Controllability of Navier-Stokes equations by means of finite-dimensional forcing

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   - The evolutionary equation
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The equations

The Navier-Stokes system in a 2D bounded domain $\Omega \subseteq \mathbb{R}^2$ with boundary Γ reads:

$$\begin{aligned} u_t + (u \cdot \nabla)u + \nabla p = -\nu \Delta u + F(x_1, x_2) + v(t, x_1, x_2) \quad \text{in} \quad \Omega; \\ \nabla \cdot u = 0 \quad \text{in} \quad \Omega; \\ u(0) = u_0. \end{aligned}$$

Nota: $\Delta \equiv -\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)$.

Boundary conditions

We consider either

- Navier boundary conditions

$$u \cdot \mathbf{n} = 0 \quad \text{on} \quad \Gamma;$$
$$\nabla^{\perp} \cdot u = \beta u \cdot \mathbf{t} \quad \text{on} \quad \Gamma$$

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- or no-slip boundary conditions

$$u = 0 \quad \text{on} \quad \Gamma.$$

Degenerated controls

We want to control the N-S equations

$$u_t = -\nu\Delta u - (u\cdot\nabla)u - \nabla p + F(x_1, x_2) + v(t, x_1, x_2) \quad \text{in} \quad \Omega;$$
$$\nabla\cdot u = 0 \quad \text{in} \quad \Omega;$$

using essentially bounded degenerated controls: $v = \sum_{i=1}^{n} v_i(t)\Phi_i(x_1,\, x_2)$ taking values in a finite-dimensional subspace of divergence free vector fields

$$span\{\Phi_i(x_1,\, x_2) \mid i = 1,\, \ldots,\, n\}.$$

- For Navier boundary conditions

$$
\begin{aligned}
H &:= \{u \in L^2(T\Omega) \mid \nabla \cdot u = 0 \;\;\&\;\; u \cdot \mathbf{n} = 0 \text{ on } \Gamma\};\\
V &:= \{u \in H^1(T\Omega) \mid \; u \in H\};\\
D(A) &:= \{u \in H^2(T\Omega) \mid \; u \in V \;\;\&\;\; \nabla^{\perp} \cdot u = \beta u \cdot \mathbf{t} \text{ on } \Gamma\}.
\end{aligned}
$$

Sobolev subspaces

- For Navier boundary conditions

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- For no-slip boundary conditions

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H &:= \{u \in L^2(T\Omega) \mid \nabla \cdot u = 0 \quad \& \quad u \cdot \mathbf{n} = 0 \text{ on } \Gamma\};\\
V &:= \{u \in H^1(T\Omega) \mid u \in H \quad \& \quad u = 0 \text{ on } \Gamma\};\\
D(A) &:= \{u \in H^2(T\Omega) \mid u \in V\}.
\end{aligned}
$$

We write the equations

$$u_t = -\nu\Delta u - (u \cdot \nabla)u - \nabla p + F(x_1, x_2) + v(t, x_1, x_2) \quad \text{in} \quad \Omega;$$
$$\nabla \cdot u = 0 \quad \text{in} \quad \Omega;$$

as an evolutionary equation in the space H of divergence free vector fields H:

$$u_t = -\nu Au - Bu + \nu Cu + F + v.$$

DEFINITION

A finite subset $g \subset V$ of vector fields, is said V-**saturating** if the sequence $(G^j)_{n\in\mathbb{N}}$ of finite-dimensional subspaces $G^j \subset V$ defined recursively by

1. $G^0 := span(g)$;
2. $G^{j+1} := \left(G^j + Conv\overline{\{BY \mid Y \in G^j\} \cap V}\right) \cap \left(G^j - Conv\overline{\{BY \mid Y \in G^j\} \cap V}\right)$

satisfies $\overline{\bigcup_{i\in\mathbb{N}} G^i} = H$.

$[\ BY = B(Y, Y)\ ]$

$L^2(T\Omega)$-approximate controllability

Proposition

Let $g \subset V$ be a V-saturating set. Then for each $u_0 \in V$ and $T > 0$, The attainable set at time T from u_0, of the N-S system controlled by controls in $span\{g\}$ is dense in H.

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Idea of proof:

Step 1: Given another point u_1, we can drive the equation from u_0 to any given small neighborhood of u_1 if, we use controls in G^N with big enough N.

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Step 2: We imitate the action of controls in G^j by the action of controls in G^{j-1}; $1 \leq j \leq N$.

Reduction to piecewise constant controls

- The solution of the N-S equation varies continuously when the control varies in the so-called relaxation metric:
$|v|_{rx} = \sup_{a,b\in[0,\,T]} \left| \int_a^b v\, dt \right|_{(\mathbb{R}^m,\, l_1)}$ [for finite-dimensional controls $v = \sum_{i=1}^m v_i(t)\Psi_i(x_1,\, x_2)$];

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- A given (essentially bounded) control taking values in $G^N \subseteq Conv\left(G^{N-1} - \overline{B(G^{N-1}) \cap V}\right)$, can be approximated, in relaxation metric, by a piecewise constant one taking values in $G^{N-1} - \overline{B(G^{N-1}) \cap V}$;

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- The next step is imitation, in each interval of constancy, of the respective (constant) control by another one taking values in G^{N-1}.

Imitation procedure

Given a constant control $\underline{e_i - c_i}$ in $G^N \subseteq G^{N-1} - \overline{B(G^{N-1}) \cap V}$ in a given interval $]t_i, t_i + \tau[$:

1. Approximate, in H, $e_i - c_i$ by a control $e_i - B\tilde{f}_i$ in $G^{N-1} - [B(G^{N-1}) \cap V]$;

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2. Approximate $\tilde{f}_i \in V$ by $f_i \in D(A)$ in V-norm; so $e_i - B\tilde{f}_i$ is approximated by $e_i - Bf_i$ (in V');

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3. $e_i + \sqrt{2}\phi_t^w f_i$ e $e_i - Bf_i$ drive the N-S system to points that are close at final instant time $t_f = t_i + \tau$ (for big enough w), where $\phi^w \sim \sin(wt)$ and $\phi^w(t_i) = 0 = \phi^w(t_f)$;

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2. Approximate $\tilde{f}_i \in V$ by $f_i \in D(A)$ in V-norm; so $e_i - B\tilde{f}_i$ is approximated by $e_i - Bf_i$ (in V');
3. $e_i + \sqrt{2}\phi_t^w f_i$ e $e_i - Bf_i$ drive the N-S system to points that are close at final instant time $t_f = t_i + \tau$ (for big enough w), where $\phi^w \sim \sin(wt)$ and $\phi^w(t_i) = 0 = \phi^w(t_f)$;
4. Finally the controls $e_i + \sqrt{2}\phi_t^w f_i$ and $e_i + \sqrt{2}\phi_t^w \tilde{f}_i \in G^{N-1}$ give close solutions (for any w).

Examples: Torus $\mathbb{T}^2$

Proposition

Agrachev-Sarychev. Periodic conditions. The set of vector fields

$$g := \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(x_1), \begin{pmatrix} 0 \\ -1 \end{pmatrix} \cos(x_1), \begin{pmatrix} -1 \\ 1 \end{pmatrix} \sin(x_1 + x_2), \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(x_1 + x_2) \right\}.$$

is V-saturating.

Proposition

Rectangle under Navier boundary conditions. The set of vector fields

$$g := \left\{ W_{(n_1,\, n_2)} \mid (n_1,\, n_2) \in \mathbb{N}_0^2;\, n_1,\, n_2 \leq 3;\, (n_1,\, n_2) \neq (3,3) \right\}$$

is V*-saturating.* $[\ \#g = 8\ ]$

$$W_k = W_{(k_1,\, k_2)} = \begin{pmatrix} \frac{-k_2\pi}{b} \sin\left(\frac{k_1\pi x_1}{a}\right) \cos\left(\frac{k_2\pi x_2}{b}\right) \\ \frac{k_1\pi}{a} \cos\left(\frac{k_1\pi x_1}{a}\right) \sin\left(\frac{k_2\pi x_2}{b}\right) \end{pmatrix}.$$

Examples: Sphere $\mathbb{S}^2$

Proposition

Agrachev-Sarychev. Sphere. The set of vector fields

$$g = \left\{ \frac{1}{2}\vec{e}_1,\ \frac{1}{2}\vec{e}_2,\ \frac{1}{2}\vec{e}_3,\ -x_3\vec{e}_3,\ \frac{3}{4}(5x_3^2 - 1)\vec{e}_3 \right\}$$

is V-saturating. $\vec{e}_i$ denotes the vector field generating rotation around the $e_i = [\delta_{1i}, \delta_{2i}, \delta_{3i}]^\top$ axis, with constant angular velocity equal to 1 (and with sense $x \wedge e_i$).

Examples: Hemisphere

Proposition

Hemisphere $\mathbb{S}^2$ with $(x_3 > 0)$ under Navier boundary conditions. The set of vector fields $g = \left\{-\frac{1}{2}\vec{e_3},\ \frac{1}{6}(x_3^2\partial_2 - x_1\vec{e_3}),\ \frac{1}{12}\left(10x_1x_3^2\partial_2 - (5x_1^2 - 1)\vec{e_3}\right)\right\}$ is V-saturating.

Remark: We are considering the chart

$$(x_1,\, x_2) \mapsto \left(x_1,\, x_2,\, \sqrt{1 - x_1^2 - x_2^2}\right).$$

Writing the N-S equation as an infinite dimensional system of ODE's:

$$\dot{u}_k := -\nu\lambda_k u_k - B_k(u, u) + \nu C_k u + F_k + v_k; \qquad k \in \mathbb{N}_0 \qquad (1)$$

Definition

The $\mathcal{G}$-**Galerkin approximation** of system (1) is the same system with the additional condition $k, n, m \in \mathcal{G} \in \mathcal{FP}(\mathbb{N}_0)$.

Corollary

Rectangle: Let $k \in \mathcal{K}^N \leftrightarrow W_k \in G^N$. *The* $\mathcal{K}^N$*-Galerkin approximations*

$$\dot{u}_k = -\nu\lambda_k u_k - B_k(u, u) + \nu C_k u + F_k + (g^0 v)_k; \quad v \in \mathbb{R}^8, \ k \in K^N, \ u \in G^N;$$

are controllable at time T. g^0 *denotes the matrix whose columns are the* 8 *elements of the saturating set* g.

Analytic Perturbation. "Generic" domain

The Navier-Stokes equations in a Riemannian manifold (Ω, g) may be written

$$u_t = -\nu\Delta(g)u - \nabla^1(g)_u u + \nabla(g)p + F + v; \quad \nabla(g)\cdot u = 0.$$

- We connect two metrics (tensors) g_0 and g_1 in Ω by an analytic homotopy g_τ; $\tau \in [0, 1]$;
- We may transfer some partial results, from (Ω, g_0) to (Ω, g_τ) with τ arbitrarily close to 1, using classic results from the (Analytic) Perturbation Theory of Linear Operators; $\tau \mapsto \Delta(g_\tau)$.

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Thanks for your attention