Unreduction

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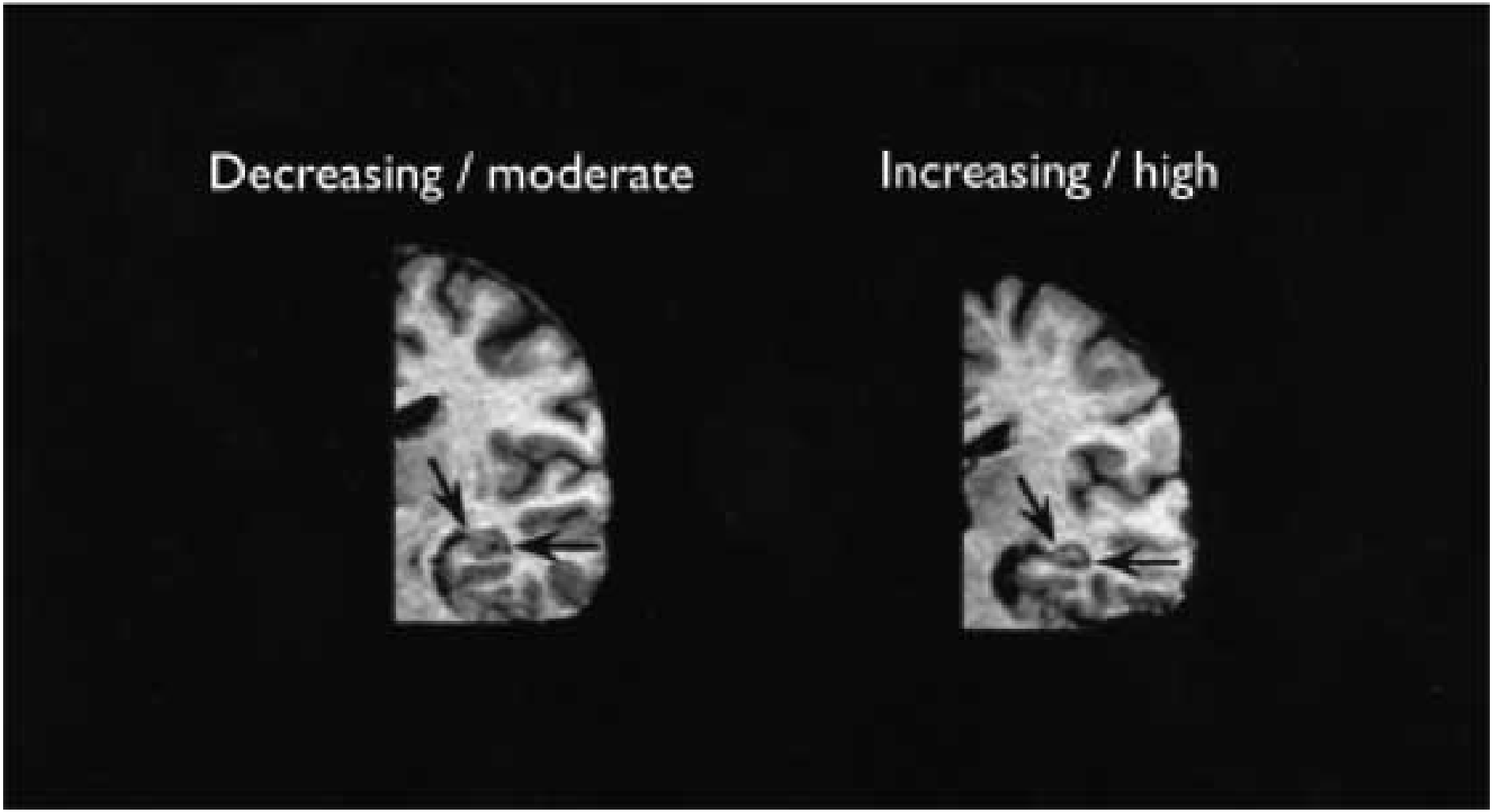
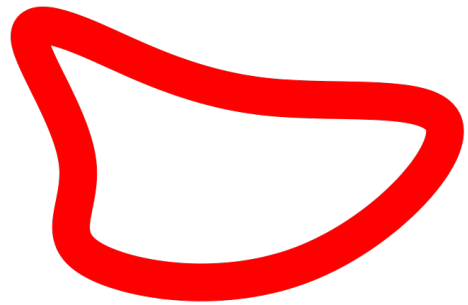


Figure: MRI image of the hippocampus (arrows on left side) comparing subjects with high/moderate cortisol levels.
Lupien et. al. *Nature Neuroscience* **1**, 69 - 73 (1998)

Shape space

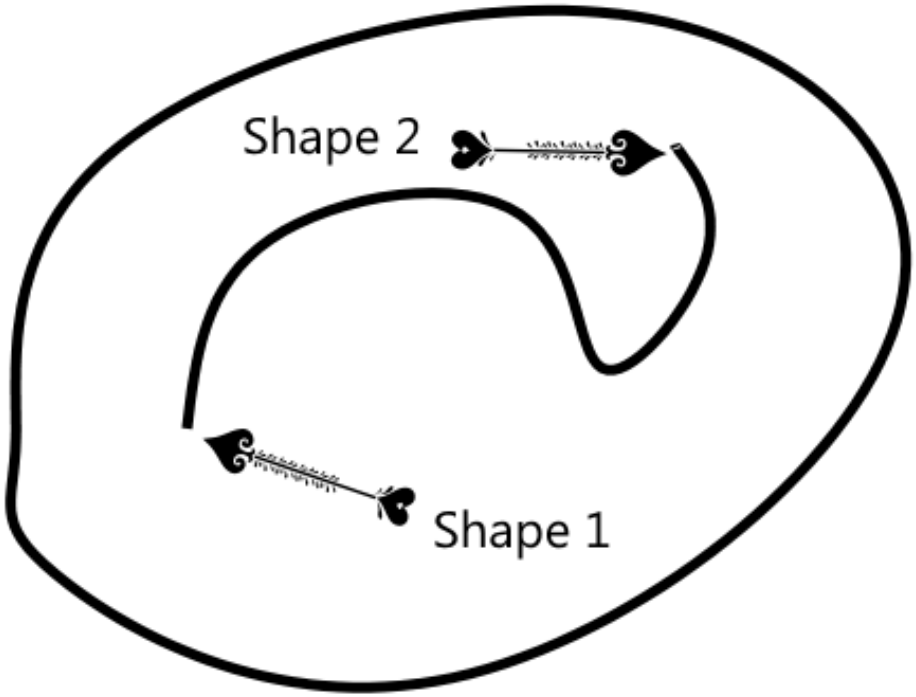


$$\pi : Q := \mathrm{Emb}\left(S^1, \mathbb{R}^2\right) \to \Sigma := \mathrm{Emb}\left(S^1, \mathbb{R}^2\right) / \mathrm{Diff}(S^1),$$

- Principal $\mathrm{Diff}(S^1)$–bundle.
- Shape only expressible as equivalence classes of embeddings

Shape morphing

- Wish to morph one shape into another geodesically in Σ.



Approach I: Naive attempt

Let $\rho : \mathbb{R} \to \Sigma$,

$$\nabla^{\Sigma}_{\dot{\rho}} \dot{\rho} = 0.$$

- No local coordinates on Σ
- No numerical implementation
- Brilliant in theory, useless in practice!

Approach II: Horizontal shooting

Developed by Bauer, Cotter, Harms, Holm, Michor, Mumford,...
Introduce connection form $\mathcal{A} : TQ \to \mathfrak{x}(S^1)$

$$\nabla^Q_{\dot{q}} \dot{q} = 0$$

$$\mathcal{A}(\dot{q}) = 0$$

- Horizontal geodesics project to geodesics
- Limited control of parametrisation dynamics

Question

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$\pi : Q \to \Sigma$ principal G-bundle

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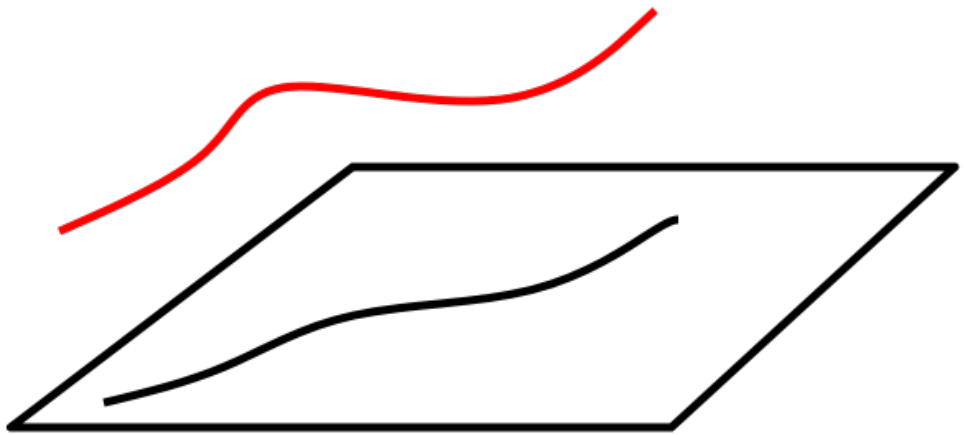

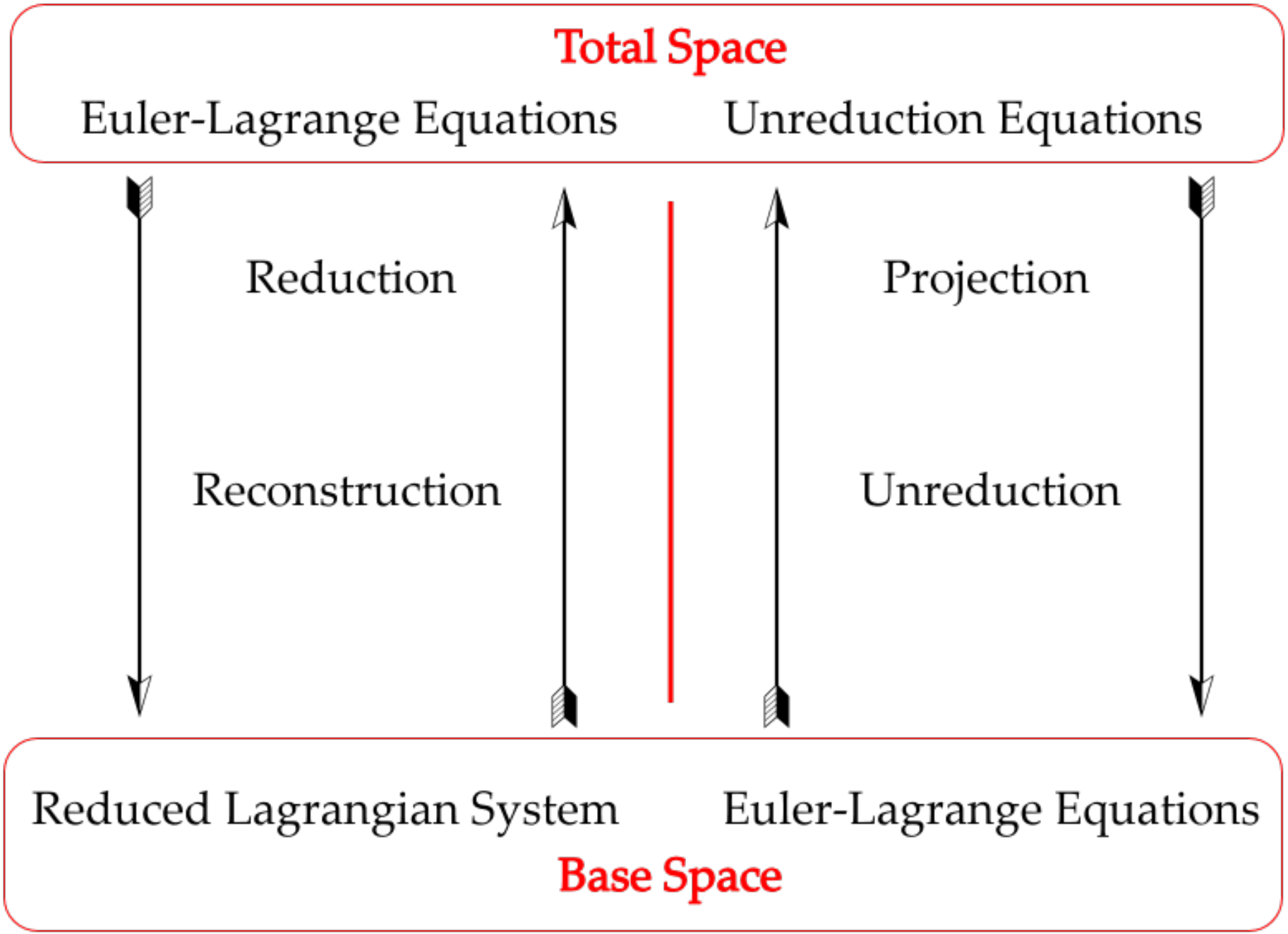
Total Space
Euler-Lagrange Equations
Unreduction Equations
Reduction
Reconstruction
Projection
Unreduction
Reduced Lagrangian System
Euler-Lagrange Equations
Base Space

Notation fixing

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We shall use the following kit:

- Lie algebra $\mathfrak{g} := \mathrm{Lie}(G)$
- Principal connection $\mathcal{A} : TQ \to \mathfrak{g}$
- Adjoint bundle $\mathrm{Ad}\, Q = Q \times_G \mathfrak{g}$
- $[\![q, \xi]\!]_{\mathfrak{g}} \in \mathrm{Ad}\, Q$ for $q \in Q$ and $\xi \in \mathfrak{g}$

Lagrangian reduction

Given Lagrangian L and G-equivariant forcing F,

$$\tilde{\nabla}_t \frac{\delta L}{\delta \dot{q}} - \frac{\delta L}{\delta q} = F(q, \dot{q})$$

implies

$$D_t \frac{\delta l}{\delta \bar{\sigma}} + \mathrm{ad}^*_{\bar{\sigma}} \frac{\delta l}{\delta \bar{\sigma}} = F^{\mathrm{Ad}}(\rho, \dot{\rho}, \bar{\sigma})$$

$$\nabla_t \frac{\delta l}{\delta \dot{\rho}} - \frac{\delta l}{\delta \rho} = F^{\Sigma}(\rho, \dot{\rho}, \bar{\sigma}) - \left\langle \frac{\delta l}{\delta \bar{\sigma}} , i_{\dot{\rho}} \bar{\mathcal{B}} \right\rangle$$

where

$$\rho = \pi(q), \qquad \bar{\sigma} = [\![q, \mathcal{A}(q, \dot{q})]\!]_{\mathfrak{g}}$$

Two distortions to EL equations

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1. Coupling between $T\Sigma$ and $\operatorname{Ad} Q$

$$I : T\Sigma \oplus \operatorname{Ad} Q \to \mathbb{R}$$

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2. Curvature terms

$$\left\langle \frac{\delta I}{\delta \bar{\sigma}} , i_{\dot{\rho}} \bar{\mathcal{B}} \right\rangle$$

Take $L^h = \pi^* l^\Sigma$, $L^v = \bar{\mathcal{A}}^* l^{\mathrm{Ad}}$, and $L = L^h + L^v$

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\begin{aligned}
D_t \frac{\delta l^{\mathrm{Ad}}}{\delta \bar{\sigma}} + \mathrm{ad}^*_{\bar{\sigma}} \frac{\delta l^{\mathrm{Ad}}}{\delta \bar{\sigma}} &= F^{\mathrm{Ad}}\left(\rho, \dot{\rho}, \bar{\sigma}\right) \\
\nabla_t \frac{\delta l^{\Sigma}}{\delta \dot{\rho}} - \frac{\delta l^{\Sigma}}{\delta \rho} &= F^{\Sigma}\left(\rho, \dot{\rho}, \bar{\sigma}\right) + \frac{\delta l^{\mathrm{Ad}}}{\delta \rho} - \left\langle \frac{\delta l^{\mathrm{Ad}}}{\delta \bar{\sigma}} \,,\, i_{\dot{\rho}} \bar{\mathcal{B}} \right\rangle
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\end{aligned}
$$

Choose

$$
F^{\Sigma} = \left\langle \frac{\delta l^{\mathrm{Ad}}}{\delta \bar{\sigma}} \,,\, i_{\dot{\rho}} \bar{\mathcal{B}} \right\rangle - \frac{\delta l^{\mathrm{Ad}}}{\delta \rho}
$$

Can not write functions on Σ

Choose $L^H, L^V : TQ \to \mathbb{R}$, fix $L^h = L^H \circ P_h$ and $L^v = L^V \circ P_v$, where P_h, P_v denote horizontal/vertical projections

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$$\frac{d}{dt} J \left(\frac{\delta \left(L^V \circ P_v \right)}{\delta \dot{q}} \right) = f(q, \dot{q})$$

$$\nabla_t \frac{\delta \left(L^H \circ P_h \right)}{\delta \dot{q}} - \frac{\delta \left(L^H \circ P_h \right)}{\delta q} = 0$$

Exogenous equivariant forcing $f : TQ \to \mathfrak{g}^*$, momentum map J

Approach III: Unreduction

Take $L^h = 1/2\|\dot{q} - \mathcal{A}(q,\dot{q})\cdot q\|^2$, and $L^v = 1/2\|\mathcal{A}(q,\dot{q})\cdot q\|^2$

Solve

$$\frac{d}{dt}J\left(\frac{\delta\left(L^V \circ P_v\right)}{\delta\dot{q}}\right) = f(q,\dot{q})$$

$$\nabla_t \frac{\delta\left(L^H \circ P_h\right)}{\delta\dot{q}} - \frac{\delta\left(L^H \circ P_h\right)}{\delta q} = 0$$

- $\pi(q)$ geodesic in Σ
- Able to choose f such that $q(t)$ parametrised uniformly $\forall t$

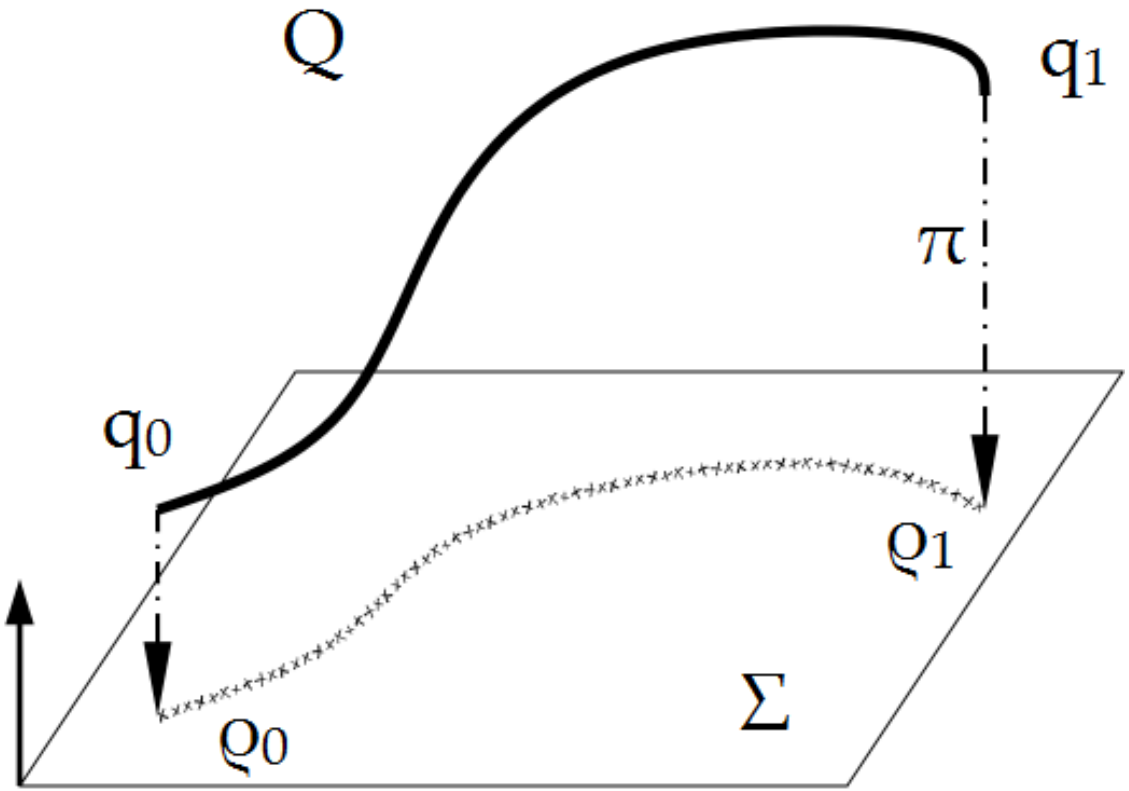

Figure: Lifting of geodesic in $\Sigma := \mathrm{Emb}(S^1, \mathbb{R}^2)/\mathrm{Diff}(S^1)$ to submanifold of uniformly parametrised curves in $Q := \mathrm{Emb}(S^1, \mathbb{R}^2)$

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1. Posed a problem from image dynamics with applications to medicine

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2. Described unreduction procedure

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2. Described unreduction procedure
3. Used unreduction to solve our problem from image dynamics

Questions