Distributed motion coordination of robotic networks

Lecture 3 – Rendezvous

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Roadmap

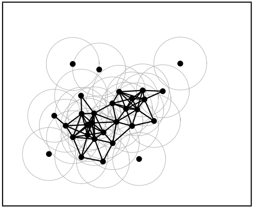
Today

1. **Rendezvous** – Basic motion coordination capability
2. **Non-deterministic discrete-time** dynamical systems – stability analysis
3. **Robustness** – against link failures in interaction topology

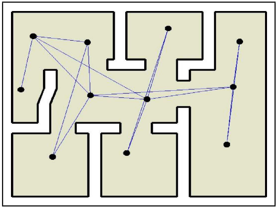
Outline

Rendezvous

Objective:
achieve rendezvous at single point, while maintaining connectivity



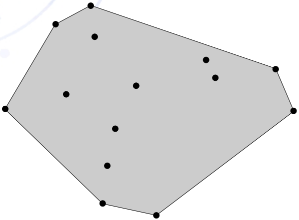
r-disk connectivity



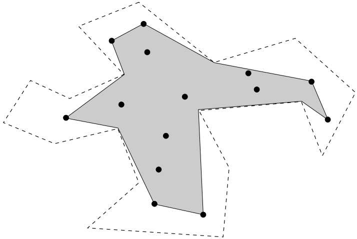
visibility connectivity

Aggregate objective functions

Coordination task formulated as function minimization

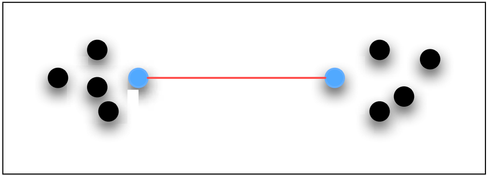


Diameter convex hull



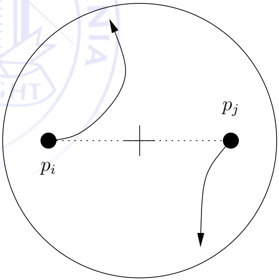
Perimeter relative convex hull

We have to be careful...

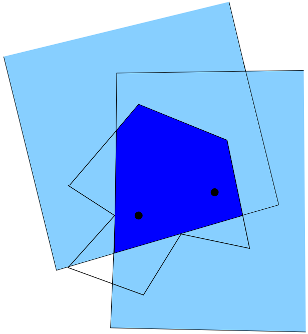


Blindly "getting closer" to neighboring agents might break overall connectivity

Constraint sets for connectivity



r-disk pair-wise constraint set



visibility pair-wise constraint set

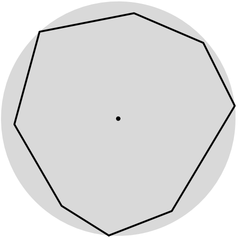
Key properties

- Constraints are flexible enough so that network does not get stuck
- Constraints change continuously with agents' position

Outline

What is the circumcenter

For $X = \mathbb{R}^d$, $X = \mathbb{S}^d$ or $X = \mathbb{R}^{d_1} \times \mathbb{S}^{d_2}$, $d = d_1 + d_2$, **circumcenter** $\mathsf{CC}(W)$ of a bounded set $W \subset X$ is center of closed ball of minimum radius that contains W. **Circumradius** $\mathsf{CR}(W)$ is radius of this ball



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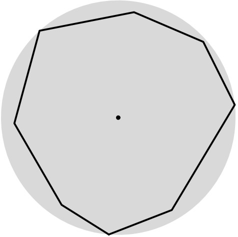

Lemma (Properties of the circumcenter in Euclidean space)

Let $S \in \mathbb{F}(\mathbb{R}^d)$. Then, the following holds:

1. $\mathsf{CC}(S) \in \mathsf{co}(S) \setminus \mathsf{Ve}(\mathsf{co}(S))$
2. *if $p \in \mathsf{co}(S) \setminus \{\mathsf{CC}(S)\}$ and $r \in \mathbb{R}_{>0}$ are such that $S \subset \overline{B}(p, r)$, then $(p, \mathsf{CC}(S))$ has a nonempty intersection with $\overline{B}(\frac{p+q}{2}, \frac{r}{2})$ for all $q \in \mathsf{co}(S)$*

Circumcenter algorithms – the basic idea

- each agent minimizes "local version" of objective function

$$\mathsf{max}\{\|p_i - p_j\| \mid p_j \text{ is neighbor of } p_i\}$$

  i.e., each agent goes toward circumcenter of neighbors and itself (which is the closest point to all these locations)
- each agent maintains connectivity by moving inside constraint set

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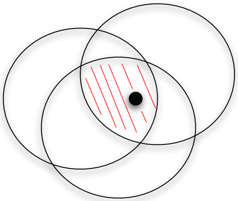
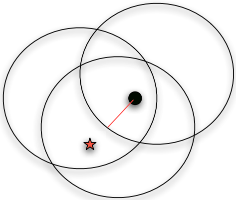
If agents i and j are neighbors then subsequent positions must belong to $\overline{B}(\frac{p_i+p_j}{2}, \frac{r}{2})$
If agent i has neighbors at locations $\{q_1, \ldots, q_l\}$ at time ℓ, then **constraint set** is

$$\mathcal{D}_r(p_i, \{q_1, \ldots, q_l\}) = \bigcap_{q \in \{q_1, \ldots, q_l\}} \overline{B}\Big(\frac{p_i + q}{2}, \frac{r}{2}\Big)$$



Circumcenter algorithms – the basic idea

To maximize the displacement toward circumcenter, each agent solves **convex optimization problem**



For q_0 and q_1 in $\mathbb{R}^d$, and for a convex closed set $Q \subset \mathbb{R}^d$ with $q_0 \in Q$, let $\lambda_{\max}(q_0, q_1, Q)$ denote the solution to the strictly convex problem

$$\begin{aligned} &\text{maximize} \;\; \lambda \\ &\text{subject to} \;\; \lambda \leq 1, \; (1-\lambda)q_0 + \lambda q_1 \in Q \end{aligned}$$

Under the stated assumptions the solution exists and is unique

Circumcenter algorithms – informally

Communication rounds take place at each natural instant of time

At each communication round each agent performs the following tasks:

1. it transmits its position and receives its neighbors' positions
2. it computes the circumcenter of the point set comprised of its neighbors and of itself, and
3. it moves toward this circumcenter while maintaining connectivity with its neighbors

Circumcenter algorithms – formally

`Robotic Network:` $\mathcal{S}_{\text{disk}}$
`Distributed Algorithm:` CIRCUMCENTER
`Alphabet:` $L = \mathbb{R}^d \cup \{\texttt{null}\}$

`function` $\text{msg}(x, w, i)$

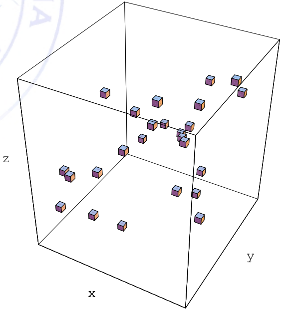
1: **return** $\text{msg}_{\text{std}}(x, w, i)$

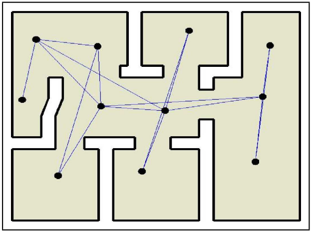
`function` $\text{ctrl}(x_{\text{smpld}}, y)$

1: $x_{\text{goal}}(x_{\text{smpld}}, y) = \mathsf{CC}(\{x_{\text{smpld}}\} \cup \{x_{\text{rcvd}} \mid \text{for all non-null } x_{\text{rcvd}} \in y\}),$
2: $\mathcal{D} := \mathcal{D}_r(x_{\text{smpld}}, \{x_{\text{rcvd}} \mid \text{for all non-null } x_{\text{rcvd}} \in y\})$
3: $\lambda^* = \lambda_{\max}\big(x_{\text{smpld}}, x_{\text{goal}}(x_{\text{smpld}}, y), \mathcal{D}\big)$
4: **return** $\lambda^*\big(x_{\text{goal}}(x_{\text{smpld}}, y) - x_{\text{smpld}}\big)$

Can also be run over any other proximity graph which is spatially distributed over $\mathcal{G}_{\text{disk}}(r)$ or over $\mathcal{G}_{\text{vis}}$

Simulations





Outline

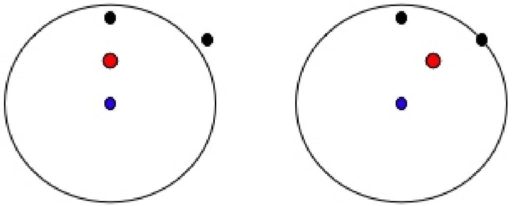
Some bad news...

Circumcenter algorithms are nonlinear discrete-time dynamical systems

$$x_{\ell+1} = f(x_\ell)$$

To analyze convergence, we need at least f continuous – to use classic Lyapunov/LaSalle results

But circumcenter algorithms are discontinuous because of changes in interaction topology



Alternative idea

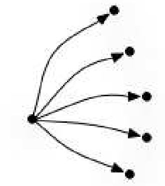
Fixed undirected graph $\mathcal{G}$, define **fixed-topology circumcenter algorithm**

$$f_{\mathcal{G}} : (\mathbb{R}^d)^n \to (\mathbb{R}^d)^n, \quad f_{\mathcal{G},i}(p_1, \dots, p_n)$$

Now, there are no topological changes in $f_{\mathcal{G}}$, hence $f_{\mathcal{G}}$ is continuous

Define set-valued map $T_{\mathcal{CC}} : (\mathbb{R}^d)^n \to \mathcal{P}((\mathbb{R}^d)^n)$

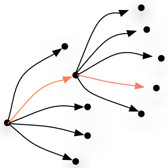
$T_{\mathcal{CC}}(p_1, \dots, p_n) = \{f_{\mathcal{G}}(p_1, \dots, p_n) \mid \mathcal{G} \text{ connected}\}$



Non-deterministic dynamical systems

Given $T : X \to \mathcal{P}(X)$, a **trajectory** of T is sequence $\{x_m\}_{m \in \mathbb{N}_0} \subset X$ such that

$$x_{m+1} \in T(x_m)\,, \quad m \in \mathbb{N}_0$$



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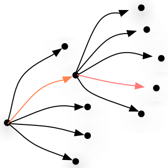

T is **closed** at x if $x_m \to x$, $y_m \to y$ with $y_m \in T(x_m)$ imply $y \in T(x)$
Every continuous map $T\colon \mathbb{R}^d \to \mathbb{R}^d$ is closed on $\mathbb{R}^d$

A set C is

- **weakly positively invariant** if, for any $p_0 \in C$, there exists $p \in T(p_0)$ such that $p \in C$
- **strongly positively invariant** if, for any $p_0 \in C$, all $p \in T(p_0)$ verifies $p \in C$

A point p_0 is a *fixed point of* T if $p_0 \in T(p_0)$

LaSalle Invariance Principle – set-valued maps

$V : X \to \mathbb{R}$ is non-increasing along T on $S \subset X$ if

$$V(x') \leq V(x) \text{ for all } x' \in T(x) \text{ and all } x \in S$$

Theorem (LaSalle Invariance Principle)

For S compact and strongly invariant with V continuous and non-increasing along closed T on S

Any trajectory starting in S converges to largest weakly invariant set contained in $\{x \in S \mid \exists x' \in T(x) \text{ with } V(x') = V(x)\}$

Proof sketch

Let $\Omega(x_n) \subset S$ be ω-limit set of sequence $\{x_n \mid n \in \mathbb{N} \cup \{0\}\}$

1. $\Omega(x_n)$ is **weakly positively invariant**

   Let $x \in \Omega(x_n)$. Then there exists a subsequence $\{x_{n_m} \mid m \in \mathbb{N} \cup \{0\}\}$ such that $x_{n_m} \to x$. Consider the sequence $\{x_{n_m+1} \mid m \in \mathbb{N} \cup \{0\}\}$. Since this sequence is bounded, it has a convergent subsequence, i.e., there exits y such that $x_{n_m+1} \to y$ (abuse of notation). By definition, $y \in \Omega(x_n)$. Because T is closed, $y \in T(x)$!

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2. V is **constant** on $\Omega(x_n)$

   Consider sequence $\{V(x_n) \mid n \in \mathbb{N} \cup \{0\}\}$. Since $\{x_n \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded and V is non-increasing along T on S, sequence is decreasing and bounded from below, and therefore convergent. Let $c \in \mathbb{R}$ such that $V(x_n) \to c$. Let us see that V on $\Omega(x_n)$ is equal to c. Take any $x \in \Omega(x_n)$. Accordingly, there exists a subsequence $\{x_{n_m} \mid m \in \mathbb{N} \cup \{0\}\}$ such that $x_{n_m} \to x$. Since V is continuous, $V(x_{n_m}) \to V(x)$. From $V(x_n) \to c$, we conclude $V(x) = c$

Proof sketch

The fact that $\Omega(x_n)$ is weakly positively invariant and V is constant on $\Omega(x_n)$ implies

$$\Omega(x_n) \subset \{x \in S \mid \exists x' \in T(x) \text{ such that } V(x') = V(x)\}$$

Proof sketch

The fact that $\Omega(x_n)$ is weakly positively invariant and V is constant on $\Omega(x_n)$ implies

$$\Omega(x_n) \subset \{x \in S \mid \exists x' \in T(x) \text{ such that } V(x') = V(x)\}$$

Therefore, $x_n \to M \cap V^{-1}(c)$, where M is the largest weakly positively invariant set contained in $\{x \in S \mid \exists x' \in T(x) \text{ such that } V(x') = V(x)\}$

In general, challenge is in identifying M explicitly

Outline

Correctness – $T_{\mathcal{CC}}$ is closed

Recall set-valued map $T_{\mathcal{CC}} : (\mathbb{R}^d)^n \to \mathcal{P}((\mathbb{R}^d)^n)$

$$T_{\mathcal{CC}}(p_1, \dots, p_n) = \{f_{\mathcal{G}}(p_1, \dots, p_n) \mid \mathcal{G} \text{ connected}\}$$

$T_{\mathcal{CC}}$ is **closed**: finite combination of individual continuous maps

In addition,

$$\mathsf{co}(P') \subset \mathsf{co}(P)$$

for all $P' \in T_{\mathcal{G}}(P)$ and $P \in (\mathbb{R}^d)^n$

Correctness – diameter as non-increasing function

$V_{\mathrm{diam}} = \mathsf{diam} \circ \mathsf{co} \colon (\mathbb{R}^d)^n \to \overline{\mathbb{R}}_+$, by

$$V_{\mathrm{diam}}(P) = \mathsf{diam}(\mathsf{co}(P)) = \max\{\|p_i - p_j\| \mid i, j \in \{1, \dots, n\}\}$$

Let $\mathsf{diag}((\mathbb{R}^d)^n) = \{(p, \dots, p) \in (\mathbb{R}^d)^n \mid p \in \mathbb{R}^d\}$

Lemma

The function $V_{\mathrm{diam}} = \mathsf{diam} \circ \mathsf{co} \colon (\mathbb{R}^d)^n \to \overline{\mathbb{R}}_+$ *verifies:*

1. V_{diam} *is continuous and invariant under permutations;*
2. $V_{\mathrm{diam}}(P) = 0$ *if and only if* $P \in \mathsf{diag}((\mathbb{R}^d)^n)$*;*
3. V_{diam} *is non-increasing along* $T_{\mathcal{CC}}$

Correctness via LaSalle Invariance Principle

To recap

1. $T_{\mathcal{CC}}$ is closed
2. $V = \mathsf{diam}$ is non-increasing along $T_{\mathcal{CC}}$
3. Evolution starting from P_0 is contained in $\mathsf{co}(P_0)$ (compact and strongly invariant)

Correctness via LaSalle Invariance Principle

To recap

1. $T_{\mathcal{CC}}$ is closed
2. $V = \mathsf{diam}$ is non-increasing along $T_{\mathcal{CC}}$
3. Evolution starting from P_0 is contained in $\mathsf{co}(P_0)$ (compact and strongly invariant)

Application of **LaSalle Invariance Principle:** trajectories starting at P_0 converge to M, largest weakly positively invariant set contained in

$$\{P \in \mathsf{co}(P_0) \mid \exists P' \in T_{\mathcal{CC}}(P) \text{ such that } \mathsf{diam}(P') = \mathsf{diam}(P)\}$$

Have to **identify** M! Ideally, $M = \mathsf{diag}((\mathbb{R}^d)^n) \cap \mathsf{co}(P_0)$

Clearly $\mathsf{diag}((\mathbb{R}^d)^n) \cap \mathsf{co}(P_0) \subset M$ – other inclusion by contradiction

LaSalle Invariance Principle – identifying M

Assume $P \in M \setminus (\mathsf{diag}((\mathbb{R}^d)^n) \cap \mathsf{co}(P_0))$, and therefore $\mathsf{diam}(P) > 0$
Let $\mathcal{G}$ be a connected directed graph and consider $T_{\mathcal{G}}(P)$

1. Trivially, all non-strictly convex vertices of $\mathsf{co}(P)$ will evolve towards a point in the $\mathsf{co}(P)$ which is not a strictly convex vertex
2. Same conclusion holds for strictly convex vertices, because graph is connected, and neighbors “pull agent out” from the vertex
3. Argument has to be conveniently extended to the case where there is more than one agent at a strictly convex vertex

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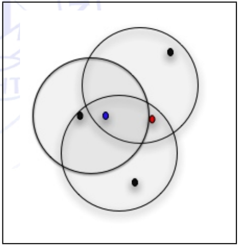
In any case, after a finite number of iterations, all agents in configuration $T_{G_1,r}(T_{G_2,r}(\dots T_{G_N,r}(P)))$ are contained in $\mathsf{co}(P) \setminus \mathsf{Ve}(\mathsf{co}(P))$
Therefore, $\mathsf{diam}(T_{G_1,r}(T_{G_2,r}(\dots T_{G_N,r}(P)))) < \mathsf{diam}(P)$, which contradicts M weakly invariant

Convergence to a point can be concluded with a little bit of extra work

Corollary: Circumcenter algorithm achieves rendezvous

Robustness of circumcenter algorithms

Push whole idea further!, e.g., for robustness against link failures



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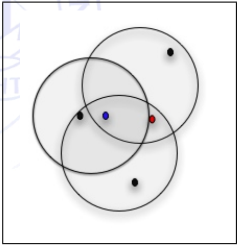

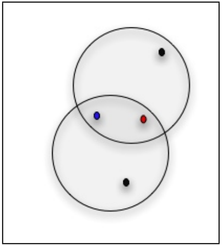

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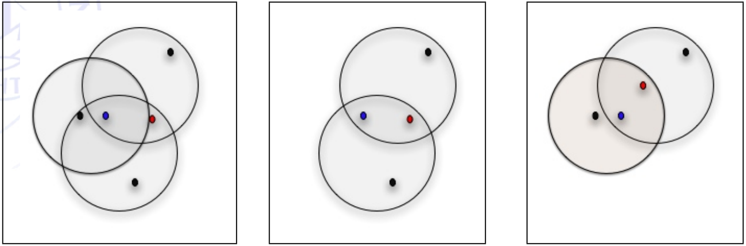

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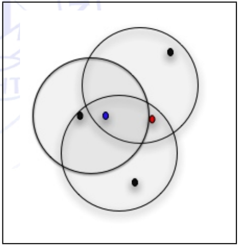
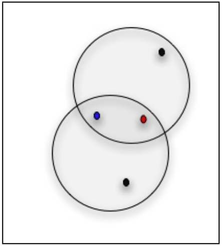
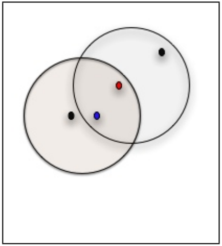


topology G_1



topology G_2



topology G_3

Look at **evolution under link failures** as outcome of nondeterministic evolution under multiple interaction topologies

$$P \longrightarrow \{\text{evolution under } G_1, \text{evolution under } G_2, \text{evolution under } G_3\}$$

Rendezvous

Corollary (Circumcenter algorithm over $\mathcal{G}_{\text{disk}}(r)$ on $\mathbb{R}^d$)

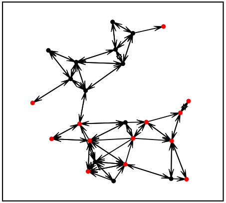
For $\{P_m\}_{m \in \mathbb{N}_0}$ synchronous execution with link failures such that union of any $\ell \in \mathbb{N}$ consecutive graphs in execution has globally reachable node

Then, there exists $(p^*, \dots, p^*) \in \mathsf{diag}((\mathbb{R}^d)^n)$ such that

$$P_m \to (p^*, \dots, p^*) \quad as \quad m \to +\infty$$

Proof uses

$$T_{\mathcal{CC},\ell}(P) = \{ f_{\mathcal{G}_\ell} \circ \dots \circ f_{\mathcal{G}_1}(P) \mid \cup_{s=1}^{\ell} \mathcal{G}_i \text{ has globally reachable node} \}$$



Beyond rendezvous: flocking

Average-heading algorithm $f_{\text{Ave}} : (\mathbb{R}^2 \times \mathbb{S}^1)^n \to (\mathbb{R}^2 \times \mathbb{S}^1)^n$ over $\mathcal{G}$

$$f_{\text{Ave},i}((p_1, \theta_1), \dots, (p_n, \theta_n)) = \big(p_i + (\cos\theta_i, \sin\theta_i), \\ \text{Average}(\theta_i \cup \{\theta_j \mid (p_j, \theta_j) \in \mathcal{N}_{\mathcal{G}}(p_i, \theta_i)\})\big)$$

Corollary (Averaging algorithm over $\mathcal{G}$ on $\mathbb{R}^2 \times \mathbb{S}^1$)

For $\{(P_m, \Theta_m)\}_{m \in \mathbb{N}_0}$ synchronous execution with link failures such that union of any $\ell \in \mathbb{N}$ consecutive graphs has globally reachable node

Then, there exists $(\theta^*, \dots, \theta^*) \in \text{diag}((\mathbb{S}^1)^n)$ such that

$$\Theta_m \to (\theta^*, \dots, \theta^*) \quad \textit{as} \quad m \to +\infty$$

$V : (\mathbb{R}^2 \times \mathbb{S}^1)^n \to \mathbb{R}$ is $\max \theta_i - \min \theta_i$

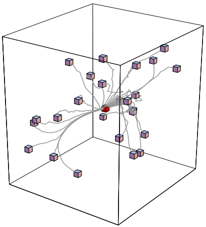
Rendezvous: example complexity analysis

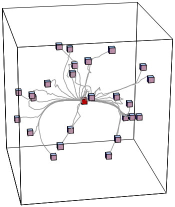
1. first-order agents with disk graph, for $d = 1$,

$$\mathsf{TC}(\mathcal{T}_{\text{rendezvous}}, \mathcal{CC}_{\text{circumcenter}}) \in \Theta(n)$$

2. first-order agents with limited Delaunay graph, for $d = 1$,

$$\mathsf{TC}(\mathcal{T}_{(r\epsilon)\text{-rendezvous}}, \mathcal{CC}_{\text{circumcenter}}) \in \Theta(n^2 \log(n\epsilon^{-1}))$$





Complexity analysis via tridiagonal Toeplitz and circulant matrices

Summary and conclusions

Rendezvous task

1. Designed circumcenter algorithms
2. Analyzed convergence via nondeterministic dynamical systems
3. Established robustness properties

Set of ideas can be further developed to provide broadly applicable tools for correctness and robustness analysis beyond rendezvous

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