Euler-Poincaré equations by several stages

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Joint work with **Hernán Cendra**.

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References

- H. Cendra, J.E. Marsden and T.S. Ratiu [2001] Lagrangian reduction by stages, *Memoirs of the AMS*, **152**, no **722**.

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- H. Cendra, J.E. Marsden and T.S. Ratiu [2001] Lagrangian reduction by stages, *Memoirs of the AMS*, **152**, no **722**.

- H. Cendra and V.A. Díaz, Lagrange-d'Alembert-Poincaré equations by several stages, *to appear*.

The talk

1. The Euler-Poincaré equations
2. The Lie bracket by stages
3. The Euler-Poincaré equations by several stages

The Euler-Poincaré equations

We consider

- a *Lie group G*

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The Euler-Poincaré equations

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- a *Lie group G*
- a left *G*-invariant *Lagrangian function* $L : TG \to \mathbb{R}$
- the *reduced Lagrangian*, $l : T\mathfrak{g} \to \mathbb{R}$ (the restriction of L to the identity)

The Euler-Poincaré theorem

- Then the following statements are equivalent:

The Euler-Poincaré theorem

- the curve $g(t) \in \Omega(G; g_0, g_1)$ is a critical point of the action functional of L for variations δg vanishing at the endpoints; i.e., the variational principle

$$\delta \int_a^b L(g(t), \dot{g}(t))dt = 0$$

holds, for variations with fixed endpoints.

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  holds, for variations with fixed endpoints.
- the curve $g(t)$ satisfies the Euler-Lagrange equations for L, that is $\mathcal{EL}(L)(g, \dot{g}, \ddot{g}) = 0$ on G;

The Euler-Poincaré theorem

- the curve $v(t) = g(t)^{-1}\dot{g}(t) \in \mathfrak{g}$ is a critical point of the reduced action functional

$$\delta \int_{t_0}^{t_1} l(v(t))dt = 0,$$

for variations of the form

$$\delta v = \dot{\eta} + [v, \eta]$$

where $\eta(t) \in \mathfrak{g}$ is an arbitrary curve that vanishes at the endpoints.

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- the curve $v(t)$ solves the **Euler-Poincaré equations**

$$\frac{d}{dt}\frac{\partial l}{\partial v} = \mathrm{ad}^*_v \frac{\partial l}{\partial v}.$$

The Lie bracket by two stages

For a given principal bundle $\pi : Q \to Q/G$, if $N \lhd G$ we have

$T_qQ = \mathrm{Ver}^N(T_qQ) \oplus (\mathrm{Ver}^N(T_qQ))^{\perp}$ ($\exists$ a G-invariant metric on Q).

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We have an identification $\mathfrak{g} \equiv \mathfrak{k}^{\mathcal{A}_N} \times \mathfrak{n} \equiv \mathfrak{k} \times \mathfrak{n}$ where $\mathfrak{k}^{\mathcal{A}_N}$ is the horizontal lift of $\mathfrak{k} = T_eK$ in the bundle $G \to K$, at $e \in \mathfrak{g}$.

The Lie bracket by two stages

Then, for a given identification $\mathfrak{g} \equiv \mathfrak{k} \oplus \mathfrak{n}$, the Lie bracket on $\mathfrak{g}$ is given by

$$[\kappa_1 \oplus \eta_1, \kappa_2 \oplus \eta_2] =$$

$$= [\kappa_1, \kappa_2] \oplus [\nabla^{(\mathcal{A}_N, V)}]_{K,\kappa_1} \eta_2 - [\nabla^{(\mathcal{A}_N, V)}]_{K,\kappa_2} \eta_1 - [\widetilde{B}^{\mathcal{A}_N}]_K(\kappa_1, \kappa_2) + [\eta_1, \eta_2] .$$

(Cendra, Marsden and Ratiu, 2001)

The Lie bracket by two stages

It can be proven that

$$[\widetilde{B}^{\mathcal{A}_N}]_K(\kappa_1,\kappa_2) = \Big[\Big[e, -\mathcal{A}_N(e)\Big([\kappa_1^{\mathcal{A}_N}, \kappa_2^{\mathcal{A}_N}]\Big)\Big]_N\Big]_K$$

and

$$[\nabla^{(\mathcal{A}_N,V)}]_{K,\,\kappa}\,\eta = \big[[e,[\kappa^{\mathcal{A}_N},\eta]]_N\big]_K$$

(via the identification of $\mathfrak{k} \equiv \widetilde{\mathfrak{k}}$ and $\mathfrak{n} \equiv \Gamma(\widetilde{\mathfrak{n}})$ thinking on $\widetilde{\mathfrak{n}}$ as a vector bundle over K).

The Lie bracket by two stages

Then the Lie bracket on $\mathfrak{g}$

$$[\kappa_1 \oplus \eta_1, \kappa_2 \oplus \eta_2] = [\kappa_1, \kappa_2] \oplus [\nabla^{(\mathcal{A}_N, V)}]_{K,\kappa_1}\eta_2 - [\nabla^{(\mathcal{A}_N, V)}]_{K,\kappa_2}\eta_1$$

$$-[\widetilde{B}^{\mathcal{A}_N}]_K(\kappa_1, \kappa_2) + [\eta_1, \eta_2]$$

The Lie bracket by two stages

Then the Lie bracket on $\mathfrak{g}$ can be written as

$$[\kappa_1 \oplus \eta_1, \kappa_2 \oplus \eta_2] = [\kappa_1, \kappa_2] \oplus \Big[[e, [\kappa_1^{\mathcal{A}_N}, \eta_2]]_N\Big]_K - \Big[[e, [\kappa_2^{\mathcal{A}_N}, \eta_1]]_N\Big]_K$$
$$- \Big[\Big[e, -\mathcal{A}_N(e)\Big([\kappa_1^{\mathcal{A}_N}, \kappa_2^{\mathcal{A}_N}]\Big)\Big]_N\Big]_K + [\eta_1, \eta_2],$$

where $\kappa_i^{\mathcal{A}_N}$ is the horizontal lift of κ_i in G at e.

Defining the bilinear forms

$$b_N : \mathfrak{k} \times \mathfrak{n} \to \widetilde{\mathfrak{n}}/K \equiv \mathfrak{n} \quad \text{and} \quad a_N : \mathfrak{k} \times \mathfrak{k} \to \widetilde{\mathfrak{n}}/K \equiv \mathfrak{n}$$

by:

$$b_N(\kappa, \eta) := \left[\left[e, [\kappa^{\mathcal{A}_N}, \eta]\right]_N\right]_K = [\nabla^{(\mathcal{A}_N, V)}]_{K,\ \kappa}\ \eta$$

and

$$a_N(\kappa, \overline{\kappa}) := \left[\left[e, -\mathcal{A}_N(e)\left([\kappa^{\mathcal{A}_N}, \overline{\kappa}^{\mathcal{A}_N}]\right)\right]_N\right]_K = \left[\widetilde{B}^{\mathcal{A}_N}\right]_K (\kappa, \overline{\kappa}),$$

we obtain the expression

$$[\kappa_1 \oplus \eta_1, \kappa_2 \oplus \eta_2] =$$

$$= [\kappa_1, \kappa_2] \oplus b_N(\kappa_1, \eta_2) - b_N(\kappa_2, \eta_1) - a_N(\kappa_1, \kappa_2) + [\eta_1, \eta_2].$$

The Lie bracket by n stages

Let G be a Lie group and a chain of normal subgroups

$$\{e\} = N_n \lhd N_{n-1} \lhd \ldots \lhd N_2 \lhd N_1 \lhd N_0 = G.$$

Consider the principal bundles

$$\pi_{N_{(j-1,j)}} : N_{j-1} \to N_{(j-1,j)} := N_{j-1}/N_j$$

with the connection $\mathcal{A}_{N_j}$ associated to a given G-invariant metric.
Consider an identification

$$\mathfrak{g} \equiv \mathfrak{k} \oplus \mathfrak{n}_{(1,2)} \oplus \ldots \oplus \mathfrak{n}_{(n-1,n)} \oplus \mathfrak{n}_{(n,n+1)} \ ,$$

where $\mathfrak{k} = \mathfrak{n}_{(0,1)} = \mathfrak{g}/\mathfrak{n}_1$.

Consider also the applications

$$b_{(N_{j-1},N_j)} : \mathfrak{n}_{(j-1,j)} \times \mathfrak{n}_j \to \widetilde{\mathfrak{n}}_j / N_{(j-1,j)} \simeq \mathfrak{n}_j$$

(where $\widetilde{\mathfrak{n}}_j$ is the adjoint bundle of the principal bundle $N_{j-1} \to N_{(j-1,j)}$ with structure group N_j) defined by

$$b_{(N_{j-1},N_j)}(\eta, \overline{\eta}) := \left[\left[e, [\eta^{\mathcal{A}_{N_j}}, \overline{\eta}] \right]_{N_j} \right]_{N_{(j-1,j)}} ; \quad \text{and}$$

$$a_{(N_j,N_{j+1})} : \mathfrak{n}_{(j,j+1)} \times \mathfrak{n}_{(j,j+1)} \to \widetilde{\mathfrak{n}}_{j+1} / N_{(j,j+1)} \simeq \mathfrak{n}_{j+1}$$ defined by

$$a_{(N_j,N_{j+1})}(\eta, \overline{\eta}) := \left[\left[e, -\mathcal{A}_{N_{j+1}}(e) \left([\eta^{\mathcal{A}_{N_{j+1}}}, \overline{\eta}^{\mathcal{A}_{N_{j+1}}}] \right) \right]_{N_{j+1}} \right]_{N_{(j,j+1)}},$$

where $\eta^{\mathcal{A}_{N_j}}$ is the horizontal lift of η in the bundle $N_{j-1} \to N_{j-1}/N_j$ with the connection $\mathcal{A}_{N_j}$. Then ...

the Lie bracket on $\mathfrak{g}$ is:

$$\left[\kappa \oplus \left(\bigoplus_{i=1}^{n} \eta^{(i,i+1)}\right), \overline{\kappa} \oplus \left(\bigoplus_{j=1}^{n} \overline{\eta}^{(j,j+1)}\right)\right] = [\kappa, \overline{\kappa}] \oplus$$

$$\left(\bigoplus_{i=1}^{n} \left([\eta^{(i,i+1)}, \overline{\eta}^{(i,i+1)}] + \sum_{k=1}^{n} \left(b^{(i,i+1)}_{(N_0,N_1)}(\kappa, \overline{\eta}^{(k,k+1)})\right.\right.\right.$$

$$\left. - b^{(i,i+1)}_{(N_0,N_1)}(\overline{\kappa}, \eta^{(k,k+1)})\right) + \sum_{j=2}^{i} \left(b^{(i,i+1)}_{(N_{j-1},N_j)} \left(\eta^{(j-1,j)}, \sum_{l=j}^{n} \overline{\eta}^{(l,l+1)}\right)\right.$$

$$\left. - b^{(i,i+1)}_{(N_{j-1},N_j)} \left(\overline{\eta}^{(j-1,j)}, \sum_{p=j}^{n} \eta^{(p,p+1)}\right)\right) - a^{(i,i+1)}_{(N_0,N_1)}(\kappa, \overline{\kappa})$$

$$\left. - \sum_{m=1}^{i-1} a^{(i,i+1)}_{(N_m,N_{m+1})}(\eta^{(m,m+1)}, \overline{\eta}^{(m,m+1)})\right)\right) .$$

Euler-Poincaré equations by several stages

As before, consider an identification $\mathcal{G} \equiv \mathfrak{k} \oplus \left(\bigoplus_{i=1}^{n} \mathfrak{n}_{(i,i+1)} \right)$ y

$$\mathfrak{g}^* \equiv \mathfrak{k}^* \oplus \left(\bigoplus_{j=1}^{n} \mathfrak{n}^*_{(j,j+1)} \right).$$

Let $v(t) = \kappa(t) \oplus \left(\bigoplus_{i=1}^{n} \eta^{(i,i+1)}(t) \right) \in \mathcal{G}$ be a curve and

$$\mu = \frac{\partial l}{\partial v} = \frac{\partial l}{\partial \kappa} \oplus \left(\bigoplus_{j=1}^{n} \frac{\partial l}{\partial \eta^{(j,j+1)}} \right) = \alpha \oplus \left(\bigoplus_{j=1}^{n} \beta_{(j,j+1)} \right) \in \mathcal{G}^*.$$

Using the obtained expression for the Lie bracket by several stages to write the Euler-Poincaré equations

$$\frac{d}{dt}\frac{\partial l}{\partial v} = \mathrm{ad}^*_v \frac{\partial l}{\partial v} \qquad \Leftrightarrow \qquad \dot{\mu} = ad^*_v\ \mu$$

we can write the **Euler-Poincaré equations by several stages** as follows:

Euler-Poincaré equations by several stages

$$
\begin{cases}
\dot{\alpha} &= \mathrm{ad}^*_{\kappa}\alpha \ , \\
\dot{\beta}_{(i,i+1)} &= \mathrm{ad}^*_{\eta^{(i,i+1)}}\beta_{(i,i+1)} + \beta_{(i,i+1)}\left(b^{(i,i+1)}_{(N_0,N_1)}(\kappa,\ .\ )\right) + \\
&\qquad \beta_{(i,i+1)}\left(\sum_{j=2}^{i} b^{(i,i+1)}_{(N_{j-1},N_j)}(\eta^{(j-1,j)},\ .\ )\right) \ , \\
0 &= \beta_{(i,i+1)}\left(b^{(i,i+1)}_{(N_0,N_1)}\left(\ .\ ,\sum_{k=1}^{n}\eta^{(k,k+1)}\right) + a^{(i,i+1)}_{(N_0,N_1)}(\kappa,\ .\ )\right) \ , \\
\alpha &= \dfrac{\partial l}{\partial \kappa} \ , \\
\beta_{(i,i+1)} &= \dfrac{\partial l}{\partial \eta^{(i,i+1)}} \ .
\end{cases}
$$

Thank you very much for your attention.