EDS and Dirac theory of constraints

Existence theorem for a field theory

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I Encuentro Iberoamericano de Geometría, Mecánica y Control, Santiago de Compostela, 2008

The talk

1. Introduction
   - An outline
   - Notations and definitions
2. A Field Theory with Singular Constraints
   - Usual Theory, from Gotay's notes.
3. Equations of motion and EDS
   - Non-standard variational problem
4. EDS analysis
   - Polar spaces associated to some integral elements
   - Solutions from the Cartan-Kähler theorem

Where we are...

Problems and an answer

Problem

The G-N algorithm applied to the field theory consisting of an electromagnetic field and a spin 1 charged particle in interaction has the following annoying behavior: It stops or not depending on the values of the fields in space-time points. That is, the number of constraints obtained via this method depends on the values of the fields.

What we want to do...

We want to consider the equations of motion of this field theory as the generators of some suitable exterior differential system (called EDS from now on.) The integral manifolds for this EDS induces solutions of the field equations, so we can use the Cartan-Kähler theorem in order to obtain sufficient conditions for the existence of solutions.

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Where we are...

Things that we need to know... (from [4, 1])

- For every bundle $Z \to M$, we will denote by $\Gamma(Z)$ its space of sections.

Definitions

- An EDS $\mathcal{I}$ on X is a differential ideal of $\Omega^\bullet(X)$.
- If $N \subset X$ satisfies $\alpha|N = 0$ for all $\alpha \in \mathcal{I}$, then N is an *integral submanifold* of $\mathcal{I}$.
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- $r(E) := \dim(H(E)) - k - 1$, where $k = \dim(E)$.
- A k-integral element E for $\mathcal{I}$ is:
  - *Ordinary* if there exists an open neighborhood $E \in \mathcal{U} \subset G_k(TX)$ such that $\mathcal{U} \cap V_k(\mathcal{I})$ is a submanifold of $G_k(TX)$; they are denoted by $V_k^o(\mathcal{I})$.
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Cartan-Kähler theorem

Theorem (Cartan-Kähler)

Let $\mathcal{I}$ be a real analytic EDS on X and suppose that $P \subset X$ is a connected, k-dimensional, real analytic and regular integral manifold for $\mathcal{I}$ such that $r(P) \geq 0$. Then there exists a (non-unique in general!) connected $k+1$-dimensional, real analytic integral manifold M for $\mathcal{I}$ such that $P \subset M$.

Where we are...

Lagrangian Side - Canonical Analysis

- Field theory on $J^1(\pi)$, where $\pi : \Lambda_1 := T^*_{\mathbb{C}} M \oplus T^* M \to M$, with lagrangian (here $\Box_\alpha = \partial_\alpha - iA_\alpha$)

$$\mathcal{L}(\omega, A, \partial\omega, \partial A) := -\frac{1}{2}(\Box_\alpha \omega_\beta)\left(\bar{\Box}^\alpha \bar{\omega}^\beta - \bar{\Box}^\beta \bar{\omega}^\alpha\right) - \\ - \frac{1}{2}(\partial_\alpha A_\beta)\left(\partial^\alpha A^\beta - \partial^\beta A^\alpha\right) - \frac{1}{2} m^2 \omega_\alpha \omega^\alpha.$$

- Admitting a decomposition $M = \Sigma \times \mathbb{R}$, then $Q := \Gamma(\pi|\Sigma \times \{\tau\})$ and

$$\begin{aligned} Q &:= \{(\omega_0, \theta, A_0, a)\} \\ TQ &:= \{(\omega_0, \theta, A_0, a; \partial_0\omega_0, \partial_0\theta, \partial_0 A_0, \partial_0 a)\} \\ T^*Q &:= \{(\omega_0, \theta, A_0, a; \Pi_0, \pi, E_0, e)\}. \end{aligned}$$

- In these coordinates the Legendre transform is given by (here $\Box = \nabla - ia$)

$$\mathbb{F}\mathcal{L} = \begin{cases} \Pi_0 = 0 \\ \pi = \frac{1}{2}\left(\bar{\Box}_0 \bar{\theta} - \bar{\Box}\bar{\omega}_0\right) \\ E_0 = 0 \\ e = \partial_0 a - \nabla A_0. \end{cases}$$

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Canonical Analysis: Constraint Submanifolds.

- Primary Constraint Submanifold:
  $M_1 := \{(\omega_0, \theta, A_0, a; \Pi_0, \pi, E_0, e) : \Pi_0 = E_0 = 0\}$.
- Secondary Constraint Submanifold:
  $M_2 := \left\{(\omega_0, \theta, A_0, a; \pi, e) : \nabla \cdot e = i\left(\pi \cdot \theta - \bar{\pi} \cdot \bar{\theta}\right) \text{ y } \bar{\Box} \cdot \pi = -m^2 \omega_0\right\}$. If $m \neq 0$ the algorithm stops here.
- Tertiary Constraint Submanifold:
  $M_3 := \left\{(\omega_0, \theta, A_0, a; \pi, e) \in M_2 : e \cdot \pi = \frac{1}{2}(\nabla \times a) \cdot \left(\bar{\Box} \times \bar{\theta}\right)\right\}$.
- Quaternary Constraint Submanifold:
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Canonical Analysis: Termination of the Algorithm?

Problem!

There will be no more constraints unless

$$\det \begin{pmatrix} \pi \cdot \bar{\pi} + \frac{1}{2} (\nabla \times A)^2 & -\pi \cdot \pi \\ \bar{\pi} \cdot \bar{\pi} & -\pi \cdot \bar{\pi} - \frac{1}{2} (\nabla \times A)^2 \end{pmatrix} = 0.$$

By Schwarz inequality, a point $x \in \Sigma \times \{\tau\}$ will obey the previous condition iff

1. $(\nabla \times A)(x) = 0$, AND
2. $\pi(x)$ and $\bar{\pi}(x)$ are linearly dependent on $\mathbb{C}$.

So M_4 is divided in two disjoint sets, a set R_4 where $\forall x \in \Sigma \times \{\tau\}$ some of the previous condition is not fulfilled, and so where the algorithm terminates, and the complementary set, where the algorithm can continue!

Where we are...

Standard variational problem for mechanics

Classical Mechanics: Formal viewpoint [3]

- **Unknowns:** Sections

$$\phi : I \to I \times Q, \qquad I \subset \mathbb{R}$$

for the trivial bundle $I \times Q \to I$.

- **Prolongations:** The derivative respect to the time variable allow us to build

$$\text{pr}\phi : I \to I \times TQ$$

for $I \times TQ \to I$, such that $\text{pr}\phi$ covers to ϕ. Let us define

$$\mathcal{K} := \left\langle \theta^i \right\rangle_{\text{diff}}, \qquad \theta^i := \mathbf{d}q^i - \dot{q}^i \mathbf{d}t;$$

$\text{pr}\phi$ can be defined as the unique section of $I \times TQ$ such that $\text{pr}^*(\phi)(\mathcal{K}) = 0$.

- **Action:** For $L \in C^\infty(I \times TQ)$, we can define the action via

$$S[\phi] := \int_I (\text{pr}\phi)^*(L\mathbf{d}t).$$

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- **Action:** For $L \in C^\infty(I \times TQ)$, we can define the action via

$$S[\phi] := \int_I (\mathrm{pr}\phi)^*(L\mathbf{d}t).$$

Standard variational problem for mechanics

Classical Mechanics: Formal viewpoint [3]

- **Unknowns:** Sections

$$\phi : I \to I \times Q, \qquad I \subset \mathbb{R}$$

for the trivial bundle $I \times Q \to I$.

- **Prolongations:** The derivative respect to the time variable allow us to build

$$\mathrm{pr}\phi : I \to I \times TQ$$

for $I \times TQ \to I$, such that $\mathrm{pr}\phi$ covers to ϕ. Let us define

$$\mathcal{K} := \left\langle \theta^i \right\rangle_{\mathrm{diff}}, \qquad \theta^i := \mathbf{d}q^i - \dot{q}^i\mathbf{d}t;$$

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Non-standard variational problem I

Our case

- **Unknowns:** Sections

$$\sigma : M \to \Lambda_1, \qquad M \text{ space-time}$$

where Λ_1 is the bundle of fields previously defined.

- **Velocity space:** Using the characterization for prolongation in terms of EDSs, let us define the bundle

$$\Lambda := T^*_{\mathbb{C}}M \oplus T^*M \oplus \bigwedge^2 (T^*_{\mathbb{C}}M) \oplus \bigwedge^2 (T^*M) \xrightarrow{p_1}$$
$$\xrightarrow{p_1} \Lambda_1 := T^*_{\mathbb{C}}M \oplus T^*M \xrightarrow{\pi} M.$$

which has the canonical forms

$$\theta_A\big|_{(\omega,A,G,F)} := A \circ (p_1 \circ \pi)_*$$
$$\Theta_F\big|_{(\omega,A,G,F)} := F \circ (p_1 \circ \pi)_*, \qquad \forall (\omega, A, G, F) \in \Lambda$$

and so on.

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Non-standard variational problem II

Our case

- **Prolongations:** Then

$$\mathcal{I} := \langle \alpha_1 := \mathbf{d}\theta_A - \Theta_F, \Omega_1 := \mathbf{d}\theta_\omega - \Theta_G - i\theta_A \wedge \theta_\omega \rangle_{\text{diff}}$$

  will be the EDS which allow us to prolong sections $\sigma \in \Gamma(\Lambda_1)$:
  1. $(\text{pr}\sigma)^* (\mathcal{I}) = 0$, and
  2. $\text{pr}\sigma$ covers to σ.

- **Action:** We can define the action for our problem via

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Admissible Variations

- **Classical mechanics:** The admissible variations to the section $\phi : I \to I \times Q$ are sections
$$\delta\phi : t \mapsto (0; \delta q^i(t), \delta\dot{q}^i(t))$$
for the pullback bundle $\phi^*(V(I \times TQ))$ such that they admits an extension $\widehat{\delta v} \in \mathfrak{X}(I \times TQ)$ which is a symmetry for the EDS $\mathcal{K}$:
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In coordinates this is equivalent to $\delta\dot{q}^i = \dot{\overline{\delta q^i}}$.
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Euler-Lagrange equations

- **Classical mechanics:** Performing admissible variations on $I \times TQ$ we obtain the Euler-Lagrange equations of motion

$$\phi^*(\delta\phi \lrcorner \mathbf{d}(L\mathbf{d}t)) = 0, \qquad \forall \delta\phi \text{ admissible variation.}$$

In coordinates

$$\begin{cases} \mathbf{d}q^i - \dot{q}^i \mathbf{d}t = 0 \\ \mathbf{d}L_{\dot{q}^i} - L_{q^i}\mathbf{d}t = 0. \end{cases}$$

- **Our case:** By doing variations in the same way, we can obtain the following set of forms which must restricts to zero on the solutions of the variational problem:

$$\begin{aligned} \Gamma_1 &:= \mathbf{d}(\Theta_{*G}) - i\theta_A \wedge \Theta_{*G}, \\ \Phi_1 &:= \mathbf{d}(\Theta_{*F}) - (i/2)(\theta_\omega \wedge \Theta_{*\bar{G}} - \theta_{\bar{\omega}} \wedge \Theta_{*G}), \\ \Omega_1 &:= \mathbf{d}\theta_\omega - \Theta_G - i\theta_A \wedge \theta_\omega, \\ \alpha_1 &:= \mathbf{d}\theta_A - \Theta_F. \end{aligned}$$

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Relation with the initial problem

Proposition

Every solution of the standard Euler-Lagrange equations of the field theory considered here induce a 4-dimensional integral submanifold for the EDS generated by the previous forms. Conversely, every 4-dimensional integral submanifold $\mathcal{S}$ for this EDS which satisfies the transversality condition

$$\mathbf{d}x^0 \wedge \cdots \wedge \mathbf{d}x^3|\mathcal{S} \neq 0$$

(where $(x^0, \cdots, x^3)$ are local coordinates for the space-time) induces a solution for the E-L equations.

Hamilton-Cartan equations (from [2])

Classical Mechanics [3]

- On $Y := C \times (TQ \oplus T^*Q)$

$$\phi^* \left(W \lrcorner \mathbf{d}\Lambda \right) = 0, W \in V(Y),$$

where

  - $H := p_i \dot{q}^i - L$, and
  - $\Lambda = L\mathbf{d}t + p_i \theta^i = -H\mathbf{d}t + p_i \mathbf{d}q^i$.

- Then

$$\begin{cases} \mathbf{d}q^i - H_{p_i}\mathbf{d}t = 0 \\ \mathbf{d}p^i + H_{q^i}\mathbf{d}t = 0 \\ H_{\dot{q}^i}\mathbf{d}t = 0. \end{cases}$$

Our system

- On $\tilde{\Lambda} := \Lambda \oplus \bigwedge^2 \left(T^*_{\mathbb{C}} M \right) \oplus \bigwedge^2 \left(T^* M \right)$ define the Cartan Form

$$\Xi := \mathcal{L} + \Theta_P \wedge \alpha_1 + \Theta_{\bar{Q}} \wedge \Omega_1 + \Theta_Q \wedge \bar{\Omega}_1.$$

- The EDS on $\tilde{\Lambda}$ is generated by $\{\alpha_1, \Omega_1\}$ and the forms

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(adding their differentials!)

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(adding their differentials!)

Equivalence of the EDS

Proposition

The projection

$$(\omega, A, G, F, Q, P) \in \tilde{\Lambda} \mapsto (\omega, A, G, F) \in \Lambda$$

maps solutions for the Hamilton-Cartan EDS to solutions of the Euler-Lagrange EDS; conversely, for every solution of the Euler-Lagrange EDS, the immersion

$$(\omega, A, G, F) \in \Lambda \mapsto (\omega, A, G, F, *G, *F) \in \tilde{\Lambda}$$

induces a solution of the Hamilton-Cartan EDS.

Where we are...

Generators of the EDS

- We are working with the EDS

$$\begin{cases} \Gamma_2' := F_{\mu\nu}\mathbf{d}G^{\mu\nu} + G_{\mu\nu}\mathbf{d}F^{\mu\nu} \\ \Gamma_1 = \mathbf{d}(\Theta_{*G}) - i\theta_A \wedge \Theta_{*G}, \\ \Phi_1 = \mathbf{d}(\Theta_{*F}) - (i/2)(\theta_\omega \wedge \Theta_{*\bar{G}} - \theta_{\bar{\omega}} \wedge \Theta_{*G}), \\ \Omega_1 = \mathbf{d}\theta_\omega - \Theta_G - i\theta_A \wedge \theta_\omega, \\ \alpha_1 = \mathbf{d}\theta_A - \Theta_F, \\ \Omega_2 = \mathbf{d}\Theta_G + i(\Theta_F \wedge \theta_\omega - \theta_A \wedge \Theta_G), \\ \alpha_2 = \mathbf{d}\Theta_F; \end{cases} \tag{1}$$

Calculation of polar spaces for some integral elements

- Let us take $E \in V_3(\mathcal{I})$ such that it is an integral element for the EDS $\langle \mathbf{d}x^0 \rangle$; using coordinates (x^0, x^i) we can write

$$\begin{aligned} F : &= (*e) \wedge \mathbf{d}x^0 + b \\ G : &= (*l) \wedge \mathbf{d}x^0 + m \\ A : &= A_0 \mathbf{d}x^0 + a \\ \omega : &= \omega_0 \mathbf{d}x^0 + \theta. \end{aligned}$$

- Let us take as element in $H(E)$ the following

$$\hat{Z} = (\partial_0; \mathcal{L}_X A_0, \mathcal{L}_X a, \mathcal{L}_X \omega_0, \mathcal{L}_X \theta, \mathcal{L}_X e, \mathcal{L}_X b, \mathcal{L}_X l, \mathcal{L}_X m);$$

we want to find the conditions that our EDS impose on it.

Equations defining the polar space associated to E

Equations defining E

$$\begin{cases} \mathbf{d}l - il \wedge a, \\ \mathbf{d}e + (i/2)(\bar{l} \wedge \theta - l \wedge \bar{\theta}), \\ \mathbf{d}\theta - ia \wedge \theta - m, \\ \mathbf{d}a - b, \\ \mathbf{d}m + i(b \wedge \theta - m \wedge a), \\ \mathbf{d}b, \\ \theta \wedge (*l \wedge *\bar{l}), \end{cases}$$

- Then $E \in V_3(\mathcal{I})$ is ordinary iff $\dim(\theta, l, \bar{l}) = 2$

Conditions on the element of $H(E)$

$$\begin{cases} \mathcal{L}_X l = i(A_0 l + a \wedge *m) - \mathbf{d}(*m), \\ \mathcal{L}_X e = -\mathbf{d}(*b) - \\ \qquad -\frac{i}{2}(\omega_0 \bar{l} - \bar{\omega_0} l - *\bar{m} \wedge \theta + *m \wedge \bar{\theta}), \\ \mathcal{L}_X \theta = *l + \mathbf{d}\omega_0 + i(A_0 \theta - \omega_0 a), \\ \mathcal{L}_X a = *e + \mathbf{d}A_0, \\ \mathcal{L}_X m = \mathbf{d}(*l) - i(\omega_0 b + *e \wedge \theta - A_0 m - *l \wedge a), \\ \mathcal{L}_X b = \mathbf{d}(*e), \\ *(\mathcal{L}_X b) \wedge m - *b \wedge \mathcal{L}_X m - \\ \qquad - *(\mathcal{L}_X e) \wedge l + *e \wedge \mathcal{L}_X l = 0. \end{cases} \tag{2}$$

Where we are...

Existence of solutions

- The integral element E will be regular iff the last equation for the polar space determines the function ω_0, because in this case the number $r(E)$ is maximal.
- So it will be the case iff $b \neq 0$ or $l, \bar{l}$ are linearly independent.
- If N is a 4-dimensional integral submanifold for $\mathcal{I}$ such that $(\mathbf{d}x^0 \wedge \cdots \wedge \mathbf{d}x^3)|N \neq 0$, it induces a local section $\sigma : U \to \Lambda$; for the given splitting $M = \Sigma \times \mathbb{R}$ and supposing that we can identify $\Lambda|\Sigma \times \{\tau\} \simeq L$ for every τ, the restriction of our solution to each slice $\Lambda|\Sigma \times \{\tau\}$ induces a time-dependent section of the bundle L.

Theorem

The application $F : \Gamma(L) \to T^*Q$ given by

$$(\omega_0, \theta, A_0, a, l, m, e, b) \mapsto (\omega_0, \theta, A_0, a, 0, -l, 0, e)$$

maps solutions of our EDS to solutions of the canonical eqs of motion on T^*Q.

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