Hamilton Jacobi equation and applications.

Paula Balseiro

(IMPA-CNPq, Rio de Janeiro, Brazil)

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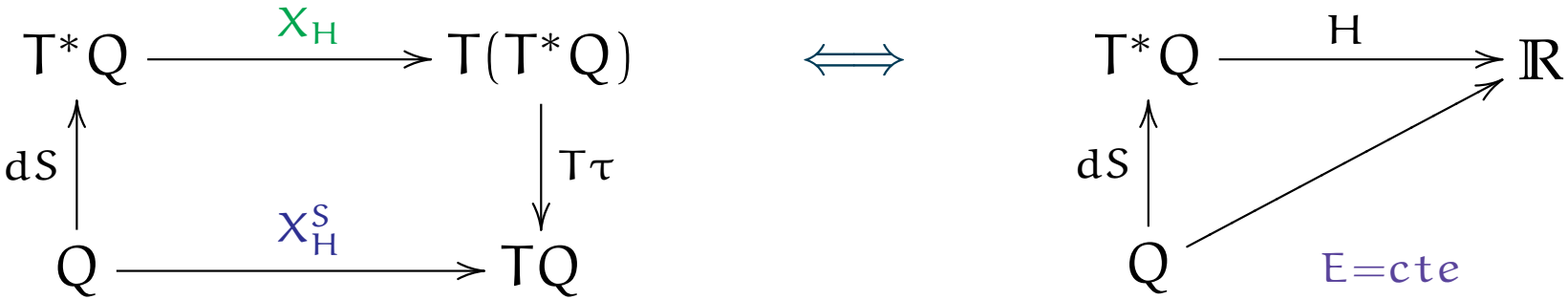
$\tau : T^*Q \to Q$, (T^*Q, Ω_Q, H) a hamiltonian system.

Theorem (Hamilton Jacobi)

Let $S : Q \to \mathbb{R}$,

1. $c(t)$ is an I.C. of $T\tau \circ X_H \circ dS \Rightarrow dS \circ c(t)$ is an I.C. of X_H.
2. *H-J equation:* $H \circ dS = E$, $d(H \circ dS) = 0$



Non Holonomic Systems

- $D \subset TQ$, $\tau_D : D \to Q$,
- $(D, [\![\cdot,\cdot]\!]_D, \rho)$ a skew-symmetric algebroid $\Leftrightarrow$ $(D^*, \{\cdot,\cdot\}_{D^*})$

linear *almost* Poisson structure.

- $H_D : D^* \to \mathbb{R}$, X_{H_D} the NH hamiltonian vector field.

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Let $\alpha : Q \to D^*$,

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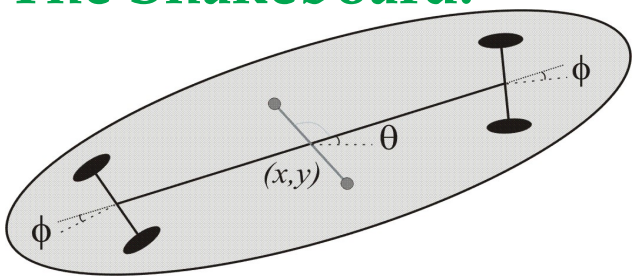
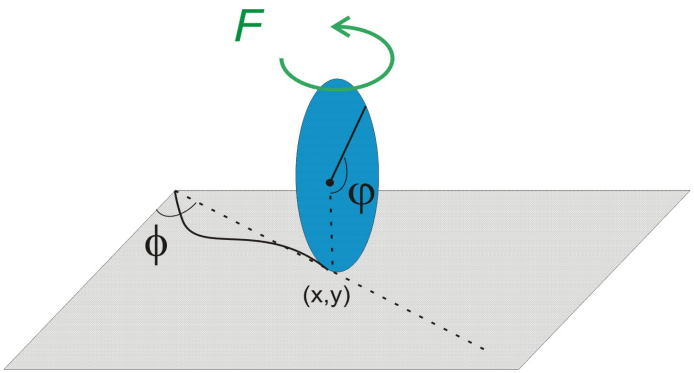
The Snakeboard:



- $Q = SE(2) \times \mathbb{T}^2$, $D \subset TQ$
- $G = SE(2)$

$$\begin{array}{ccc} D/G & \subset & T\mathbb{T}^2 \times \mathfrak{se}(2) \\ \downarrow & & \downarrow \\ Q/G & & \mathbb{T}^2 \end{array}$$

Rolling Disk with forces



- $Q = \mathbb{R}^2 \times S^1 \times S^1$, $\sim (x, y, \phi, \theta)$.
- Constraints: $\begin{cases} \dot{x} = R\cos\phi\dot{\theta} \\ \dot{y} = R\sin\phi\dot{\theta} \end{cases}$
- $D = span\{X_1, X_2\} \rightsquigarrow (D, [\![\cdot,\cdot]\!]_D, \rho)$ a skew-symm. algebroid.
- $X_{H_D} \in \mathfrak{X}(D^*)$ assoc. to $\{\cdot,\cdot\}_{D^*}$.

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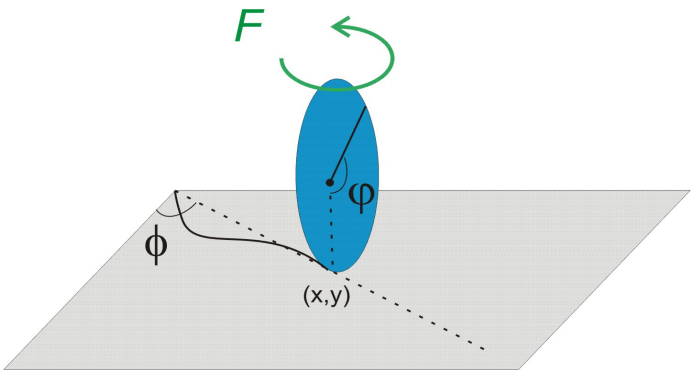

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$$F(q, \dot{q}) = (K\cos\phi\,\dot{\phi})\,d\phi \rightsquigarrow F = (\tfrac{K}{J}\cos\phi\,\nu^2)\,X^2$$

$$\Downarrow$$

$\tau : \mathbb{R} \times D \to Q$ skew-symmetric algebroid.

Idea: Basis: $\{(0, X_1), (0, X_2), (1, 0)\}$

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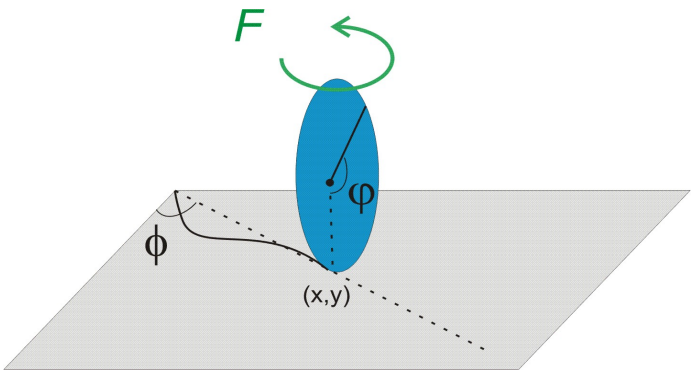

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The *new* hamiltonian vector field: $R_{H_D} = X_{H_D} - (F^*)^\vee \in \mathfrak{X}(D^*)$

$$\dot{q}^i = \rho^i_a \frac{\partial H}{\partial p_a} \qquad \dot{p}_b = -\rho^i_b \frac{\partial H}{\partial q^i} + \mathcal{C}^c_{ab} p_c \frac{\partial H}{\partial p_a} - F^a_b p_a$$

Theorem. (Hamilton Jacobi)

Let $\alpha : Q \to D^*$,

1. $c(t)$ is an I.C. of $T\tau_D \circ X_{H_D} \circ \alpha \Rightarrow \alpha \circ c(t)$ is an I.C. of $R_{H_D} = X_{H_D} - (F^*)^{\vee}$.
2. *H-J equation:*

$$d^D(H_D \circ \alpha) + F^*(\alpha) = 0 \qquad \text{if } d^D\alpha = 0$$

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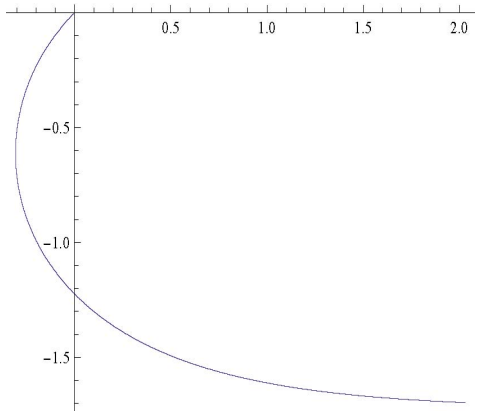
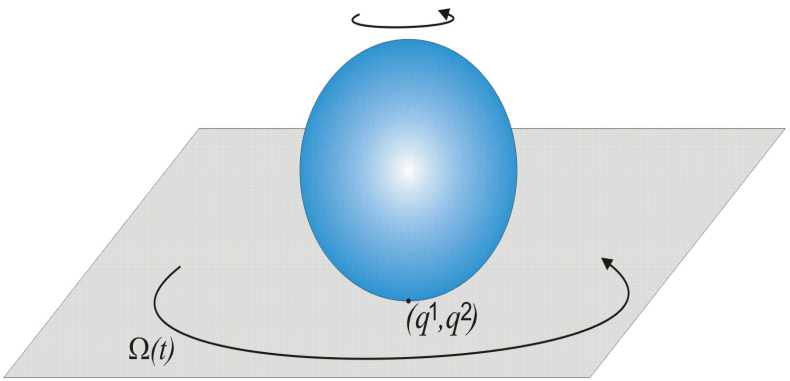


Affine Constraints



- $(t, q^1, q^2, \dot{q}^1, \dot{q}^2, \omega_1, \omega_2, \omega_3)$
- Constraints:

$$\begin{cases} \dot{q}^1 - r\omega_2 = -\Omega(t)q^2 \\ \dot{q}^2 + r\omega_1 = \Omega(t)q^1. \end{cases}$$

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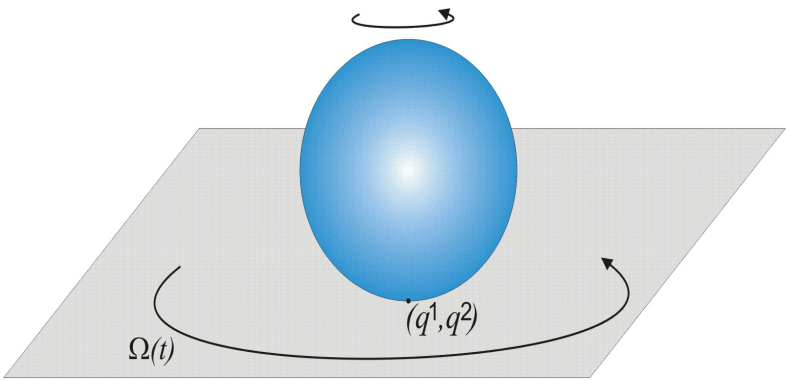

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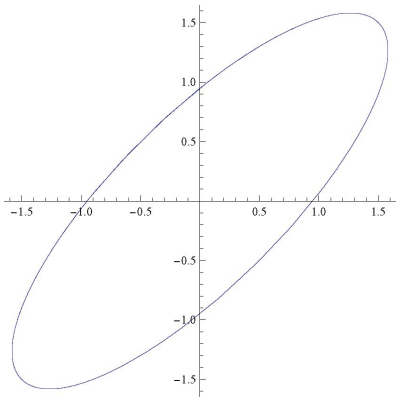

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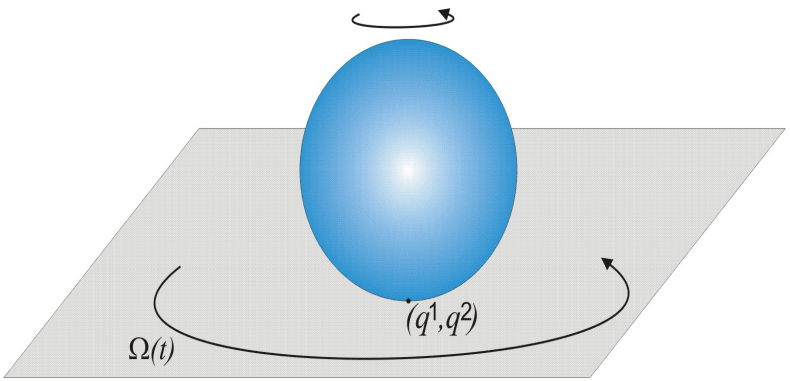

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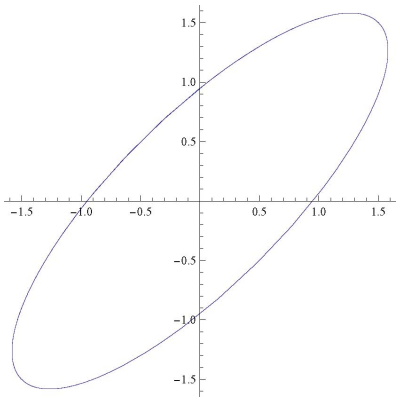

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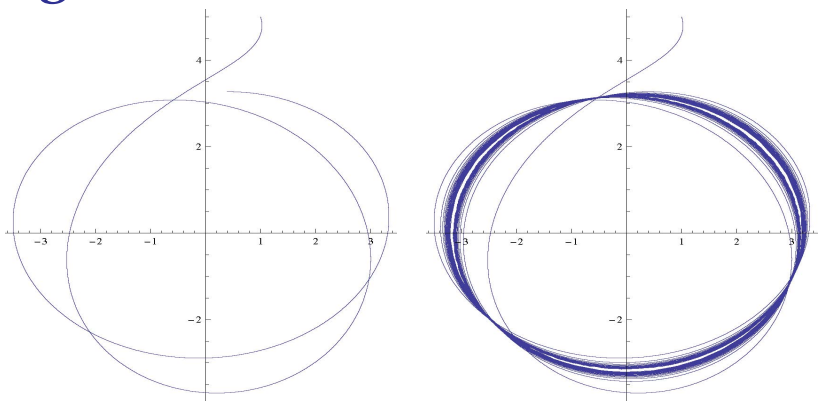

$\Omega(t) = t$

Time dependent systems:

- $Q = \mathbb{R} \times P, \quad W : Q \to \mathbb{R}$
- $H : \mathbb{R} \times T^*P \to \mathbb{R}$

$$\frac{\partial W}{\partial t} + H(t, q^i, \frac{\partial W}{\partial q^i}) = \text{constant on } P,$$

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Theorem. (Hamilton Jacobi)

Let $(E, [\![\cdot,\cdot]\!], \rho, \phi, h)$ be an almost hamiltonian system on E. If $\alpha \in \Gamma(V^*)$,

1. $c(t)$ is an I.C. of $T\tau_{V^*} \circ R_h \circ \alpha \Rightarrow \alpha \circ c(t)$ is an I.C. of $R_h \in \mathfrak{X}(V^*)$.
2. *H-J equation:* $\mu \circ i_{\xi^\alpha_{H_D}}(d^E(h \circ \alpha)) = 0$

Thank you!