Generalized nonholonomic systems and control of servomechanisms

Sergio D. Grillo

Instituto Balseiro - Centro Atómico Bariloche

Argentina

SYSTEMS OF N PARTICLES WITH CONSTRAINTS

Let us call $\mathbf{r}_i \in \mathbb{R}^3$ the position of the i-th particle of the system.
Of course, if we apply on it a ***known*** force $\mathbf{f}_i \in \mathbb{R}^3$, then (if $m_i = 1$)

$$\ddot{\mathbf{r}}_i(t) = \mathbf{f}_i(\mathbf{r}_1(t), ..., \mathbf{r}_N(t), \dot{\mathbf{r}}_1(t), ..., \dot{\mathbf{r}}_N(t)),$$

i.e. curves $\mathbf{r}_i(t)$'s satisfy **Newton equations**.

SYSTEMS OF N PARTICLES WITH CONSTRAINTS

Let us call $\mathbf{r}_i \in \mathbb{R}^3$ the position of the i-th particle of the system.
Of course, if we apply on it a *known* force $\mathbf{f}_i \in \mathbb{R}^3$, then (if $m_i = 1$)

$$\ddot{\mathbf{r}}_i(t) = \mathbf{f}_i(\mathbf{r}_1(t), ..., \mathbf{r}_N(t), \dot{\mathbf{r}}_1(t), ..., \dot{\mathbf{r}}_N(t)),$$

i.e. curves $\mathbf{r}_i(t)$'s satisfy **Newton equations**. Writing

$$\mathbf{R} = (\mathbf{r}_1, ..., \mathbf{r}_N) \in \mathbb{R}^{3N} \quad and \quad \mathbf{F} = (\mathbf{f}_1, ..., \mathbf{f}_N) \in \mathbb{R}^{3N}$$

such equations reduce to

$$\ddot{\mathbf{R}}(t) = \mathbf{F}\left(\mathbf{R}(t), \dot{\mathbf{R}}(t)\right).$$

SYSTEMS OF N PARTICLES WITH CONSTRAINTS

Let us call $\mathbf{r}_i \in \mathbb{R}^3$ the position of the i-th particle of the system. Of course, if we apply on it a *known* force $\mathbf{f}_i \in \mathbb{R}^3$, then (if $m_i = 1$)

$$\ddot{\mathbf{r}}_i(t) = \mathbf{f}_i\left(\mathbf{r}_1(t), ..., \mathbf{r}_N(t), \dot{\mathbf{r}}_1(t), ..., \dot{\mathbf{r}}_N(t)\right),$$

i.e. curves $\mathbf{r}_i(t)$'s satisfy **Newton equations**. Writing

$$\mathbf{R} = (\mathbf{r}_1, ..., \mathbf{r}_N) \in \mathbb{R}^{3N} \qquad and \qquad \mathbf{F} = (\mathbf{f}_1, ..., \mathbf{f}_N) \in \mathbb{R}^{3N}$$

such equations reduce to

$$\ddot{\mathbf{R}}(t) = \mathbf{F}\left(\mathbf{R}(t), \dot{\mathbf{R}}(t)\right).$$

- $3N$ unknowns $\mathbf{R}(t)$, $3N$ normal ODE $\Longrightarrow$ existence and uniqueness of solutions.

Let us consider for this system a set of **constraints**

$$\omega_k\left(\mathbf{R},\dot{\mathbf{R}}\right)=0,\qquad k=1,...,K.$$

Let us consider for this system a set of **constraints**

$$\omega_k\left(\mathbf{R},\dot{\mathbf{R}}\right)=0,\qquad k=1,...,K.$$

Note: We say such constraints are **holonomic** if there exist functions ϕ_k such that

$$\omega_k\left(\mathbf{R},\dot{\mathbf{R}}\right)=\frac{\partial\phi_k}{\partial\mathbf{R}}\left(\mathbf{R}\right)\cdot\dot{\mathbf{R}}\qquad\left(i.e.\ \omega_k=\frac{d\phi_k}{dt}\right).$$

In this case last conditions can be derived from equations

$$\phi_k\left(\mathbf{R}\right)=cte,\qquad k=1,...,K.$$

Otherwise, we say the constraints are **nonholonomic**.

If we ask that $\mathbf{R}(t)$ satisfies the constraints, i.e.

$$\omega_k\left(\mathbf{R}(t),\dot{\mathbf{R}}(t)\right)=0, \qquad k=1,...,K,$$

we must add a force on the system, **the constraint force**, which we shall denote $\mathbf{F}_C$.

If we ask that $\mathbf{R}(t)$ satisfies the constraints, i.e.

$$\omega_k\left(\mathbf{R}(t), \dot{\mathbf{R}}(t)\right) = 0, \qquad k = 1, ..., K,$$

we must add a force on the system, **the constraint force**, which we shall denote $\mathbf{F}_C$. Then, a trajectory $\mathbf{R}(t)$ of the system must satisfy

$$\ddot{\mathbf{R}}(t) = \mathbf{F}\left(\mathbf{R}(t), \dot{\mathbf{R}}(t)\right) + \mathbf{F}_C(t) \quad and \quad \omega_k\left(\mathbf{R}(t), \dot{\mathbf{R}}(t)\right) = 0,$$

for some force $\mathbf{F}_C(t)$.

If we ask that $\mathbf{R}(t)$ satisfies the constraints, i.e.

$$\omega_k\left(\mathbf{R}(t),\dot{\mathbf{R}}(t)\right)=0,\qquad k=1,...,K,$$

we must add a force on the system, **the constraint force**, which we shall denote $\mathbf{F}_C$. Then, a trajectory $\mathbf{R}(t)$ of the system must satisfy

$$\ddot{\mathbf{R}}(t)=\mathbf{F}\left(\mathbf{R}(t),\dot{\mathbf{R}}(t)\right)+\mathbf{F}_C(t)\quad and\quad \omega_k\left(\mathbf{R}(t),\dot{\mathbf{R}}(t)\right)=0,$$

for some force $\mathbf{F}_C(t)$.

- $3N+3N=6N$ unknowns $\mathbf{R}(t)$ and $\mathbf{F}_C(t)$; $3N+K$ equations. It is not a good model for the physical system. We need more information.

D'ALEMBERT PRINCIPLE

Assume that constraints are linear in velocities, i.e.

$$\omega_k\left(\mathbf{R},\dot{\mathbf{R}}\right) = \boldsymbol{\omega}_k\left(\mathbf{R}\right)\cdot\dot{\mathbf{R}} = 0.$$

D'Alembert principle says that

$$\mathbf{F}_C\big|_{\mathbf{R}}\cdot\delta\mathbf{R} = \mathbf{0}, \quad \forall\delta\mathbf{R}\ /\ \boldsymbol{\omega}_k\left(\mathbf{R}\right)\cdot\delta\mathbf{R} = 0.$$

D'ALEMBERT PRINCIPLE

Assume that constraints are linear in velocities, i.e.

$$\omega_k\left(\mathbf{R},\dot{\mathbf{R}}\right)=\boldsymbol{\omega}_k\left(\mathbf{R}\right)\cdot\dot{\mathbf{R}}=0.$$

D'Alembert principle says that

$$\mathbf{F}_C\big|_{\mathbf{R}}\cdot\delta\mathbf{R}=\mathbf{0},\quad\forall\delta\mathbf{R}\ /\ \boldsymbol{\omega}_k\left(\mathbf{R}\right)\cdot\delta\mathbf{R}=0.$$

In other words, defining the distribution

$$\mathcal{C}_{\mathbf{R}}=\left\{\boldsymbol{\omega}_1\left(\mathbf{R}\right),...,\boldsymbol{\omega}_K\left(\mathbf{R}\right)\right\}^{\perp},\quad\mathbf{R}\in\mathbb{R}^{3N},$$

then $\mathbf{F}_C\big|_{\mathbf{R}}\in\mathcal{C}_{\mathbf{R}}^{\perp}\ \rightsquigarrow\ 3N-K$ equations.

D'ALEMBERT PRINCIPLE

Assume that constraints are linear in velocities, i.e.

$$\omega_k\left(\mathbf{R},\dot{\mathbf{R}}\right)=\boldsymbol{\omega}_k\left(\mathbf{R}\right)\cdot\dot{\mathbf{R}}=0.$$

D'Alembert principle says that

$$\mathbf{F}_C\big|_{\mathbf{R}}\cdot\delta\mathbf{R}=\mathbf{0},\quad \forall\delta\mathbf{R}\ /\ \boldsymbol{\omega}_k\left(\mathbf{R}\right)\cdot\delta\mathbf{R}=0.$$

In other words, defining the distribution

$$\mathcal{C}_{\mathbf{R}}=\left\{\boldsymbol{\omega}_1\left(\mathbf{R}\right),...,\boldsymbol{\omega}_K\left(\mathbf{R}\right)\right\}^{\perp},\quad \mathbf{R}\in\mathbb{R}^{3N},$$

then $\mathbf{F}_C\big|_{\mathbf{R}}\in\mathcal{C}_{\mathbf{R}}^{\perp}\quad\rightsquigarrow\quad 3N-K$ equations.

Note. Equation $\boldsymbol{\omega}_k\left(\mathbf{R}\left(t\right)\right)\cdot\dot{\mathbf{R}}\left(t\right)=0$ can be replaced by $\dot{\mathbf{R}}\left(t\right)\in\mathcal{C}_{\mathbf{R}(t)}$.

Equations of motions would be given by

$$\begin{cases} \ddot{\mathbf{R}}(t) = \mathbf{F}\left(\mathbf{R}(t), \dot{\mathbf{R}}(t)\right) + \mathbf{F}_C(t), \\ \dot{\mathbf{R}}(t) \in \mathcal{C}_{\mathbf{R}(t)}, \\ \mathbf{F}_C(t) \in \mathcal{C}^{\perp}_{\mathbf{R}(t)}. \end{cases}$$

Equations of motions would be given by

$$\begin{cases} \ddot{\mathbf{R}}(t) = \mathbf{F}\left(\mathbf{R}(t), \dot{\mathbf{R}}(t)\right) + \mathbf{F}_C(t), \\ \dot{\mathbf{R}}(t) \in \mathcal{C}_{\mathbf{R}(t)}, \\ \mathbf{F}_C(t) \in \mathcal{C}^{\perp}_{\mathbf{R}(t)}. \end{cases}$$

- $6N$ unknowns $\mathbf{R}(t)$ and $\mathbf{F}_C(t)$, and $3N + K + (3N - K) = 6N$ equations.

In Lagrangian terms we have a so-called **nonholonomic system**: a trajectory $\gamma : I \rightarrow Q$ of the system is given by

$$\mathcal{EL}\left(L\right)\left(\gamma^{(2)}\left(t\right)\right) = f\left(t\right), \quad \gamma'\left(t\right) \in \mathcal{C}_{\gamma(t)}, \qquad f\left(t\right) \in \mathcal{C}^{o}_{\gamma(t)},$$

where

$$L : TQ \rightarrow \mathbb{R}, \quad \mathcal{EL}\left(L\right) : T^{(2)}Q \rightarrow T^{*}Q$$

and $\mathcal{C} \subset TQ$ is a distribution on Q.

In Lagrangian terms we have a so-called **nonholonomic system**: a trajectory $\gamma : I \to Q$ of the system is given by

$$\mathcal{E}\mathcal{L}\left(L\right)\left(\gamma^{(2)}\left(t\right)\right) = f\left(t\right), \quad \gamma'\left(t\right) \in \mathcal{C}_{\gamma(t)}, \quad f\left(t\right) \in \mathcal{C}^{o}_{\gamma(t)},$$

where

$$L : TQ \to \mathbb{R}, \quad \mathcal{E}\mathcal{L}\left(L\right) : T^{(2)}Q \to T^{*}Q$$

and $\mathcal{C} \subset TQ$ is a distribution on Q.

- If L is regular we have existence and uniqueness of solutions.

Some questions:

- Is this principle always valid?

Some questions:

- Is this principle always valid?
  Systems with constraints implemented by the contact between punctual masses and rigid bodies usually satisfy the principle. But it is easy to build up systems which do not: for instance, using servomechanisms.

- What about more general constraints?

- What about more general constraints?
  For nonlinear constraints in velocities* and/or involving higher order derivatives

$$\omega_k\left(\mathbf{R},\dot{\mathbf{R}},\ddot{\mathbf{R}},...,\mathbf{R}^{(l)}\right)=0,\qquad k=1,...,K,$$

Chetaev's principle says that†

$$\mathbf{F}_C|_{\mathbf{R}}\cdot\delta\mathbf{R}=0,\qquad \forall\delta\mathbf{R}\ /\ \frac{\partial\omega_k}{\partial\mathbf{R}^{(l)}}\cdot\delta\mathbf{R}=\mathbf{0}.$$

Unfortunately, there do not exist interesting mechanical systems fulfilling this principle.

---

*Appell (**1911**). †Chetaev (**1934**); Valcovici (**1958**); Pironneau (**1983**).

- Why insist on deriving, by a universal procedure, the space of constraint forces from constraint itself?

- Why insist on deriving, by a universal procedure, the space of constraint forces from constraint itself?
  This is, probably, a consequence of a misunderstanding of the concept of virtual displacement, mainly in relation with variational principles.

GENERALIZED NONHOLONOMIC SYSTEMS

Idea: Consider the constraints and the space where constraint forces take their values as independent data, and do not attempt to derive one from another by a universal procedure [Dazord (1994); Marle (1998); Cendra et al (2004)].

Nonholonomic systems:

1. Data: $(L, \mathcal{C})$, $\mathcal{C} \subset TQ$ a distribution.
2. Equations of motion:

$$\mathcal{E}\mathcal{L}\left(L\right)\left(\gamma^{(2)}\left(t\right)\right) = f\left(t\right), \quad \gamma'\left(t\right) \in \mathcal{C}_{\gamma(t)}, \qquad f\left(t\right) \in \mathcal{C}^{o}_{\gamma(t)}.$$

GENERALIZED NONHOLONOMIC SYSTEMS

Idea: Consider the constraints and the space where constraint forces take their values as independent data; and do not attempt to derive one from another by a universal procedure [Dazord (1994); Marle (1998); Cendra et al (2004)].

Generalized nonholonomic systems (GNHS):

1. Data: $(L, \mathcal{C}, \mathcal{F})$, $\mathcal{C} \subset TQ$, $\mathcal{F} \subset T^*Q$ a codistribution.
2. Equations of motion:

$$\mathcal{EL}\left(L\right)\left(\gamma^{(2)}\left(t\right)\right) = f\left(t\right), \quad \gamma'\left(t\right) \in \mathcal{C}_{\gamma(t)}, \qquad f\left(t\right) \in \mathcal{F}_{\gamma(t)}.$$

GENERALIZED NONHOLONOMIC SYSTEMS

Idea: Consider the constraints and the space where constraint forces take their values as independent data; and do not attempt to derive one from another by a universal procedure [Dazord (1994); Marle (1998); Cendra et al (2004)].

Generalized nonholonomic systems (GNHS):

1. Data: $(L, \mathcal{C}, \mathcal{F})$, $\mathcal{C} \subset TQ$, $\mathcal{F} \subset T^*Q$ a codistribution.
2. Equations of motion:

$$\mathcal{EL}\left(L\right)\left(\gamma^{(2)}\left(t\right)\right) = f\left(t\right), \quad \gamma'\left(t\right) \in \mathcal{C}_{\gamma(t)}, \quad f\left(t\right) \in \mathcal{F}_{\gamma(t)}.$$

D'Alembert principle: $\mathcal{F} = \mathcal{C}^o$.

In general,*

1. Data: $(L, \mathcal{C}, \mathcal{F})$, $\mathcal{C} \subset T^{(k)}Q$, $\mathcal{F} \subset T^{(l)}Q \times_Q T^*Q$.
2. Equations of motion:

$$\mathcal{EL}\left(L\right)\left(\gamma^{(2)}\left(t\right)\right) = f\left(t\right), \quad \gamma^{(k)}\left(t\right) \in \mathcal{C}_{\gamma(t)}, \quad \left(\gamma^{(l)}\left(t\right), f\left(t\right)\right) \in \mathcal{F}_{\gamma(t)}.$$

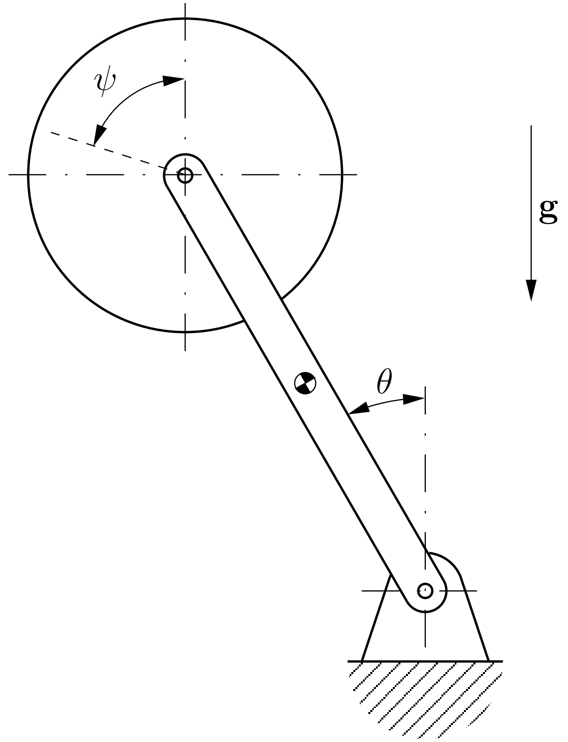
Examples: Elastic rolling bodies (like pneumatic tires), systems with friction forces, servomechanisms.

---

*Cendra et al (**2004**); Cendra & Grillo (**2007**).

CONTROL OF SERVOMECHANISMS

Inertia wheel pendulum: $Q = S^1 \times S^1$



- Lagrangian: $L = \frac{1}{2}\, I\, \dot{\theta}^2 + \frac{1}{2}\, J\, \left(\dot{\theta} + \dot{\psi}\right)^2 - M\, g\, (1 + \cos\theta)$.
- Space of actuators: $\mathcal{F}_{(\theta,\psi)} = \{(f_\theta, f_\psi) : f_\theta = 0\} \subset T^*_{(\theta,\psi)}Q$.

Every triple $(L, \mathcal{C}, \mathcal{F})$ defines a Lagrangian system with external forces (L, u), with $u : TQ \to T^*Q$ such that $\mathrm{Im}\,(u) \subset \mathcal{F}$. The pair (L, u) can be interpreted as a **closed-loop mechanical system (CLMS)**, being u the *control law*. Then, given an underactuated system $(L, \mathcal{F})$ and a set of constraints $\mathcal{C}$, we have a CLMS:

$$(L, \mathcal{F}) \oplus \mathcal{C} \rightsquigarrow (L, \mathcal{C}, \mathcal{F}) \rightsquigarrow (L, u)\,, \quad \mathrm{Im}\,(u) \subset \mathcal{F}.$$

Every triple $(L, \mathcal{C}, \mathcal{F})$ defines a Lagrangian system with external forces (L, u), with $u : TQ \to T^*Q$ such that $\mathrm{Im}\,(u) \subset \mathcal{F}$. The pair (L, u) can be interpreted as a **closed-loop mechanical system (CLMS)**, being u the *control law*. Then, given an underactuated system $(L, \mathcal{F})$ and a set of constraints $\mathcal{C}$, we have a CLMS:

$$(L, \mathcal{F}) \oplus \mathcal{C} \rightsquigarrow (L, \mathcal{C}, \mathcal{F}) \rightsquigarrow (L, u)\,, \quad \mathrm{Im}\,(u) \subset \mathcal{F}.$$

Idea: In order to stabilize a given underactuated system $(L, \mathcal{F})$, fix a set of constraint $\mathcal{C}$ that make stabilization possible, and derive control law u as the related constraint force [Marle (1998); Shiriaev, Perram, Canudas-de-Wit (2005)].

Example (inertia wheel pendulum): We can consider the kinematic constraints

$$\mathcal{C}_{(\theta,\psi)} = \left\{ \left(\dot\theta, \dot\psi \right) : \dot\theta + a\,\dot\psi = b\,\sin\theta \right\},$$

and implement it by using the actuators (i.e. a torque on the disc), so

$$\mathcal{F}_{(\theta,\psi)} = \{ (f_\theta, f_\psi) : f_\theta = 0 \}\,.$$

Example (inertia wheel pendulum): We can consider the kinematic constraints

$$\mathcal{C}_{(\theta,\psi)} = \left\{ \left(\dot{\theta}, \dot{\psi}\right) : \dot{\theta} + a\,\dot{\psi} = b\,\sin\theta \right\},$$

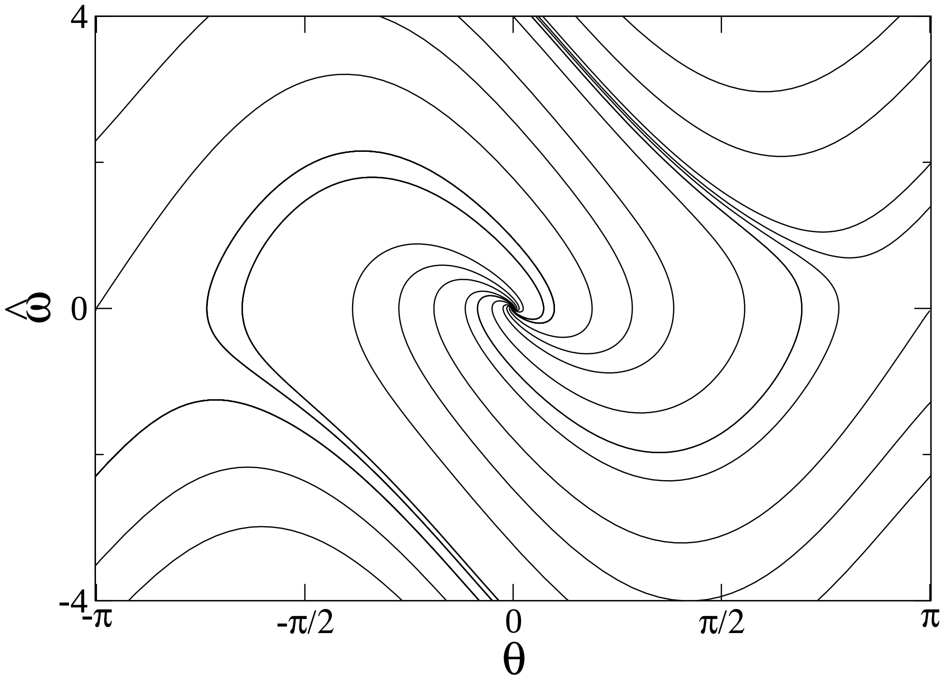
and implement it by using the actuators (i.e. a torque on the disc), so

$$\mathcal{F}_{(\theta,\psi)} = \{(f_\theta, f_\psi) : f_\theta = 0\}\,.$$

This gives a CLMS with $u = (u_\theta, u_\psi) = (0, u_\psi)$, where

$$u_\psi\left(\theta, \psi, \dot{\theta}, \dot{\psi}\right) = \frac{(I\,J\,b/a)\;\dot{\theta}\;\cos\theta + (1 - 1/a)\;M\,g\,J\;\sin\theta}{I + J - J/a}\,.$$

For certain values of a and b we have quasi-global asymptotic stability:



Here, $\widehat{\omega}$ is a variable proportional to $\dot{\theta}$ [D. Pérez (2006)].

Some comments:

- Is this method systematic?

Some comments:

- Is this method systematic?

It is by now just an alternative.

Some comments:

- Is this method systematic?

It is by now just an alternative.

- Any CLMS can be constructed from a GNHS; i.e. any control law can be seen as the constraint force related to a given set of constraints?

Some comments:

- Is this method systematic?

It is by now just an alternative.

- Any CLMS can be constructed from a GNHS; i.e. any control law can be seen as the constraint force related to a given set of constraints?

It can be shown that every CLMS is equivalent to a second order GNHS. So, there is a deep connection between CLMS and constrained mechanical systems.