_2.$$

Δ is involutive, so Δ is integrable.

$$2x(t)^\top \dot{x}(t) = \frac{d}{dt}\left[x(t)^\top x(t)\right] = 0.$$

Consequently, the maximal integral manifold of Δ, at a given $x \in M$, is the sphere centered at the origin, passing through x.

The reachable set from $x \in M$ is contained in a 2-dimensional sphere. The system is not controllable.

Integrability of control distribution rules out controllability!?

Control distribution (example)

Back to the car model

$$g_1(x) = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}, \qquad g_2(x) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The control distribution Δ is 2-dimensional.

$$[g_1, g_2](x) = \begin{bmatrix} \sin\theta \\ -\cos\theta \\ 0 \end{bmatrix} \notin \Delta(x).$$

The control distribution is not involutive. So, Δ is not integrable.

The reachable sets are not restricted to 2-dimensional submanifolds.

Is the system controllable? (Our experience says yes!)

Bracket generating property

If the system

$$\dot{x} = \sum_{i=1}^{m} u_i g_i(x), \quad x \in M$$

is controllable, its control distribution

$$\Delta = \text{span}\{g_1, \cdots, g_m\}$$

should satisfy a property that is intuitively opposite to integrability.

The distribution $\Delta = \text{span}\{g_1, \cdots, g_m\}$ on M is said to be **bracket generating** if the iterated Lie brackets

$$g_i, [g_i, g_j], [g_i, [g_j, g_k]], \cdots, 1 \leq i, j, k \leq m,$$

span the tangent space of M at every point.

Rashevsky-Chow Theorem

In other words:

Bracket generating property

$\Delta = \mathrm{span}\{g_1, \cdots, g_m\}$ is bracket generating iff

$$\mathrm{Lie}_x(\mathcal{F}) = T_x M, \text{ for every } x \in M.$$

So, if M is connected and Δ is bracket generating, then

$$\mathcal{O}(x) = M, \text{ for every } x \in M.$$

Since, for drift free systems $\mathcal{O}(x) = \mathcal{R}(x)$, one gets:

Theorem (Rashevsky-Chow)

If M is connected and the control distribution Δ is bracket generating, then the (drift free) system is controllable.

Heisenberg system (Brockett integrator)

$M = \mathbb{R}^3$

$$\begin{array}{rcl} \dot{x}_1 & = & u_1 \\ \dot{x}_2 & = & u_2 \\ \dot{x}_3 & = & x_1 u_2 - x_2 u_1 \end{array} \qquad \Leftrightarrow \qquad \dot{x} = u_1 g_1(x) + u_2 g_2(x)$$

$$g_1 = \begin{bmatrix} 1 \\ 0 \\ -x_2 \end{bmatrix}, \quad g_2 = \begin{bmatrix} 0 \\ 1 \\ x_1 \end{bmatrix}, \quad g_3 = [g_1, g_2] = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}.$$

$$\text{span}\{g_1(x), g_2(x), g_3(x)\} = \mathbb{R}^3, \forall x.$$

So, the Heisenberg system is controllable.

Control systems with drift

$$\dot{x} = \underbrace{g_0(x)}_{\text{drift}} + \sum_{i=1}^{m} u_i g_i(x), \quad x \in M$$

The presence of drift significantly complicates the question of controllability!

- It is possible that the trajectories of a system can't be restricted to a lower dimensional sub-manifold, and yet the system is uncontrollable, as this example shows.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2^2 \\ 0 \end{bmatrix} + u \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad M = \mathbb{R}^2.$$

The reachable set from a point $z \in \mathbb{R}^2$ is:

$$\mathcal{R}(z) = \left\{ w \in \mathbb{R}^2 \middle| \, w_1 > z_1 \right\} \cup \{z\}.$$

Thus, the system is accessible but not controllable!

Control systems with drift (cont.)

One case of interest where accessibility implies controllability, is the linear control system evolving in $\mathbb{R}^n$:

$$\dot{x} = Ax + \sum_{i=1}^{m} u_i b_i = Ax + Bu, \qquad Be_i = b_i.$$

Since $\quad [Ax, b_i] = -Ab_i, \quad [Ax, [Ax, b_i]] = A^2 b_i, \quad \cdots$

$$\begin{aligned} &\mathrm{Lie}_x\{Ax, b_i,\ i = 1, \cdots, m\} \\ &\qquad = \mathrm{span}\{Ax, b_i, Ab_i, A^2 b_i, \cdots, A^{n-1} b_i,\ i = 1, \cdots, m\}. \end{aligned}$$

So, the bracket generating condition at $x = 0$ is:

$$\mathrm{rank} \begin{bmatrix} B & AB & A^2 B & \cdots & A^{n-1} B \end{bmatrix} = n.$$

In the linear case, accessibility is equivalent to controllability. But the drift vector field is involved in the condition!

- Lecture II -

Control systems on Lie groups

Control systems on Lie groups

G is a (matrix) Lie group. Right-invariant vector fields on G form a Lie algebra.

$\mathcal{L}(G) \equiv T_I G$ (the Lie algebra of G).

Consider the right-invariant control system evolving on G:

$$\dot{X} = \underbrace{A_0 X}_{g_0(X)} + \sum_{i=1}^{m} u_i \underbrace{A_i X}_{g_i(X)} \qquad (\Sigma)$$

$$X \in G, \quad A_0, A_1, \cdots, A_m \in \mathcal{L}(G), \quad m \leq dim(G).$$

$$(\Sigma) \qquad \equiv \qquad \mathcal{F} = \{A_0 + \sum_{i=1}^{m} u_i A_i\}_{u_i \in \mathcal{U}} \subset \mathcal{L}(G)$$

A control system on G is identified with a subset of $\mathcal{L}(G)$.

Control systems on Lie groups (cont.)

The flow of a vector field $f(X) = AX$ is defined through the matrix exponential:

$$\begin{array}{rccc} t \mapsto \exp tf: & G & \to & G \\ & X & \mapsto & e^{tA}X = \left(\displaystyle\sum_{k=0}^{+\infty} \frac{t^k}{k!}A^k\right)X. \end{array}$$

Orbit through a point $X \in G$:

$$\mathcal{O}(X) = \left\{ e^{t_k B_k} \cdots e^{t_1 B_1} X \mid k \in \mathbb{N},\ B_i \in \mathcal{F},\ t_i \in \mathbb{R} \right\}$$

Control systems on Lie groups (cont.)

$$\mathcal{O}(X) = \left\{ e^{t_k B_k} \cdots e^{t_1 B_1} X \,|\, k \in \mathbb{N},\ B_i \in \mathcal{F},\ t_i \in \mathbb{R} \right\}$$

Lemma (Properties of orbits)

(1) $\mathcal{O}(X) = \mathcal{O}(I)\,X$
(2) $\mathcal{O}(I)$ is a connected Lie subgroup of G with Lie algebra Lie$(\mathcal{F})$

Sketch of proof:

(1) Follows from the definition of orbit.
(2)

- Any point in the orbit $\mathcal{O}(I)$ is *connected* to the identity by a continuous curve.
- $\mathcal{O}(I)$ is a *subgroup* of G:
  - $I \in \mathcal{O}(I)$.
  - If $Y, Z \in \mathcal{O}(I)$, then $YZ \in \mathcal{O}(I)$ (as a product of exponentials).
  - If $Y = e^{t_k B_k} \cdots e^{t_1 B_1} \in \mathcal{O}(I)$, then $Y^{-1} = e^{-t_1 B_1} \cdots e^{-t_k B_k} \in \mathcal{O}(I)$.

Reachable sets

$$\mathcal{R}(X) = \left\{ e^{t_k B_k} \cdots e^{t_1 B_1} X \,|\, k \in \mathbb{N},\, B_i \in \mathcal{F},\, t_i \geq 0 \right\}$$

Properties of reachable sets

- $\mathcal{R}(X) \subset \mathcal{O}(X)$
- $\mathcal{R}(X) = \mathcal{R}(I)\, X$
- $\mathcal{R}(I)$ is a sub-semigroup of G.

Proposition

If the system is symmetric, i.e. $\mathcal{F} = -\mathcal{F}$, then $\mathcal{R}(I) = \mathcal{O}(I)$.

Proposition

A right-invariant system is controllable on G if and only if $\mathcal{R}(I) = G$.

Controllability on Lie groups

Most of the results about "Control systems on Lie groups" appeared in 1972, in a remarkable paper by Jurdjevic and Sussmann.

In the meanwhile, several refinements have been done by several authors. Fundamental literature include:
V. Jurdjevic, Geometric Control Theory (1997),
Y. Sachkov, Controllability on Lie groups (2004).

Controllability on Lie groups

Theorem

A right-invariant system is controllable in G if and only if

(1) G is connected

(2) $\text{Lie}(\mathcal{F}) = \mathcal{L}(G)$ (controllability rank condition)

(3) The reachable set $\mathcal{R}(I)$ is a subgroup of G.

Sketch of proof:

($\Rightarrow$) Necessary conditions for controllability are obvious!
($\Leftarrow$) If $\mathcal{R}(I)$ is a subgroup of G, then $e^{-t_iB_i} \in \mathcal{R}(I)$ whenever $e^{t_iB_i} \in \mathcal{R}(I)$. So $\mathcal{R}(I) = \mathcal{O}(I)$. By previous result $\mathcal{O}(I)$ is a connected Lie subgroup of G with Lie algebra $\text{Lie}(\mathcal{F}) = \mathcal{L}(G)$. Consequently $\mathcal{R}(I) = \mathcal{O}(I) = G$.

□

In practice, the crucial question is:

When is $\mathcal{R}(I)$ a subgroup?

Controllability on Lie groups (cont.)

Theorem

A right-invariant system is controllable in G if and only if

(1) G is connected

(2) $\mathrm{Lie}(\mathcal{F}) = \mathcal{L}(G)$ (controllability rank condition)

(3) The reachable set $\mathcal{R}(I)$ is a subgroup of G.

For systems without drift, (3) follows from (1)+(2). So,

For driftless systems, controllability is easy to check!

Controllability on Lie groups (cont.)

Assume G is connected. $H \subset G$ a connected subgroup whose Lie algebra is $\mathrm{Lie}(\mathcal{F})$.

Lemma

If $\mathcal{R}(I)$ in dense in H, then $\mathcal{R}(I) = H$.

Theorem (Closure test)

A right-invariant system is controllable in G if and only if the following holds:

1. $\mathrm{Lie}(\mathcal{F}) = \mathcal{L}(G)$;
2. $\mathrm{closure}(\mathcal{R}(I)) = G$.

Consequence of the previous theorem:

In the study of controllability of systems that satisfy $\mathrm{Lie}(\mathcal{F}) = \mathcal{L}(G)$, one can replace the reachable set by its closure!

Controllability on compact Lie groups

Lemma

If G is connected and compact and $A \in \mathcal{L}(G)$, then

$$\text{closure}\{e^{-tA},\ t \geq 0\} \subset \text{closure}\{e^{tA},\ t \geq 0\},$$

and, consequently, the orbits of $\mathcal{F}$ agree with the reachable sets.

Theorem

If G is connected and compact, a right-invariant systems is controllable in G if and only if the controllability rank condition $\text{Lie}(\mathcal{F}) = \mathcal{L}(G)$ holds.

For systems evolving on compact Lie groups, controllability is easy to check!

Next example illustrates that if G is connected but not compact, the controllability rank condition is not enough for controllability.

Controllability on Lie groups (cont.)

$G = SL(2), \mathcal{L}(G) = sl(2) = \{2 \times 2 \text{ real traceless matrices}\}$.

$$A_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad [A_0, A_1] = A_2 = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}.$$

$$\dot{X} = (A_0 + uA_1)X \Leftrightarrow \begin{cases} \dot{x}_{11} = x_{11} + ux_{21} \\ \dot{x}_{21} = -x_{21} + ux_{11} \\ \dot{x}_{12} = x_{12} + ux_{22} \\ \dot{x}_{22} = -x_{22} + ux_{12} \end{cases}$$

$\mathrm{Lie}\{A_0, A_1\} = sl(2)$, so the controllability rank condition holds.

$$x_{11}\dot{x}_{11} - x_{21}\dot{x}_{21} = x_{11}^2 + x_{21}^2.$$

$$\frac{1}{2}\frac{d}{dt}(x_{11}^2 - x_{21}^2) = x_{11}^2 + x_{21}^2 > 0 \Rightarrow (x_{11}^2 - x_{21}^2 \text{ is increasing}).$$

Which points $X \in G$ can be reached from $I \in G$ (in positive time)?

Only those for which $x_{11}^2 > x_{21}^2 + 1$. System not controllable!

Controllability on noncompact Lie groups

One of the consequences of the closure test is that, if the controllability rank condition holds, in the study of controllability one may replace the reachable set by its closure.

- Equivalent classes for families of vector fields

$$\mathcal{F}_1 \sim \mathcal{F}_2 \Leftrightarrow \text{closure}\{\mathcal{R}_{\mathcal{F}_1}\} = \text{closure}\{\mathcal{R}_{\mathcal{F}_2}\}$$

A canonical representative:

Lie saturate

Let $\mathcal{F}$ be the family of vector fields for a right-invariant control system. Lie saturate of $\mathcal{F}$, denoted $\mathcal{LS}(\mathcal{F})$, is the largest subset of $\text{Lie}\{\mathcal{F}\}$ which is equivalent to $\mathcal{F}$.

$$\boxed{\mathcal{LS}(\mathcal{F}) = \{A \in \text{Lie}\{\mathcal{F}\} : \ e^{\mathbb{R}^+ A} \subset \text{closure}\{\mathcal{R}_{\mathcal{F}}\}\}.}$$

Controllability on noncompact Lie groups (cont.)

Theorem (Lie saturate test)

A right-invariant system is controllable in G if and only if $\mathcal{LS}(\mathcal{F}) = \mathcal{L}(G)$.

How to compute $\mathcal{LS}(\mathcal{F})$? Usually, it is difficult!

The controllability problem on noncompact Lie groups is much harder then on compact Lie groups!

Controllability on Lie groups

- If G is semisimple and $\dot{X} = (A + uB)X$, most known results make substantial use of Lie algebraic structure of $\mathcal{L}(G)$ (root-system, root-space decomposition, ...).

  Jurdjevic + Kupka (1981), Crouch + SL (1988), El Assoudi + Gauthier + Kupka (1995), Sachkov (2000), ...

- Controllability issues for semisimple Lie groups have also been tackled using a more geometric approach, based on decompositions of Lie groups (into products of one-parameter subgroups).

  SL (1983), Crouch + SL (1991).

Examples (our car - the unicycle)

This vehicle moves on the plane. A description of its position and orientation requires 3 coordinates: (x, y) for the position of the center of mass, θ for the orientation of the vehicle.

$X(t) \in SE(2)$ describes simultaneously position (of center of mass) and orientation of the unicycle at the instant of time t.

$$X(t) = \begin{bmatrix} \cos\theta(t) & \sin\theta(t) & 0 \\ -\sin\theta(t) & \cos\theta(t) & 0 \\ x(t) & y(t) & 1 \end{bmatrix}$$

There are two degrees of freedom (playing the role of the controls):

$$u_1 = \sqrt{\dot{x}^2 + \dot{y}^2}, \quad u_2 = \dot{\theta}$$

The unicycle (cont.)

$$\dot{X}(t) = (u_1(t)A_1 + u_2(t)A_2)X(t), \quad X \in SE(2)$$

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$A_1, A_2 \in se(2) = \left\{ \begin{bmatrix} 0 & a & 0 \\ -a & 0 & 0 \\ b & c & 0 \end{bmatrix}, \quad a, b, c \in \mathbb{R} \right\}$$

$$[A_1, A_2] = A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \text{ and } \operatorname{span}\{A_1, A_2, A_3\} = se(2).$$

So, $\mathrm{Lie}\{A_1, A_2\} = se(2)$.

Finally, we have proved what we already new empirically!

Our car is controllable!

Examples (the satellite)

- $\{C, i_1, i_2, i_3\}$ is a fixed orthonormal frame with origin at the center of mass C of the satellite.
- $\{C, r_1, r_2, r_3\}$ is an orthonormal frame fixed in the satellite with origin at its center of mass.
- The orientation (angular position) of the satellite is described by a rotation matrix X that transforms $\{C, i_1, i_2, i_3\}$ into $\{C, r_1, r_2, r_3\}$, i.e. $Xi_j = r_j$, $j = 1, 2, 3$.

Evolution of X:

$$\dot{X}(t) = \begin{bmatrix} 0 & -w_3(t) & w_2(t) \\ w_3(t) & 0 & -w_1(t) \\ -w_2(t) & w_1(t) & 0 \end{bmatrix} X(t),$$

where w_j, $j = 1, 2, 3$, are the components of the angular velocity of the satellite (with respect to the frame $\{C, r_1, r_2, r_3\}$).

The satellite (cont.)

Assuming it is possible to control directly the angular velocities (for instance, by applying 3 opposing pairs of gas jets), the equation describing the attitude of the satellite is a right-invariant control system:

$$\dot{X}(t) = \sum_{i=1}^{3} u_i A_j X(t), \quad X \in SO(3),$$

where

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathrm{Span}\{A_1, A_2, A_3\} = so(3).$$

The system is controllable!

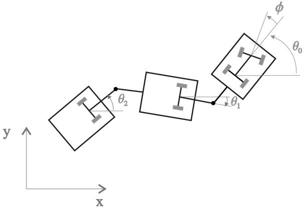
The satellite (cont.)

$$\text{Lie}\{A_i, A_j\} = so(3), \quad \text{for } 1 \leq i \neq j \leq 3.$$

- If one loses control in one direction and are left with two controls only, the attitude of the satellite is still controllable.
- Euler's theorem for rotations ensures that if the two control vector fields are orthogonal, one can go from one configuration to another by rotating around 2 orthogonal axis in, at most 2, switchings.
- If the 2 control vector fields are not orthogonal, the number of switchings depends on the angle between them, increasing as the angle decreases.

- Sometimes, having a minimal number of controls is good!
- Sometimes, minimizing the number of switchings (changes in direction) is better!

A car pulling n trailers



A control system evolving on the special unipotent group $SUP(n+4)$, consisting of all matrices of the form $(k = n+4)$:

$$X = \begin{bmatrix} 1 & & & & & \\ x_2 & 1 & & & & \\ x_3 & x_1 & 1 & & & \\ x_4 & \frac{{x_1}^2}{2} & x_1 & 1 & & \\ \vdots & \vdots & & \ddots & \ddots & \\ x_k & \frac{{x_1}^{k-2}}{(k-2)!} & \cdots & \frac{{x_1}^2}{2} & x_1 & 1 \end{bmatrix}, \qquad x_1, \cdots x_k \in \mathbb{R}.$$

A car pulling n trailers (cont.)

Its Lie algebra $sup(n+4)$ consists of special nilpotent matrices of the form:

$$A = \begin{bmatrix} 0 & & & \cdots & & 0 \\ a_2 & 0 & & & & \\ a_3 & a_1 & 0 & & & \vdots \\ a_4 & 0 & a_1 & & & \\ \vdots & \vdots & \ddots & \ddots & \ddots & \\ a_k & 0 & \cdots & 0 & a_1 & 0 \end{bmatrix}, a_1, \cdots, a_k \in \mathbb{R}$$

A car pulling n trailers (cont.)

This vehicle also moves on the plane.

Its behavior may be modeled by the following nonlinear system evolving in $SUP(n+4)$, with 2 controls only.

$$\dot{X}(t) = (u_1 A_1 + u_2 A_2)X(t), \qquad \text{(no drift)}$$

$$A_1 = E_{32} + E_{43} + \cdots + E_{k,k-1} \quad \text{and} \quad A_2 = E_{21}$$

$\dim SUP(n+4) = n+4$. So, underactuated system.

$\mathrm{Lie}\{A_1, A_2\} = sup(n+4)$.

The system is controllable, no matter how many trailers!

Controllability with just one control!

Theorem (Kuranishi 1951)

Any semisimple Lie algebra can be generated by a pair of its elements.

- For instance, for $G = SO(n)$ (connected, compact, semisimple), no matter how big the dimension is, there always exists a pair (A, B) such that $\text{Lie}\{A, B\} = \mathcal{L} = so(n)$ and, consequently, the control system

$$\dot{X} = (A + uB)X,$$

is always controllable, even though we have control on one direction only!

One possible pair of generators for $so(n)$ is:

$$A = A_{12} + A_{23} + \cdots + A_{n-2,n-1}, \quad B = A_{n-1,n}.$$

The Marriage problem

It is known that the set of pairs (A, B) such that $\text{Lie}\{A, B\} = \mathcal{L}$ is open and dense in $\mathcal{L} \times \mathcal{L}$.

Given $A \in \mathcal{L}$ semisimple, find $B \in \mathcal{L}$ so that $\text{Lie}\{A, B\} = \mathcal{L}$ and, consequently, the following systems are controllable:

$$\begin{array}{lll} \dot{X} = (A + uB)X, \quad \dot{X} = (uA + vB)X, & & G \text{ compact} \\ \dot{X} = (uA + vB)X, & & G \text{ non compact.} \end{array}$$

Known results for the marriage problem use Lie algebraic tools.

Controllability on semi-direct products of Lie groups

Controllability properties for compact Lie groups carry over to semi-direct products:

$G = V \ltimes K$, where V is a vector space and K is a compact Lie group acting on V, so that $Kx = x$ only for $V \ni x = 0$, the following holds.

Theorem

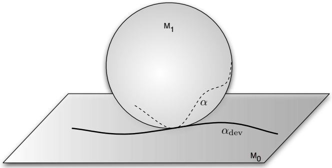
A right-invariant control system is controllable in $G = V \ltimes K$ if and only if $\mathrm{Lie}\{\mathcal{F}\} = \mathcal{L}(G)$.

Example - The group of orientation-preserving rigid motions in $\mathbb{R}^n$:

$$SE(n) = \mathbb{R}^n \ltimes SO(n).$$

The rolling sphere

The rolling sphere consists of a sphere in 3-space, rolling without slip or twist over the tangent space at a point.



This rigid motion is described by the action of $SE(3)$, but has 2 types of constraints:

- Holonomic constraints (sphere keeps tangent to the plane during motion)
- Nonholonomic constraints (sphere can't twist or slip)
  - No twist (performing spins not allowed!)
  - No slip (performing slidding not allowed!)

Kinematic equations for the rolling sphere

The motion of S^2, when rolling over the south tangent plane is described by the following right-invariant control system evolving on $SE(3) = \mathbb{R}^3 \ltimes SO(3)$:

$$\dot{s} = \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix} \qquad \text{(translational velocity)}$$

$$\dot{R} = \begin{bmatrix} 0 & 0 & u_1 \\ 0 & 0 & u_2 \\ -u_1 & -u_2 & 0 \end{bmatrix} R \qquad \text{(rotational velocity)}$$

This system is controllable.

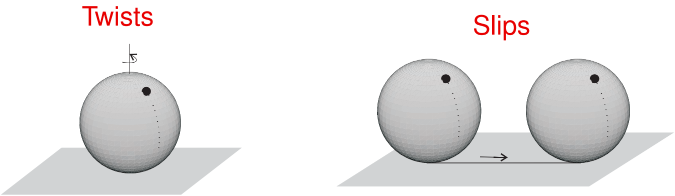
The rolling sphere is a complete nonholonomic system.

Forbidden motions

Question:

How to steer the rolling sphere from one initial configuration to any other admissible configuration, without violating the nonholonomic constraints (i.e, avoiding forbidden motions)?

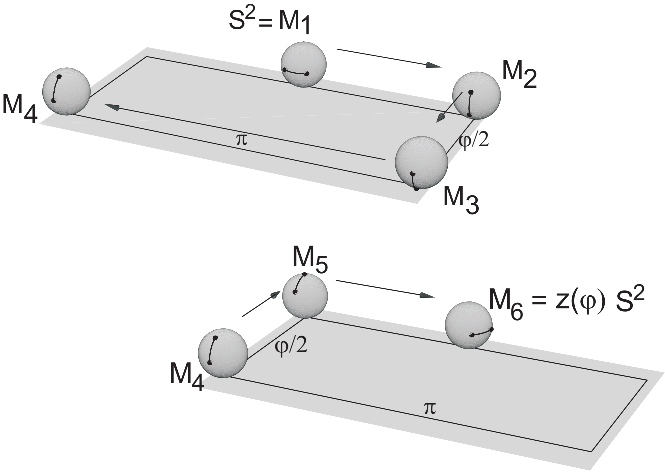
Forbidden motions



Answer: **Realizing the forbidden motions by rolling the sphere without slip or twist!**

Realizing a twist

Realizing a rotation of an angle φ around the z-axis:



Realizing a twist

A twist is a rotation around the z-axis.

$$z(\varphi) = e^{-\varphi A_{12}} = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Euler's theorem guarantees that any rotation around the z-axis decomposes as rotations around the x-axis and the y-axis.

But to perform a twist, such decomposition has to be carefully chosen, so that the angles of rotation around these 2 orthogonal axis add up to zero.

Decomposition corresponding to the previous picture:

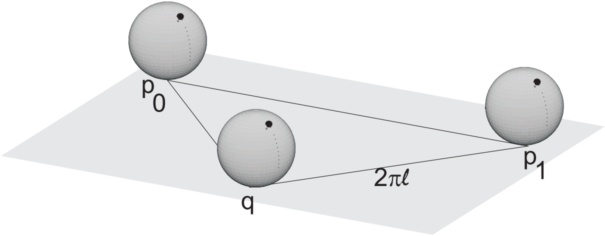
$$z(\varphi) = x(\frac{\pi}{2})\ y(\frac{\varphi}{2})\ x(-\pi)\ y(-\frac{\varphi}{2})\ x(\frac{\pi}{2})$$

Realizing a slip

A slip is a pure translation.

- $\mathrm{d}(p_0, p_1) =$ multiple of 2π
  Roll along the segment $\overline{p_0 p_1}$.
- $\mathrm{d}(p_0, p_1) \neq$ multiple of 2π
  Roll along the sides of the isosceles triangle in the picture.

l is the smallest integer satisfying $2\pi l > \mathrm{d}(p_0, p_1)$.



4 Representation of trajectories

Representation of system trajectories

After considering the general problem of controllability, other questions arise.

Constructing trajectories

How to construct specific trajectories?

We will describe two different ways to construct trajectories, using two types of coordinates:

- Coordinates of first kind (Magnus representation);
- Coordinates of second kind (Wei-Norman representation).

Single exponential representation (Magnus, 1954)

Let $X \in G \subset GL(n), \quad A_0, A_1, \cdots, A_m \in \mathcal{L}(G), \quad m \leq dim(G)$.

If X is the solution of the right-invariant control system

$$\dot{X} = \underbrace{\sum_{i=1}^{m} u_i(t) A_i}_{A(t)} X, \quad X(0) = I$$

then, $\exists T > 0$ s.t. for $|t| < T$,

$$X(t) = e^{W(t)},$$

where $W(t) \in \mathcal{L}(G)$ satisfies $W(0) = 0$ and

$$\dot{W} = A(t) - \tfrac{1}{2}[W, A(t)] + \textstyle\sum_{j=1}^{+\infty} \frac{\beta_{2j}}{(2j)!} ad_W^{2j} A(t) \text{ (Magnus equation)}$$

($\beta_{2j}, \ j = 1, 2, \cdots$ are Bernoulli numbers).

Magnus representation (cont.)

Using series expansion, one may find an expression for $W(t)$:

Define $\widetilde{A}(t) := \int_0^t A(\tau)\, d\tau$. Then

$$\begin{aligned} W(t) &= \widetilde{A}(t) + \frac{1}{2}\int_0^t [\dot{\widetilde{A}}(\tau_1), \widetilde{A}(\tau_1)]\, d\tau_1 \\ &+ \frac{1}{12}\int_0^t [[\dot{\widetilde{A}}(\tau_1), \widetilde{A}(\tau_1)], \widetilde{A}(\tau_1)]]\, d\tau_1 \\ &+ \frac{1}{4}\int_0^t \int_0^{\tau_1} [[\dot{\widetilde{A}}(\tau_1), [\dot{\widetilde{A}}(\tau_2), \widetilde{A}(\tau_2)]]\, d\tau_2 d\tau_1 \\ &+ \text{(terms involving 3 brackets)} \end{aligned}$$

(Convergence properties of this series and more details in: Nonholonomic Mechanics and Control, A. Bloch, 2003)

Product of exponentials represent. (Wei, Norman, 1964)

If X is the solution of the right-invariant control system

$$\dot{X} = \underbrace{\sum_{i=1}^{m} u_i(t)A_i}_{A(t)} X, \quad X(0) = I$$

then, $\exists T > 0$ s.t. for $|t| < T$,

$$X(t) = e^{h_1(t)A_1} \cdots e^{h_d(t)A_d},$$

where $\{A_1, \cdots, A_m, A_{m+1}, \cdots, A_d,\}$ is a basis of $\mathcal{L}(G)$ and $h_i, i = 1, \cdots, d$, are solutions of a differential equation depending only on the Lie algebra $\mathcal{L}(G)$ and controls $u_1, \cdots u_m$.

If the Lie algebra is solvable, this representation is global.

$\mathcal{L}$ is solvable if $\mathcal{L}^{(k)} = \{0\}$, for some $k \in \mathbb{N}$, where $\mathcal{L}^{(0)} = \mathcal{L}$ and $\mathcal{L}^{(j+1)} = [\mathcal{L}^{(j)}, \mathcal{L}^{(j)}],\ j \geq 0$.

Product of exponentials represent. (Wei, Norman, 1964)

Sketch of proof:

Compute the derivative of $X(t) = e^{h_1(t)A_1} \cdots e^{h_d(t)A_d}$, and use the Campbell-Hausdorf formula $\boxed{e^A B e^{-A} = e^{ad_A} B, \quad A, B \in \mathcal{L}}$, to obtain

$$\begin{aligned} \dot{X} &= \Big(\dot{h}_1 A_1 + \dot{h}_2 e^{h_1 ad_{A_1}} A_2 + \cdots \\ &\quad + \dot{h}_d e^{h_1 ad_{A_1}} \cdots e^{h_{d-1} ad_{A_{d-1}}} A_d \Big) X \\ &= \Big(\sum_{i=1}^{d} a_i(t) A_i \Big) X. \end{aligned}$$

Compare with the given control system $\dot{X} = \sum_{i=1}^{m} u_i(t) A_i X$.

One obtains a differential equation for the functions h_i involving the control functions and the structure constants of $\mathcal{L}$ only.

Wei-Norman representation for the Unicycle

$$\dot{X} = (u_1 A_1 + u_2 A_2)X, \quad (X \in SE(2),\, A_1 = A_{12},\, A_2 = E_{31}).$$

$$A_3 = [A_1, A_2] = E_{32},\, [A_1, A_3] = -A_2,\, [A_2, A_3] = 0.$$

$\mathrm{span}\{A_1, A_2, A_3\} = se(2)$.

$$X(t) = e^{h_1(t)A_1} e^{h_2(t)A_2} e^{h_3(t)A_3}$$

$$\begin{aligned} \dot{X}X^{-1} &= \dot{h}_1 A_1 + \dot{h}_2 e^{h_1 ad_{A_1}} A_2 + \dot{h}_3 e^{h_1 ad_{A_1}} e^{h_2 ad_{A_2}} A_3 \\ &= A_1 \dot{h}_1 + A_2(\dot{h}_2 \cos h_1 + \dot{h}_3 \sin h_1) \\ &\quad + A_3(-\dot{h}_2 \sin h_1 + \dot{h}_3 \cos h_1) \end{aligned} .$$

Comparing with the system above:

Wei-Norman representation for the Unicycle (cont.)

$$\begin{cases} \dot{h}_1 = u_1 \\ \dot{h}_2 \cos h_1 + \dot{h}_3 \sin h_1 = u_2 \\ -\dot{h}_2 \sin h_1 + \dot{h}_3 \cos h_1 = 0 \end{cases} \Leftrightarrow A(h_1) \begin{bmatrix} \dot{h}_1 \\ \dot{h}_2 \\ \dot{h}_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix},$$

where

$$A(h_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos h_1 & \sin h_1 \\ 0 & -\sin h_1 & \cos h_1 \end{bmatrix} \qquad \text{is invertible.}$$

So, the functions h_i satisfy:

$$\begin{cases} \dot{h}_1 = u_1 \\ \dot{h}_2 = u_2 \cos h_1 , \\ \dot{h}_3 = u_2 \sin h_1 \end{cases} \qquad h_i(0) = 0$$

These equations are integrable by quadratures (the solution can be found by algebraic operations and calculating integrals of functions on one variable).

Note that $se(2)$ is solvable ($\mathcal{L}^{(2)} = \{0\}$).

Wei-Norman equations for the Satellite

Proceeding as before, we arrive at:

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 & \sin h_2 & \sin h_2 \\ 0 & \cos h_1 & -\sin h_1 \cos h_2 \\ 0 & \sin h_1 & \cos h_1 \cos h_2 \end{bmatrix} \begin{bmatrix} \dot{h}_1 \\ \dot{h}_2 \\ \dot{h}_3 \end{bmatrix}, \qquad h_i(0) = 0$$

This representation is not global! ($\cos h_2 \neq 0$).

Magnus equation for the Unicycle

$$\begin{cases} \dot{w}_1 = u_1 \\ \dot{w}_2 = u_2 + \frac{1}{2} w_3 u_1 + \frac{1}{12}\left(w_1^2 u_2 - w_1 w_2 u_1\right) \\ \dot{w}_3 = \frac{1}{2}\left(w_2 u_1 - w_1 u_2\right) + \frac{1}{12} w_1 w_3 u_1 \end{cases} \qquad w_i(0) = 0$$

Wei-Norman equations for the car pulling N trailers

$(n = N + 4)$

$$\begin{cases} \dot{h}_1 &= u_1 \\ \dot{h}_2 &= u_2 \\ \dot{h}_3 &= h_2 u_1 \\ & \vdots \\ \dot{h}_n &= h_{n-1} u_1 \end{cases} \qquad h_i(0) = 0$$

These equations may be solved by quadratures:

$$\begin{aligned} h_1(t) &= \int_0^t u_1(\tau_1) d\tau_1, \\ h_2(t) &= \int_0^t u_2(\tau_2) d\tau_2, \\ h_j(t) &= \int_0^t u_1(\tau_1) \left[\int_0^{\tau_1} u_1(\tau_2) \cdots \right. \\ & \qquad \left. \int_0^{\tau_{j-3}} u_1(\tau_{j-2}) \int_0^{\tau_{j-2}} u_2(\tau_{j-1}) d\tau_{j-1} d\tau_{j-2} \cdots d\tau_2 \right] d\tau_1, \end{aligned}$$

$j = 3, 4, \cdots, n.$

Reduction to the "simple" case

Assume that $X \in G \subset GL(n)$, $Lie\{A_1, \cdots, A_m\} = \mathcal{L} = \mathcal{L}(G)$.

We want to solve the following:

$$\begin{cases} \dot{X} = \sum_{i=1}^{m} u_i(t) A_i X \\ \\ X(0) = I \end{cases} \qquad (*)$$

One can show that solving $(*)$ reduces to solving (at least locally) a similar problem when $\mathcal{L}$ is a simple Lie algebra (non Abelian and only trivial ideals).

This is, in part, a consequence of Levi decomposition

$$\boxed{\begin{array}{c} \mathcal{L} = \mathcal{R} \oplus \mathcal{S} \\ \\ \text{where } \mathcal{R} \text{ is the radical of } \mathcal{L} \text{ (solvable ideal)} \\ \text{and } \mathcal{S} \text{ is a semisimple Lie subalgebra of } \mathcal{L}, \end{array}}$$

Reduction to the "simple" case (cont.)

Using the Levi decomposition, $A(t) = A_1(t) + A_2(t)$. Simple computations show that If X_2 and X_1 are respectively solutions of

$$\begin{cases} \dot{X}_2 = A_2(t)\, X_2 \\ X_2(0) = I \end{cases} \qquad \begin{cases} \dot{X}_1 = (X_2^{-1} A_1(t) X_2)\, X_1 \\ X_1(0) = I \end{cases},$$

then $X = X_2 X_1$ is the solution of

$$\begin{cases} \dot{X} = A(t)\, X \\ X(0) = I \end{cases}$$

Reduction to the "simple" case (cont.)

Now, the exponential mapping is a local diffeomorphism. So, in a neighborhood of $t = 0$, $X_2(t) = e^{B_2(t)}$, with $B_2(t) \in \mathcal{S}$. Also, $A_1(t) \in \mathcal{R}$. As a consequence

$$X_2^{-1} A_1(t) X_2 = e^{-ad_{B_2}} A_1(t) \in \mathcal{R}.$$

So,

$$\dot{X}_2 = A_2(t)\, X_2 \qquad \text{(evolves on a semisimple Lie group)}$$

$$\dot{X}_1 = (X_2^{-1} A_1(t) X_2)\, X_1 \qquad \text{(evolves on a solvable Lie group).}$$

Using a convenient change of coordinates, the last system may be put in triangular form (Lie theorem) and then integrated by quadratures.

The first system may still be split into k subsystems evolving on simple Lie groups.

Reduction to the "simple" case (cont.)

This is due to the decomposition of a semisimple Lie algebra:

$$\mathcal{S} = \mathcal{S}_1 \oplus \cdots \oplus \mathcal{S}_k$$

where the $\mathcal{S}_i$'s are simple ideals of $\mathcal{S}$ which commute pairwise.

So,

$$\mathcal{S} \ni A_2 = A_{2,1} + \cdots + A_{2,k}, \quad \text{with} \quad [A_{2,i}, A_{2,j}] = 0.$$

Simple computations show that if $X_{2,i}$ is the solution of

$$\begin{cases} \dot{X}_{2,i} = A_{2,i}(t)\, X_{2,i} & (\text{ evolving on a simple Lie group}) \\ X_{2,i}(0) = I \end{cases}$$

then $X_2 = \prod_{i=1}^{k} X_{2,i}$ is the solution of

$$\begin{cases} \dot{X}_2 = A_2(t)\, X_2 \\ X_2(0) = I \end{cases}$$

■

- Lecture III -

Optimal Control

Calculus of variations / optimal control

Calculus of variations deals with optimization problems in a function space

$$J(x) = \int_a^b L(t, x(t), \dot{x}(t))\, dt \to \min_x$$

subject to:

$$x(a) = x_0, \quad x(b) = x_1.$$

In contrast, optimal control problems involve minimization over a set of curves which is determined by some dynamical constraints:

$$\dot{x}(t) = f(x(t), u(t)),$$

for some choice of the control function $t \mapsto u(t)$.

Optimal control problems

$$J(u) = \int_0^{t_1} \varphi(x(t), u(t))\, dt \to \min \quad \text{(cost functional)}$$

subject to:

$$\dot{x} = f(x, u), \quad x \in M, \quad u \in \mathcal{U} \quad \text{(a control system)}$$

$$x(0) = x_0, \quad x(t_1) = x_1 \quad \text{(boundary conditions)}$$

φ is called the cost function. It may be thought of a Lagrangian, or a metric, or some optimality criterion.

- Here, we also have to assume some integrability conditions for the cost function.
- Admissible controls assumed to be measurable locally bounded.

Geodesic problem on a Riemannian manifold $M, < .,. >$

$$\int_0^{t_1} \sum_{i=1}^m {u_i}^2(t)\, dt \;\rightarrow\; \min_u$$

subject to

$$\dot{x} = \sum_{i=1}^m u_i g_i(x),$$

$$x(0) = x_0, \quad x(t_1) = x_1.$$

If $m < n = \dim M$, this is the Sub-Riemannian geodesic problem.

Assume that M is parallelizable, and

$$< g_i(x), g_j(x) >= \delta_{ij},\ \forall x \in M,\ 1 \leq i, j \leq n.$$

This optimal control problem may be posed as a Lagrange problem on M.

Lagrangian formulation

$$\int_0^{t_1} < \dot{x}, \dot{x} > \, dt \;\to\; \min_x$$

subject to

$$\dot{x} = \sum_{i=1}^m u_i g_i(x),$$

$$x(0) = x_0, \quad x(t_1) = x_1 .$$

If $m < n =$, the constraint on velocities is simply

$$< g_k, \dot{x} >= 0, \quad m < k \leq n.$$

The control system is non-holonomic (under-actuated).

Lagrangian formulation

This Lagrange problem may be solved using a variational approach.

Solutions of the optimal control problem may be obtained by a Hamiltonian approach.

The foundation of the optimal control theory was formulated in Hamiltonian form around middle 20th century (Pontryagin ...). Hamiltonian systems play a very important role in optimal control.

Optimal control problems (cont.)

$$J(u) = \int_0^{t_1} \varphi(x(t), u(t))\, dt \to \text{min} \quad \text{(cost functional)}$$

subject to:

$$\dot{x} = f(x, u), \quad x \in M, \quad u \in \mathcal{U} \quad \text{(a control system)}$$

$$x(0) = x_0, \quad x(t_1) = x_1 \quad \text{(boundary conditions)}$$

A solution $\tilde{u}$ is called optimal control.
The corresp. solution $\tilde{x}$ of $\dot{x} = f(x, \tilde{u})$ is an optimal trajectory.

The study of OCP basically reduces to the study of reachable sets of an extended control system.

Optimal control problems and reachable sets

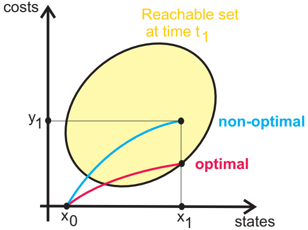
$$\widehat{M} = \mathbb{R} \times M = \{\hat{x} = (y, x) : \ y \in \mathbb{R}, x \in M\} \quad \text{(extended manifold).}$$

$$\dot{\hat{x}} = \hat{f}(\hat{x}, u) \Leftrightarrow \begin{cases} \dot{y} = \varphi(x, u) \\ \dot{x} = f(x, u) \end{cases} \qquad \text{(extended system)}$$

Proposition

If $\tilde{u}$ is an optimal solution for the OCP, the corresp. trajectory $\hat{x}_{\tilde{u}}$ of the extended system comes to the boundary of the reachable set $\widehat{\mathcal{R}}_{(0,x_0)}(t_1)$.

Proof:



Optimal control problems and reachable sets

To obtain optimal trajectories for the optimal control problem (or their lifts to the extended system):

1. Find trajectories of the extended control system that come to $\partial\widehat{\mathcal{R}}_{(0,x_0)}(t_1)$
2. Select optimal among them.

The first step is the most important and the most difficult!

Existence of optimal solutions

Existence of optimal solutions of the optimal control problem reduces to compactness of reachable sets.

Theorem (Filippov)

Assume

- space of control parameters is compact,
- the vector fields are complete,
- $\{f_u(x)\} \subset T_xM$ is convex.

Then, the reachable set $\widehat{\mathcal{R}}_{x_0}(t_1)$ is compact, $\forall x_0 \in M,\ t > 0$.

Hamiltonian mechanics

▷ Hamiltonian measures the total energy of a system (in which energy changes over time)

▷ Hamiltonian equations give the time evolution of the system.

- $H \in \mathcal{C}^\infty(T^*M)$ Hamiltonian
- $\overrightarrow{H} \in \mathrm{Vec}(T^*M)$ Hamiltonian vector field ($dH = \omega(\overrightarrow{H}, .)$)
- $\dot{\lambda} = \overrightarrow{H}(\lambda),\ \lambda \in T^*M$ Hamiltonian system

In canonical coordinates $\lambda = (p, x) \in T^*M$, the Hamiltonian system has the form

$$\begin{cases} \dot{p} = -\frac{\partial H}{\partial x} \\ \dot{x} = +\frac{\partial H}{\partial p} \end{cases} \qquad \text{(Hamiltonian equations on } T^*M\text{)}$$

Optimal control problems (cont.)

$$J(u) = \int_0^{t_1} \varphi(x(t), u(t))\, dt \to \text{min} \quad \text{(cost functional)}$$

subject to:

$$\dot{x} = f(x, u), \quad x \in M, \quad u \in \mathcal{U} \qquad \text{(a control system)}$$

$$x(0) = x_0, \quad x(t_1) = x_1 \qquad \text{(boundary conditions)}$$

Solutions of this optimal control problem are described by trajectories of a family of Hamiltonian systems, intrinsically associated to the problem by a procedure that is a geometric elaboration of the Lagrange multipliers rule.

Pontryagin maximum principle

Let $\lambda \in T^*M$, ν a real parameter, u a control parameter.

Consider the following family of Hamiltonians:

$$H_u^\nu(\lambda) = <\lambda, f(x,u)> + \nu\varphi(x,u).$$

Pontryagin maximum principle

If $\tilde{u}$ is an optimal control for the problem above, then there exists a curve $\lambda_t \in T^*_{\tilde{x}(t)}M$, $t \in [0, t_1]$ and a number $\nu \in \mathbb{R}$ such that:

$$\dot{\lambda}_t = \overrightarrow{H^\nu_{\tilde{u}(t)}}(\lambda_t), \quad \text{(Hamiltonian system)}$$

$$H^\nu_{\tilde{u}(t)}(\lambda_t) = \max_{u\in\mathcal{U}} H^\nu_u(\lambda_t), \quad \text{(maximality condition)}$$

$$(\lambda_t, \nu) \not\equiv (0,0), \quad t \in [0, t_1],$$

$$\nu \leq 0.$$

PMP (cont.)

- If $u(t)$ is an admissible control and λ_t satisfies the conditions in PMP, the curve λ_t is called an extremal and its projection on M (horizontal projection) is an extremal trajectory.

- The PMP gives necessary conditions for extremals.

- If a pair $(\tilde{u}(t), \lambda_t)$ satisfies the PMP, one has

$$\boxed{H^{\nu}_{\tilde{u}(t)}(\lambda_t) = \text{constant}, \quad t \in [0, t_1].}$$

PMP (cont.)

In problems where terminal time t_1 is a variable (free time), PMP has an extra condition

$$H^{\nu}_{\tilde{u}(t)}(\lambda_t) \equiv 0$$

For the proof of the PMP on manifolds:

"Geometric Control Theory, Agrachev + Sachkov, 2004"

For the proof of the PMP on Lie groups:

$\rightarrow$ "Geometric Control Theory, Jurdjevic, 1997"

Normal and abnormal extremals

$$H_u^{\nu}(\lambda) =< \lambda, f(x,u) > +\nu\,\varphi(x,u).$$

- $\nu = 0$

  The extremals don't depend on the cost function (depend on the constraints only) and are called abnormal.

  (Interesting history and details in "Abnormal minimizers, Montgomery, 1994")

- $\nu < 0$

  The extremals are called normal.

Optimal control problems (cont.)

In terms of canonical coordinates, the solutions of the optimal control problem have to satisfy:

$$\begin{cases} \dot{p} = -\frac{\partial H}{\partial x}(p, x, u) \\ \dot{x} = +\frac{\partial H}{\partial p}(p, x, u) \end{cases} \qquad \text{(Hamiltonian equations on } T^*M\text{)}$$

$+$ one extra equation

$$\frac{\partial H}{\partial u}(p, x, u) = 0 \qquad \text{(Maximality condition)}$$

If the last equation can be solved for u, replacing above gives classical hamiltonian equations.

Optimal control on Lie groups

The cotangent bundle T^*G of a Lie group G has a natural trivialization : $T^*G \cong \mathcal{L}^* \times G = \{(a, g) \mid a \in \mathcal{L}^*, g \in G\}$.

For a right-invariant optimal control problem on a Lie group G, the Hamiltonian $H = H(a)$ doesn't depend on $g \in G$. Denoting $dH = \frac{\partial H}{\partial a} \in (\mathcal{L}^*)^* = \mathcal{L}$.

The Hamiltonian system on $\mathcal{L}^* \times G$ takes the form:

$$\begin{cases} \dot{a} = \left(\operatorname{ad} \dfrac{\partial H}{\partial a}\right)^* a, \quad a \in \mathcal{L}^* & \text{(vertical subsystem)} \\ \dot{g} = \dfrac{\partial H}{\partial a} g, \quad g \in G & \text{(horizontal subsystem)} \end{cases}$$

Optimal control on Lie groups (cont.)

If $G \subset GL(n)$ is compact and connected, its Lie algebra has an bi-invariant inner product $< ., . >$: (capital letters here!)

$$< A, B >= -\mathrm{tr}(AB)$$

and this allows the identification $\mathcal{L} \equiv \mathcal{L}^*$:

$$\begin{array}{ccc} \mathcal{L} & \leftrightarrow & \mathcal{L}^* \\ A & \leftrightarrow & a =< A, . > \end{array}$$

in particular,

$$\left(\mathrm{ad}\,\frac{\partial H}{\partial a}\right)^* : \mathcal{L}^* \to \mathcal{L}^* \quad \text{is identified with} \quad -\mathrm{ad}\,\frac{\partial H}{\partial A} : \mathcal{L} \to \mathcal{L}.$$

$$\begin{cases} \dot{A} = \left[A, \dfrac{\partial H}{\partial A}\right], & A \in \mathcal{L} \\ \dot{X} = \dfrac{\partial H}{\partial A} X, & X \in G. \end{cases}$$

(Hamiltonian equations on $\mathcal{L} \times G$)

Examples

- Geodesic problem
- Sub-Riemannian geodesic problem
- Sub-Riemannian Heisenberg problem
- Dido's problem
- Euler Elastica problem
- Rolling optimaly

Main source of information: "Control theory on Lie groups, Y. Sachkov, 2009".

Other important references include "Geometric Control Theory, V. Jurdjevic, 1997".

Geodesic problem

Assume that $G \subset GL(n)$ is compact and connected.

Problem:

Given any pair of points $X_0, X_1 \in G$, find the shortest curve in G that connects X_0 to X_1.

The Riemannian length of a curve $X : [0, t_1] \to G$ is defined as:

$$l = \int_0^{t_1} \sqrt{< \dot{X}(t), \dot{X}(t) >}\, dt.$$

Corresponding optimal control problem:

$$\begin{array}{l} l(U) = \int_0^{t_1} |U(t)|\, dt \ \to \ \min \\ \text{subject to} \\ \dot{X} = UX, \quad X \in G, \quad U \in \mathcal{L}, \\ X(0) = X_0, \quad X(t_1) = X_1, \\ X_0, X_1 \in G \quad \text{fixed.} \end{array}$$

Geodesic problem (cont.)

Due to Cauchy-Schwartz inequality

$$I(U)^2 = \left(\int_0^{t_1} |U(t)|\, dt \right)^2 \leq \int_0^{t_1} t_1 |U(t)|^2\, dt,$$

replace the functional $I(U)$ by $J(U) = \int_0^{t_1} |U(t)|^2\, dt$ (J is smooth).

Equivalent optimal control problem:

$$J(U) = \frac{1}{2} \int_0^{t_1} |U(t)|^2\, dt \;\rightarrow\; \min$$

subject to

$$\begin{array}{ll} \dot{X} = UX, & X \in G, U \in \mathcal{L}, \\ X(0) = X_0, & X(t_1) = X_1, \\ X_0, X_1 \in G & \text{fixed.} \end{array}$$

Geodesic problem (cont.)

Geometric nonlinear control
.
What is a control system?
Dynamical systems / Control systems
Typical problems of control theory
Examples
Accessibility / Controllability
Reachable sets
Accessibility and controllability
Distributions and integrability
Control systems on Lie groups
Main theorems on controllability
Examples
Representation of trajectories
Reduction to the "simple" case

Hamiltonian for this problem:

$$H_u^\nu(A,X) = < AX, UX > + \tfrac{\nu}{2}|U|^2 = < A, U > + \tfrac{\nu}{2} < U, U > = H_u^\nu(A)$$

According to PMP, if a pair $(U(t), X(t)) \in \mathcal{L} \times G$ is optimal, there exists a curve $t \mapsto A(t) \in \mathcal{L}$ and $\nu \leq 0$ such that:

$$\begin{cases} \dot{A} = \left[A, \dfrac{\partial H}{\partial A}\right] = [A, U] \\ \dot{X} = \dfrac{\partial H}{\partial A} X = UX \end{cases} \qquad \text{(Hamiltonian equations)}$$

$$H^\nu_{U(t)}(A(t)) = \max_{U \in \mathcal{L}} H^\nu_U(A(t)) \qquad \text{(maximality condition).}$$

$$(A(t), \nu) \neq 0$$

Geodesic problem (cont.)

Looking for solutions:

1. The abnormal case: $\nu = 0$

$$H_U^0(A) = < A, U > \rightarrow \max_{U \in \mathcal{L}}$$

Since $\frac{\partial}{\partial U} < A, U > = A$, the maximality condition reduces to

$$A(t) = 0, \qquad \text{(maximality condition).}$$

This contradicts the PMP $((A(t), \nu) \neq 0)$!

So, there are no abnormal extremal trajectories.

Geodesic problem (cont.)

2. The normal case: $\nu < 0$ (assume $\nu = -1$)

$$H_U^{-1}(A) = < A, U > - \frac{1}{2} < U, U > \rightarrow \max_{U \in \mathcal{L}}$$

$$U(t) \equiv A(t), \qquad \text{(maximality condition).}$$

$$\begin{cases} \dot{A} = [A, A] = 0 \\ \dot{X} = \dfrac{\partial H}{\partial A} X = AX \end{cases} \qquad \text{(Hamiltonian equations)}$$

So, normal optimal trajectories for the geodesic problem are right-translations of one-parameter subgroups in G:

$$X(t) = e^{tA} X_0, \quad A \in \mathcal{L},$$

where A is defined by $X_1 X_0^{-1} = e^{t_1 A}$.

Sub-Riemannian geodesic problem

A sub-Riemannian problem:

Given any pair of points $X_0, X_1 \in G$, find the shortest curve in G that connects X_0 to X_1, but now the admissible velocities $\dot{X}$ should be tangent to a certain distribution Δ of rank $k < n$ on G.

Optimal control problem on $G = SO(3)$:

$$J(U) = \frac{1}{2}\int_0^{t_1} < U(t), U(t) > \, dt \;\to\; \min$$

subject to

$$\dot{X} = UX, \quad X \in SO(3), \quad U \in \mathrm{span}\{A_{12}, A_{23}\} \subset so(3),$$

$$X(0) = X_0, \quad X(t_1) = X_1, \quad X_0, X_1 \in SO(3) \quad \text{fixed.}$$

$\Delta(X) = \{UX \,|\, U \in \mathrm{span}\{A_{12}, A_{23}\}\}$, $k = 2$, $n = 3$.
$\{A_{ij},\ 1 \leq i < j \leq 3\}$ is the canonical basis for $so(3)$.

Sub-Riemannian geodesic problem (cont.)

Hamiltonian (as before): $H_u^\nu(A) =< A, U > + \frac{\nu}{2} < U, U >$.

1. Looking for abnormal trajectories: $\nu = 0$

$$H_U^0(A) =< A, U > \to \max_{U \in \text{span}\{A_{12}, A_{23}\}}$$

$$A = a_1 A_{12} + a_2 A_{23} + a_3 A_{13} \quad \text{and} \quad A_{13} \perp \text{span}\{A_{12}, A_{23}\}.$$

So,

$$H_U^0(A) =< a_1 A_{12} + a_2 A_{23}, U > .$$

Maximality condition:

$$\frac{\partial}{\partial A} H_U^0(A) = 0 \Leftrightarrow A(t) = a_3(t) A_{13}, \qquad a_3(t) \neq 0.$$

Sub-Riemannian geodesic problem (cont.)

Note that $A = a_3 A_{13}$ implies $[A, U] \in \mathrm{span}\{A_{12}, A_{23}\}$. So,

$$\dot{A} = \left[A, \frac{\partial H}{\partial A}\right] \Leftrightarrow \dot{a}_3 A_{13} = [A, U], \qquad \text{(vertical subsystem)}$$

implies that $\dot{a}_3 \equiv 0$ and, therefore, $U \equiv 0$. Consequently,

$$\dot{X} = \frac{\partial H}{\partial A} X = UX = 0, \qquad \text{(horizontal subsystem)}$$

So, the abnormal extremal trajectories are constant, giving trivial solutions to the problem when $X_0 = X_1$.

The normal case: $\nu = -1$

Define $A_\perp := a_1 A_{12} + a_2 A_{23}$, so that $A = A_\perp + a_3 A_{13}$. Thus

$$H_U^{-1}(A) =< A_\perp, U > -\frac{1}{2} < U, U > \text{ and } \frac{\partial}{\partial U} H_U^{-1}(A) = A_\perp - U.$$

$$U = A_\perp = A - a_3 A_{13}, \qquad (\text{normal optimal controls}).$$

The vertical part of the Hamiltonian system takes the form

$$\dot{A} = [A, \frac{\partial H}{\partial A}] \Leftrightarrow \dot{A} = [A, U] = [A, A - a_3 A_{13}] = a_3 [A_{13}, A].$$

$$< \dot{A}, A_{13} >= a_3 < [A_{13}, A], A >= 0 \Rightarrow < A, A_{13} >= 2a_3 \text{ constant.}$$

$$\dot{A} = ad_C\, A, \text{ where } \quad C := a_3 A_{13}, \quad (\text{vertical subsystem})$$

is immediately solved:

$$A(t) = e^{t ad_C} A(0) = e^{tC} A(0) e^{-tC}.$$

Sub-Riemannian geodesic problem (cont.)

Replacing $A(t) = e^{tC}A(0)e^{-tC}$ in $U = A - C$ one gets:

$$U = e^{tC}A(0)e^{-tC} - C.$$

So, the horizontal Hamiltonian subsystem becomes

$$\dot{X} = UX = \left(e^{tC}A(0)e^{-tC} - C\right) X = e^{tC}(A(0) - C)e^{-tC}X.$$

Making the change of variable $Y = e^{-tC}X$, one gets

$$\dot{Y} = A(0)Y \quad \Rightarrow \quad Y(t) = e^{tA(0)}Y(0).$$

Finally

$$X(t) = e^{tC}e^{tA(0)}X(0) = e^{t<A(0),A_{13}>A_{13}}e^{tA(0)}X(0).$$

Normal sub-Riemannian geodesics on $SO(3)$ are products of one-parameter subgroups.

Variational approach for sub-Riemannian geodesic problem

$$\int_0^{t_1} < \dot{x}, \dot{x} > \, dt \, \rightarrow \, \min_x$$

subject to

$$\dot{x} = \sum_{i=1}^m u_i g_i(x), \quad x \in M, \quad m < n,$$

$$x(0) = x_0, \quad x(t_1) = x_1.$$

The constraints on velocities is simply

$$< \dot{x}, g_k >= 0, \quad m < k \leq n.$$

Take $Z_t = \sum_{k=m+1}^n \lambda_k(t) g_k$, where the λ_k are Lagrange multipliers. One can use the fundamental theorem of calculus of variations for the extended Lagrangian

$$\int_0^{t_1} (< \dot{x}, \dot{x} > + < Z_t, \dot{x} >) \, dt.$$

Dido's problem

According to Virgil (epic poem "Aeneid", 1st century, BC), Dido, the daughter of a Phoenician king, had to escape across the Mediterranean sea after her brother Pygmalion killed her husband and took her possessions.

She arrived at the Kingdom of Jarbas, in African coast, and begged for land, only what she could demarcate with the skin of an ox.

The king agreed to what seemed a small portion of land.

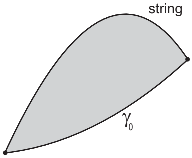
Dido’s problem

Dido asked her servants to split the hide into extremely thin strips of leather and tie them together.

She then used this string to construct a giant semicircle that, when combined with the natural boundary imposed by the Mediterranean sea, turned out to encompass far more area than King Jarbas could have ever imagined.

(Details in Montgomery, "A tour of sub-Riemannian geometries ...", AMS (2002)

Dido's problem (cont.)



Dido's problem:

Given a string of fixed length l and a fixed curve γ_0 (the Mediterranean coastline), place the ends of the string on γ_0 and determine the shape of the curve γ formed by the string for which the figure enclosed by γ and γ_0 has the maximum possible area.

Dido found the solution and founded the semicircular city of Carthage (Tunisia).

Dido's problem (cont.)

Dido's problem is a variant of the

Isoperimetric problem:

Among all plane figures with a given perimeter, find the one that encompasses the greatest area.

The Heisenberg group

$$G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad x, y, z \in \mathbb{R} \right\} \qquad \text{(non-compact, nilpotent)}$$

$$\mathcal{L} = \mathcal{L}(G) = \left\{ \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}, \quad a, b, c \in \mathbb{R} \right\} = span\{E_1, E_2, E_3\},$$

where $E_1 = E_{12},\ E_2 = E_{23},\ E_3 = E_{13}$.

$$[E_1, E_2] = E_3, \quad [E_1, E_3] = [E_2, E_3] = 0 \quad \text{(structure constants)}$$

$$\mathcal{L}^* = span\{\omega_1, \omega_2, \omega_3\}, \quad <\omega_i, E_j>= \delta_{ij}$$

$$A = \sum_{i=1}^{3} A_i E_i \in \mathcal{L} \Leftrightarrow \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \qquad a = \sum_{i=1}^{3} a_i \omega_i \in \mathcal{L}^* \Leftrightarrow \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix}$$

The Heisenberg group (cont.)

The scalar product in $\mathcal{L}$:

$$< E_i, E_j >= \delta_{ij}$$

generates a right-invariant scalar product in $T_X G$:

$$< E_i X, E_j X >=< E_i, E_j >= \delta_{ij}.$$

Sub-Riemannian lenght problem on the Heisenberg group

Optimal control problem on the Heisenberg group:

$$J(U) = \frac{1}{2}\int_0^{t_1} (u_1^2 + u_2^2)\, dt \;\rightarrow\; \min$$

subject to

$$\dot{X} = (u_1 E_1 + u_2 E_2)X \quad X \in G, \quad (u_1, u_2) \in \mathbb{R}^2$$

$$X(0) = X_0, \quad X(t_1) = X_1, \quad X_0, X_1 \in G \quad \text{fixed.}$$

$$\Delta(X) = \mathrm{span}\{E_1 X, E_2 X\} \subset T_X G, \;\; (k = 2, n = 3).$$

The constraints form a controllable system
(G connected, $\mathrm{Lie}\{E_1, E_2\} = \mathcal{L}$, no drift.)

Sub-Riemannian problem on the Heisenberg group (cont.)

Hamiltonian: $H_u^\nu(a) = < a, u_1 E_1 + u_2 E_2 > + \frac{\nu}{2}(u_1^2 + u_2^2)$

$$\frac{\partial H}{\partial a} = u_1 E_1 + u_2 E_2 = \begin{pmatrix} 0 & u_1 & 0 \\ 0 & 0 & u_2 \\ 0 & 0 & 0 \end{pmatrix}, \quad ad\,\frac{\partial H}{\partial a} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -u_2 & u_1 & 0 \end{pmatrix}$$

Vertical Hamiltonian subsystem of PMP: $\dot{a} = \left(ad\,\frac{\partial H}{\partial a}\right)^* a$

$$\begin{pmatrix} \dot{a}_1 & \dot{a}_2 & \dot{a}_3 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -u_2 & u_1 & 0 \end{pmatrix}$$

$$\Updownarrow$$

$$\begin{cases} \dot{a}_1 = -a_3 u_2 \\ \dot{a}_2 = a_3 u_1 \\ \dot{a}_3 = 0 \end{cases}$$

1. Abnormal trajectories: $\nu = 0$

Maximality condition:

$$H_u^0(a) = < a, u_1 E_1 + u_2 E_2 > \rightarrow \max_{(u_1, u_2) \in \mathbb{R}^2}$$

implies $a_1 = a_2 = 0$, thus $a_3 \neq 0$.

Then, the vertical subsystem (previous page) implies

$$\begin{aligned} \dot{a}_1 &= -a_3 u_2 = 0 \\ \dot{a}_2 &= a_3 u_1 = 0 \end{aligned} .$$

So, the abnormal optimal controls are $u_1 \equiv u_2 \equiv 0$.
The horizontal Hamiltonian subsystem

$$\dot{X} = \frac{\partial H}{\partial a} X = (u_1 E_1 + u_2 E_2) X$$

gives $X(t) = X_0$. There are no nonconstant abnormal optimal trajectories.

Sub-Riemannian problem on the Heisenberg group (cont.)

The normal case: $\nu = -1$

Maximality condition:

$$H_u^{-1}(a) = < a, u_1 E_1 + u_2 E_2 > -\frac{1}{2}(u_1^2 + u_2^2) \ \rightarrow \max_{(u_1, u_2) \in \mathbb{R}^2}$$

implies $u_1 = a_1, u_2 = a_2$. So, the Hamiltonian system has the form:

$$\begin{aligned} \dot{a}_1 &= -a_3 a_2 \\ \dot{a}_2 &= a_3 a_1 \qquad\qquad \text{(vertical subsystem)} \\ \dot{a}_3 &= 0 \end{aligned}$$

$$\dot{X} = (a_1 E_1 + a_2 E_2) X \quad \text{(horizontal subsystem)}$$

The first 2 equations imply $\dot{a}_1 a_1 + \dot{a}_2 a_2 = 0$, so

$$a_1^2 + a_2^2 = r^2, \quad r \text{ constant}$$

The optimal controls are sinusoids.

A convenient change of coordinates:

$$a_1 = r\cos\theta, \quad a_2 = r\sin\theta$$

gives

$$\begin{aligned} \dot r &= 0 \\ \dot\theta &= a_3 \qquad \text{(vertical subsystem)} \\ \dot a_3 &= 0 \end{aligned}$$

with solutions

$$\begin{aligned} \theta &= \theta_0 + a_3 t \\ a_1 &= r\cos(\theta_0 + a_3 t) \\ a_2 &= r\sin(\theta_0 + a_3 t) \end{aligned}$$

Sub-Riemannian problem on the Heisenberg group

The vertical Hamiltonian subsystem $\dot{X} = (a_1 E_1 + a_2 E_2)X$:

$$\begin{pmatrix} 0 & \dot{x} & \dot{z} \\ 0 & 0 & \dot{y} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a_1 & 0 \\ 0 & 0 & a_2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & a_1 & ya_1 \\ 0 & 0 & a_2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Leftrightarrow \qquad \begin{cases} \dot{x} = a_1 = r\cos(\theta_0 + a_3 t) \\ \dot{y} = a_2 = r\sin(\theta_0 + a_3 t) \\ \dot{z} = ya_1 \end{cases}$$

Assume that $x(0) = y(0) = z(0) = 0$ (trajectories start from the origin) and integrate ...

Sub-Riemannian problem on the Heisenberg group

For $a_3 = 0$, one gets:

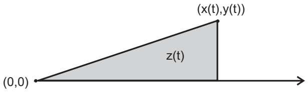
$$x(t) = t\, r \cos\theta_0$$

$$y(t) = t\, r \sin\theta_0$$

$$z(t) = \tfrac{1}{2} t^2 r^2 \cos\theta_0 \sin\theta_0$$

The projections (to the *xy*-plane) of extremal trajectories are straight lines.

$z(t)$ measures the area of a triangle



Sub-Riemannian problem on the Heisenberg group

For $a_3 \neq 0$, one gets:

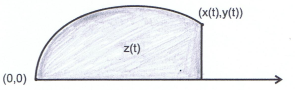
$$x(t) == \tfrac{r}{a_3}(\sin(\theta_0 + a_3 t) - \sin\theta_0)$$

$$y(t) = \tfrac{r}{a_3}(\cos\theta_0 - \cos(\theta_0 + a_3 t))$$

$$z(t) = \textstyle\int_0^t y(t)\dot{x}(t)\,dt = \ldots$$

The projections (to the xy-plane) of extremal trajectories are arcs of circles (they are optimal for $t \in [0, 2\pi/|a_3|]$).

$z(t)$ measures de area of the plane region bounded by the curve $(x(t), y(t))$, the x-axis and the straight line perpendicular to this axis.



Heisenberg meets Dido

The Heisenberg problem has the following geometric interpretation:

Geometric interpretation of Heisenberg problem

Given a plane curve γ_0, connecting two points $P_0 = (x_0, y_0)$, $P_1 = (x_1, y_1)$ and a number $A = z_1 - z_0 > 0$, find a plane curve γ connecting P_0 to P_1, so that the domain bounded by γ and γ_0 has area equal to A and γ is as short as possible.

Heisenberg problem is dual to Dido's problem.

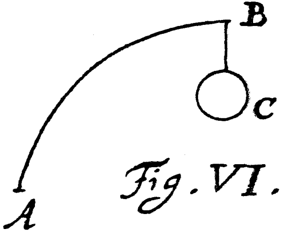
Geometric nonlinear control

Elastic curves of Euler (1744)

An elastic rod is a 1-dimensional object which is flexible (bendable but not stretchable), which looks like a portion of a straight line in its natural state.

Which form takes an elastic rod when subject to external forces applied to its ends?

Jacob Bernoulli posed this problem in 1691 and showed that the elastic energy of a deformed elastic rod is proportional to $\int \kappa^2(t)\, dt$, where $\kappa(t)$ is the geodesic curvature. He also arrived to the correct solution.



Elastic curves of Euler (1744)

Euler (1744) also studied the variational principle

$$\int \kappa^2 \, dt \quad \rightarrow \text{ min}$$

that gives rise to the shape that minimizes the elastic energy.

The solution is most commonly attributed to Euler because of his compelling mathematical treatment and illustrations.

Elastic curves

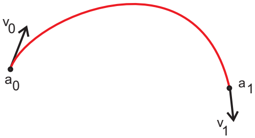
Let $\gamma(t) = (x(t), y(t)),\ t \in [0, t_1]$ be the unit speed arc-length parametrization of a plane curve with fixed length t_1.

$$\frac{1}{2}\int_0^{t_1} \kappa^2(t)\, dt \quad \to \min$$

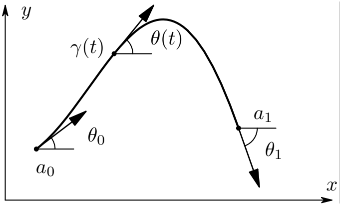
subject to

$$\begin{array}{ll} \gamma(0) = a_0, & \gamma(t_1) = a_1, \\ \dot\gamma(0) = v_0, & \dot\gamma(t_1) = v_1, \end{array}$$

for given a_0, a_1 points in the plane, $v_0,\ v_1$ unit tangent vectors at a_0 and a_1, respectively, $t_1 \in \mathbb{R}$.



Elastic curves (cont.)



Let $\theta(t)$ be the angle between the velocity vector and the positive direction of the x-axis, so that

$$v_0 = (\cos\theta_0, \sin\theta_0), \qquad v_1 = (\cos\theta_1, \sin\theta_1).$$

Assume we can control the angular velocity $\dot\theta(t)$. Since arc-length parameterized curves have unit speed and $\kappa^2 = \dot\theta^2$, the elastica problem can be reformulated as:

Elastic curves (cont.)

Elastica problem

$$\frac{1}{2}\int_0^{t_1} u^2(t)\,dt \quad \to \min$$

subject to

$$\begin{cases} \dot{x} = \cos\theta \\ \dot{y} = \sin\theta \\ \dot{\theta} = u \\ \\ (x,y,\theta)(0) = (x_0,y_0,\theta_0), \quad (x,y,\theta)(1) = (x_1,y_1,\theta_1) \end{cases}$$

Boundary data: $\begin{array}{ll} a_0 = (x_0, y_0), & v_0 = (\cos\theta_0, \sin\theta_0), \\ a_1 = (x_1, y_1), & v_1 = (\cos\theta_1, \sin\theta_1). \end{array}$

This problem may be formulated as a right-invariant optimal control problem on $SE(2)$.

Elastic curves (cont.)

Geometric nonlinear control

What is a control system?
Dynamical systems / Control systems
Typical problems of control theory
Examples

Accessibility / Controllability
Reachable sets
Accessibility and controllability
Distributions and integrability

Control systems on Lie groups
Main theorems on controllability
Examples

Representation of trajectories
Reduction to the "simple" case

$$(x, y, \theta) \in \mathbb{R}^3 \quad \leftrightarrow \quad \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ x & y & 1 \end{bmatrix} \in SE(2).$$

Elastica problem

$$\frac{1}{2}\int_0^{t_1} u^2(t)\, dt \quad \rightarrow \min$$

subject to

$$\dot{X} = (E_1 + uE_2)X, \quad X \in SE(2), \quad u \in \mathbb{R}$$

$$X(0) = X_0, X(t_1) = X_1,$$

$E_1 = E_{31}$, $E_2 = A_{12} \in se(2)$. Since $\mathrm{Lie}\{E_1, E_2\} = se(2)$, the dynamics are controllable (even though the system has drift).

Elastic curves (cont.)

$SE(2)$ is not compact, so proceed as in the Heisenberg problem.

Hamiltonian: $H_u^\nu(a) = < a, E_1 + uE_2 > + \frac{\nu}{2} u_1^2, \quad a \in se(2)^*.$

$$\frac{\partial H}{\partial a} = \begin{pmatrix} 0 & u & 0 \\ -u & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad ad\,\frac{\partial H}{\partial a} = \begin{pmatrix} 0 & 0 & u \\ 0 & 0 & 0 \\ -u & 1 & 0 \end{pmatrix}.$$

Vertical Hamiltonian subsystem of PMP:

$$\dot{a} = \left(ad\,\frac{\partial H}{\partial a}\right)^* a \quad \Leftrightarrow \quad \begin{aligned} \dot{a}_1 &= -a_3 u \\ \dot{a}_2 &= a_3 \\ \dot{a}_3 &= a_1 u \end{aligned}$$

Elastic curves (cont.)

1. Abnormal trajectories: $\nu = 0$

Maximality condition:

$$H_u^0(a) =< a, E_1 + uE_2 >= a_1 + ua_2 \;\rightarrow\; \max_{u\in\mathbb{R}}$$

implies a_2. Then, the vertical subsystem (previous page) implies that $a_3 = 0$, $a_1 \neq 0$ and $u = 0$.

So, abnormal extremal trajectories are produced by zero controls. And because $|u| = \kappa$, the projections of such extremal trajectories on the xy-plane are geodesics (straight lines), as seen from integrating the horizontal Hamiltonian subsystem

$$\begin{aligned} \dot{x} &= \cos\theta \\ \dot{y} &= \sin\theta \\ \dot{\theta} &= u \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} x &= t\cos\theta_0 + x_0 \\ y &= \sin\theta_0 + y_0 \\ \theta &= \theta_0 \end{aligned} \quad .$$

Elastic curves (cont.)

The normal case: $\nu = -1$

Maximality condition:

$$H_u^{-1}(a) = a_1 + ua_2 - \frac{1}{2}u^2 \rightarrow \max_{u \in \mathbb{R}}$$

implies $\boxed{u = a_2}$. So the vertical subsystem has the form:

$$\begin{aligned} \dot{a}_1 &= -a_2 a_3 \\ \dot{a}_2 &= a_3 \\ \dot{a}_3 &= a_1 a_2 \end{aligned}$$

Taking a short cut to the solutions ...

Notice that $a_1^2 + a_3^2 = r^2$ (constant). Define an angle ϕ by

$$a_1 = r\cos\phi, \; a_3 = r\sin\phi$$

Then,

The elastica meets the pendulum!

$$\begin{aligned} \dot{r} &= 0 \\ \dot{\phi} &= a_2 \\ \dot{a}_2 &= r \sin\phi \end{aligned} \qquad \text{vertical subsystem}$$

Equation of the mathematical pendulum

$$\ddot{\phi} = r \sin\phi.$$

This is the basis for Kirchhoff's analogy of the elastica with the motion of the pendulum.

Heisenberg meets Euler!

Since

$$\dot{\phi} = a_2 = u = \dot{\theta},$$

the two angles ϕ and θ differ by a constant angle. We can apply a rotation in the plane so that these angles coincide. So

$$\begin{aligned} \dot{x} &= \cos\theta \\ \dot{y} &= \sin\theta \qquad\qquad \text{vertical subsystem} \\ \ddot{\theta} &= -r\sin\theta \end{aligned}$$

- $r = 0$

  θ is a linear function and the optimal trajectories are straight lines and circles as in the sub-Riemannian problem of Heisenberg.

- $r > 0$

The extremal trajectories, when parameterized by θ, are given by:

$$\frac{dx}{d\theta} = \frac{\frac{dx}{dt}}{\frac{d\theta}{dt}} = \frac{\cos\theta}{\pm\sqrt{2(H - r\cos\theta)}}$$

$$\frac{dy}{d\theta} = \frac{\frac{dy}{dt}}{\frac{d\theta}{dt}} = \frac{\sin\theta}{\pm\sqrt{2(H - r\cos\theta)}}.$$

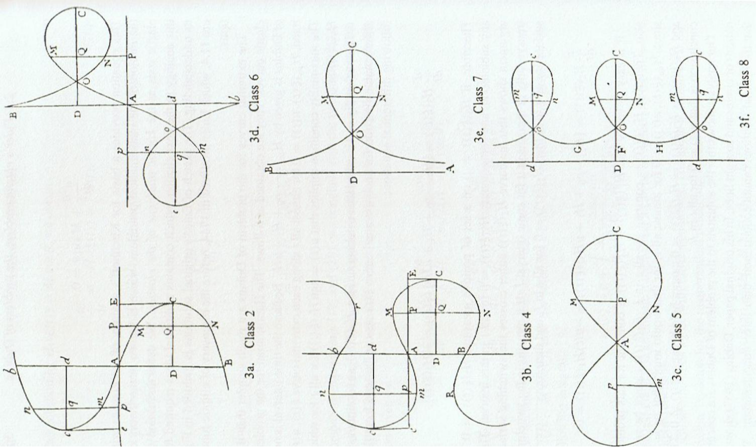
The second equation immediately integrates, the first can be integrated by means of elliptic functions:

$$x(\theta) = c_1 + \int \frac{\cos\theta}{\pm\sqrt{2(H - r\cos\theta)}}\, d\theta$$

$$y(\theta) = c_2 + \frac{1}{r}\sqrt{2(H - r\cos\theta)}.$$

Euler's Elastica

In 1744, Euler sketched these beautiful curves even before the discover of the elliptic functions (Carl Jacobi) was borne!



From Jurdjevic's book

Elastic curves on Riemannian manifolds

Jurdjevic, Non-Euclidean elastica (1995) - Maximum principal
Camarinha, Crouch + SL (1996, 2000) - Variational approach

$$\frac{1}{2}\int_0^{t_1} \kappa^2(t)\,dt \quad \to \min$$

subject to

$$\begin{array}{ll} x(0) = x_0, & x(t_1) = x_1, \\ \dot{x}(0) = v_0, & \dot{x}(t_1) = v_1, \end{array}$$

- x_0, x_1 are given points in M,
- $v_0,\ v_1$ are given unit tangent vectors at x_0 and x_1.

Elastic curves on Riemannian manifolds

$\frac{D}{dt}$ denotes covariant derivative along a curve in M.

Geodesic curvature of a curve x:

$$\kappa(t) = \frac{< \frac{D\dot{x}}{dt}, \frac{D\dot{x}}{dt} >^{1/2}}{< \dot{x}, \dot{x} >} - \frac{< \frac{D\dot{x}}{dt}, \dot{x} >}{< \dot{x}, \dot{x} >^{3/2}}$$

If $< \dot{x}, \dot{x} >= 1$, then $< \frac{D\dot{x}}{dt}, \dot{x} >= 0$ and, consequently,

$$\kappa^2(t) =< \frac{D\dot{x}}{dt}, \frac{D\dot{x}}{dt} >.$$

Elastic curves on Riemannian manifolds

Ω is the family of smooth curves in M.

Given x_0, x_1 points in M, $v_0 \in T_{x_0}$, $v_1 \in T_{x_1}$,

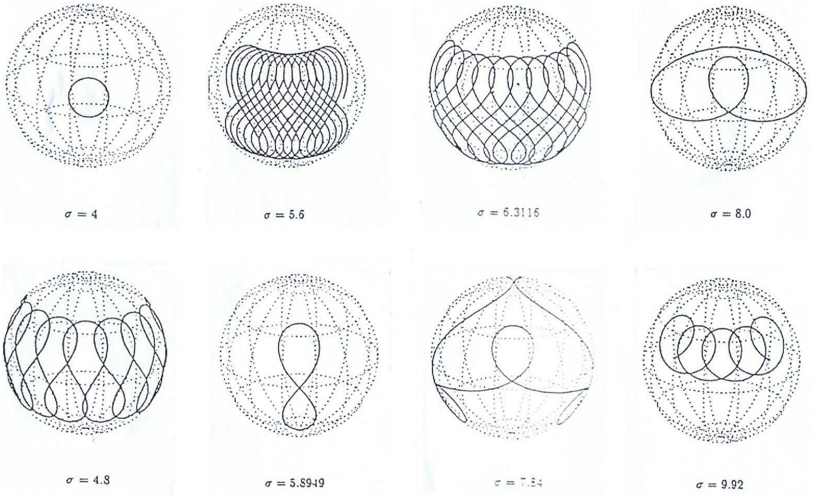
$$J(x) = \int_0^1 \left\langle \frac{D\dot{x}}{dt}, \frac{D\dot{x}}{dt} \right\rangle dt \quad \rightarrow \text{min}$$

subject to:

$$\langle \dot{x}, \dot{x} \rangle = 1,$$

$$\begin{aligned} x(0) &= x_0, & x(1) &= x_1, \\ \dot{x}(0) &= v_0, & \dot{x}(1) &= v_1, \end{aligned}$$

Illustration - Elastic curves on the S^2 sphere



The rolling sphere (again)

Given 2 admissible configurations, roll the sphere upon the tangent plane from the first configuration to the second, so that the curve traced in the plane by the contact point be the shortest possible.

$$J(u) = \frac{1}{2}\int_0^{t_1} (u_1^2 + u_2^2)\, dt \to \min \qquad \text{(cost functional)}$$

subject to:

$$\dot{s}(t) = \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix}$$

(Kinematic equations)

$$\dot{R}(t) = R(t) \begin{bmatrix} 0 & 0 & u_1 \\ 0 & 0 & u_2 \\ -u_1 & -u_2 & 0 \end{bmatrix}$$

$$X(0) = X_0 = (s_0, R_0) \quad X(t_1) = X_1 = (s_1, R_1)$$

The rolling sphere meets Euler

The point of contact of the sphere rolling optimally traces Euler elastica on the plane!

This may be observed when applying the PMP to the rolling sphere problem.

Higher order problems on Lie groups

$$J(x) = \int_0^{t_1} < \frac{D\dot{x}}{dt}, \frac{D\dot{x}}{dt} > dt \quad \to \min_x$$

subject to

$$\frac{D\dot{x}}{dt} = \sum_{i=1}^{m} u_i(t) X_i(x),$$

$$\begin{aligned} x(0) &= x_0, & x(t_1) &= x_1, \\ \dot{x}(0) &= v_0, & \dot{x}(t_1) &= v_1. \end{aligned}$$

- $m = n$ Riemannian cubic polynomial
- $m < n$ sub-Riemannian cubic polynomial

Assume $\{X_1, \cdots, X_n\}$ orthonormal basis of right-invariant vector fields.

Higher order problems

$$\int_0^{t_1} \sum_{i=1}^{m} u_i^2(t)\, dt \to \min_u$$

subject to

$$\frac{D\dot{x}}{dt} = \sum_{i=1}^{m} u_i(t) X_i(x),$$

$$x(0) = x_0, \qquad x(t_1) = x_1,$$
$$\dot{x}(0) = v_0, \qquad \dot{x}(t_1) = v_1.$$

Higher-order optimal control problems are extremely hard to tackle ...

The optimal trajectories are projections of a Hamiltonian flow in TT^*TM!

Higher order problems on Lie groups

But, using a variational approach, the Euler-Lagrange equations have been derived, when G is connected and compact and the metric is bi-invariant.

Euler-Lagrange equations

$$\dddot{V} + [V, \ddot{V}] = 0,$$

where

$$V = dx(t)/dt = \sum_{i=1}^{n} v_i(t) X_i(x(t)),$$

$$\ddot{V} = \sum_{i=1}^{n} \ddot{v}_i(t) X_i(x(t)),$$

$$\dddot{V} = \sum_{i=1}^{n} \dddot{v}_i(t) X_i(x(t)).$$

Crouch + SL (1991)

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