THE RELATION BETWEEN COVARIANT AND FUNCTIONAL HAMILTON EQUATIONS

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Abstract

It is well-known that a decomposition of space-time into space + time allows us to pass from the covariant Euler-Lagrange equations to the functional Euler-Lagrange equations. By means of the functional Legendre transformation, these in turn give rise to the functional Hamilton equations. But it is also possible to use the covariant Legendre transformation to get covariant Hamilton equations (also known as the DeDonder-Weyl equations). In this work, we propose to complete the quadrangle by presenting a procedure for passing directly from the covariant to the functional Hamilton equations. Examples of this passage in field theory and continuum mechanics are discussed.

1 The Four Formulations of Field Theory

The relations between the four formulations of classical field theory are illustrated by the following diagram:

$$\begin{array}{ccccc}
\text{covariant} & & \text{covariant} & & \text{covariant} \\
\text{Lagrangian} & \longrightarrow & \text{Legendre} & \longrightarrow & \text{Hamiltonian} \\
\text{formalism} & & \text{transformation} & & \text{formalism} \\
\downarrow & & & & \downarrow \\
\text{space} + \text{time} & & & & \text{space} + \text{time} \\
\downarrow & & & & \downarrow \\
\text{functional} & & \text{functional} & & \text{functional} \\
\text{Lagrangian} & \longrightarrow & \text{Legendre} & \longrightarrow & \text{Hamiltonian} \\
\text{formalism} & & \text{transformation} & & \text{formalism}
\end{array}$$

What is missing in this picture is the passage from the covariant Hamilton equations

$$\partial_\mu \varphi^i \;=\; \frac{\partial H}{\partial \pi_i^\mu} \quad , \quad \partial_\mu \pi_i^\mu \;=\; -\,\frac{\partial H}{\partial \varphi^i}$$

to the functional Hamilton equations through a decomposition of space-time into space + time

$$\partial_t \varphi^i \;=\; \frac{\partial h}{\partial \pi_i} \quad , \quad \partial_t \pi_i \;=\; -\left(\frac{\partial h}{\partial \varphi^i} - \partial_a \left(\frac{\partial h}{\partial\, \partial_a \varphi^i} \right) \right)$$

where $H(x^\mu, \varphi^i, \pi_i^\mu)$ is the covariant Hamiltonian and $h(t, x^a, \varphi^i, \partial_a \varphi^i, \pi_i)$ is the functional Hamiltonian density.

2 Covariant Lagrangian Formalism

The space-time is $\mathbb{R}^{n+1}$ with coordinates x^μ. The space where the fields take their values is $\mathbb{R}^k$ with coordinates q^i. The space of linear maps from $\mathbb{R}^{n+1}$ to $\mathbb{R}^k$ is $\mathrm{Hom}(\mathbb{R}^{n+1}, \mathbb{R}^k)$ with coordinates q^i_μ.

A field is a smooth map $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}^k$ with components φ^i. A covariant Lagrangian density is a smooth function

$$L : \mathbb{R}^{n+1} \times \mathbb{R}^k \times \mathrm{Hom}(\mathbb{R}^{n+1}, \mathbb{R}^k) \longrightarrow \mathbb{R} \,.$$

Given a field φ we can define its action by

$$\mathcal{S}[\varphi] \;=\; \int_{\mathbb{R}^{n+1}} L(x^\mu, \varphi^i, \partial_\mu \varphi^i)\, d^{n+1}x \,.$$

The field φ is a critical point of $\mathcal{S}$ if and only if it satisfies the covariant Euler-Lagrange Equations, that is,

$$\frac{\delta\, \mathcal{S}}{\delta \varphi} \;=\; \frac{\partial L}{\partial \varphi^i} - \partial_\mu \left(\frac{\partial L}{\partial\, \partial_\mu \varphi^i} \right) \;=\; 0$$

where

$$\frac{\partial L}{\partial \varphi^i} \;=\; \frac{\partial L}{\partial q^i}(x, \varphi, \partial\varphi) \quad , \quad \frac{\partial L}{\partial\, \partial_\mu \varphi^i} \;=\; \frac{\partial L}{\partial q^i_\mu}(x, \varphi, \partial\varphi).$$

3 Covariant Hamiltonian Formalism

The covariant Legendre transformation of L is defined by

$$(x^\mu, q^i, q^i_\mu) \mapsto (x^\mu, q^i, p^\mu_i \;=\; \frac{\partial L}{\partial q^i_\mu}) \;.$$

The set of variables p^μ_i is independent whenever the the Lagrangian L is regular that is,

$$\det\left(\frac{\partial^2 L}{\partial q^i_\mu\, \partial q^j_\nu}\right) \;\neq\; 0 \;.$$

Granted this, we can use the implicit function theorem to write $q^i_\mu = q^i_\mu(x^\mu, q^j, p^\nu_j)$ and introduce the covariant Hamiltonian as

$$H(x^\mu, q^i, p^\mu_i) \;=\; p^\mu_i\, q^i_\mu - L(x^\mu, q^i, q^i_\mu) \;.$$

A simple computation gives

$$\frac{\partial H}{\partial p^\mu_i} \;=\; q^i_\mu + p^\nu_j \frac{\partial q^j_\nu}{\partial p^\mu_i} - \frac{\partial L}{\partial q^j_\nu}\frac{\partial q^j_\nu}{\partial p^\mu_i} \;=\; q^i_\mu$$

$$\frac{\partial H}{\partial q^i} \;=\; p^\nu_j \frac{\partial q^j_\nu}{\partial q^i} - \frac{\partial L}{\partial q^i} - \frac{\partial L}{\partial q^j_\nu}\frac{\partial q^j_\nu}{\partial q^i} \;=\; -\frac{\partial L}{\partial q^i} \;.$$

Given φ we set

$$\pi_i^\mu = \frac{\partial L}{\partial\, \partial_\mu \varphi^i},$$

and also

$$\frac{\partial H}{\partial \pi_i^\mu} = \frac{\partial H}{\partial p_i^\mu}(x, \varphi, \pi) \quad , \quad \frac{\partial H}{\partial \varphi^i} = \frac{\partial H}{\partial q^i}(x, \varphi, \pi) \; .$$

So the deduced formulas become

$$\frac{\partial H}{\partial \pi_i^\mu} = \partial_\mu \varphi^i \quad , \quad \frac{\partial H}{\partial \varphi^i} = -\frac{\partial L}{\partial \varphi^i} \; .$$

If φ satisfies the Euler-Lagrange equations

$$\frac{\partial L}{\partial \varphi^i} - \partial_\mu \left(\frac{\partial L}{\partial\, \partial_\mu \varphi^i} \right) = 0$$

then the pair (φ, π) satisfies the covariant Hamilton equations

$$\partial_\mu \varphi^i = \frac{\partial H}{\partial \pi_i^\mu} \quad , \quad \partial_\mu \pi_i^\mu = -\frac{\partial H}{\partial \varphi^i} \; .$$

4 The Functional Lagrangian Formalism

A $(n+1)$-splitting of $\mathbb{R}^{n+1}$ is an identification $\mathbb{R}^{n+1} \cong \mathbb{R} \times \mathbb{R}^n$, which amounts to a choice of coordinates of the form $(t = x^0, x^a)$.
The functional Lagrangian density $\mathcal{L}$ associated to the covariant Lagrangian density L is defined as

$$l(t, x^a, q^i, \underbrace{\dot{q}^i, q^i_a}_{q^i_\mu}) \;=\; L(x^\mu, q^i, q^i_\mu) \;,$$

where $\dot{q}^i = q^i_0$.
The Euler-Lagrange equations are obtained by considering the critical points of the Lagrangian functional

$$\mathcal{L}[\varphi, \dot{\varphi}] \;=\; \int_{\mathbb{R}^n} l(t, x^a, \varphi^i, \dot{\varphi}^i, \partial_a \varphi^i)\, \mathrm{d}^n x \;,$$

that is,

$$\frac{\delta \mathcal{L}}{\delta \varphi^i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\delta \mathcal{L}}{\delta \dot{\varphi}^i}\right) \;=\; 0 \;.$$

These equations are just the covariant Euler-Lagrange equations in disguise:

$$\begin{aligned} \frac{\delta \mathcal{L}}{\delta \varphi^i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\delta \mathcal{L}}{\delta \dot{\varphi}^i}\right) \;&=\; \frac{\partial l}{\partial \varphi^i} - \partial_a\left(\frac{\partial l}{\partial\, \partial_a \varphi^i}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial l}{\partial \dot{\varphi}^i}\right) \\ &=\; \frac{\partial l}{\partial \varphi^i} - \partial_a\left(\frac{\partial l}{\partial\, \partial_a \varphi^i}\right) - \partial_t\left(\frac{\partial l}{\partial\, \partial_t \varphi^i}\right) \;. \end{aligned}$$

5 Functional Hamiltonian Formalism

Assume that the temporal Hessian matrix of l is non-singular, that is,

$$\det\left(\frac{\partial^2 l}{\partial \dot{q}^i \, \partial \dot{q}^j}\right) \neq 0 \; .$$

Now we can apply the implicit function theorem to solve the equations

$$p_i = \frac{\partial l}{\partial \dot{q}^i}$$

for $\dot{q}^i$ as function of the variables t, x^a, q^j, q^j_b and p_i. Define the functional Hamiltonian density by

$$h(t, x^a, q^i, p_i, q^i_a) = p_i \, \dot{q}^i - l(t, x^a, q^i, \dot{q}^i, q^i_a) \; .$$

Now we compute

$$\frac{\partial h}{\partial q^i_a} = p_j \frac{\partial \dot{q}^j}{\partial \dot{q}^i} - \frac{\partial l}{\partial \dot{q}^j} \frac{\partial \dot{q}^j}{\partial q^i_a} - \frac{\partial l}{\partial q^i_a} = -\frac{\partial l}{\partial q^i_a}$$

$$\frac{\partial h}{\partial p_i} = \dot{q}^i + p_j \frac{\partial \dot{q}^j}{\partial p_i} - \frac{\partial l}{\partial \dot{q}^j} \frac{\partial \dot{q}^j}{\partial p_i} = \dot{q}^i$$

$$\frac{\partial h}{\partial q^i} = p_j \frac{\partial \dot{q}^j}{\partial q^i} - \frac{\partial l}{\partial \dot{q}^j} \frac{\partial \dot{q}^j}{\partial q^i} - \frac{\partial l}{\partial q^i} = -\frac{\partial l}{\partial q^i} \; .$$

If we set

$$\pi_i = \frac{\partial l}{\partial \dot{\varphi}^i} ,$$

then the previous formulas become

$$\frac{\partial h}{\partial \partial_a \varphi^i} = \frac{\partial l}{\partial \partial_a \varphi^i} \quad , \quad \frac{\partial h}{\partial \pi_i} = \dot{\varphi}^i \quad , \quad \frac{\partial h}{\partial \varphi^i} = -\frac{\partial l}{\partial \varphi^i} .$$

Given a solution φ of the functional Euler-Lagrange equations, that is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial l}{\partial \dot{\varphi}^i}\right) = \frac{\partial l}{\partial \varphi^i} - \partial_a\left(\frac{\partial l}{\partial\, \partial_a \varphi^i}\right)$$

then we have also a solution of the functional Hamilton equations

$$\partial_t \pi_i = -\left(\frac{\partial h}{\partial \varphi^i} - \partial_a\left(\frac{\partial h}{\partial\, \partial_a \varphi^i}\right)\right) \quad , \quad \partial_t \varphi^i = \frac{\partial h}{\partial \pi_i}.$$

The functional Hamiltonian is given by

$$\mathcal{H}[\varphi, \pi] = \int_{\mathbb{R}^n} h(t, x^a, \varphi^i, \pi_i, \partial_a \varphi^i)\, \mathrm{d}^n x .$$

so that the functional Hamilton equations can be obtained by taking the variational derivatives of $\mathcal{H}$ with respect to π_i and φ^i, respectively:

$$\partial_t \pi_i = -\frac{\delta\, \mathcal{H}}{\delta \varphi^i} = -\left(\frac{\partial h}{\partial \varphi^i} - \partial_a\left(\frac{\partial h}{\partial\, \partial_a \varphi^i}\right)\right)$$

$$\partial_t \varphi^i = \frac{\delta\, \mathcal{H}}{\delta \pi_i} = \frac{\partial h}{\partial \pi_i} .$$

6 The Relation Between the Covariant and the Functional Hamiltonian Formalisms

Let us consider again a $(n+1)$-splitting of $\mathbb{R}^{n+1}$, that is, an identification of coordinates of the form $(t = x^0, x^a)$. We write the covariant Hamiltonian equations in these coordinates:

$$\begin{aligned}\partial_t \pi_i^0 &= -\partial_a \pi_i^a - \frac{\partial H}{\partial \varphi^i} \\ \partial_t \varphi^i &= \frac{\partial H}{\partial \pi_i^0} \\ \partial_a \varphi^i &= \frac{\partial H}{\partial \pi_i^a}\end{aligned}$$

The last equations do not have time derivatives and so they are just algebraic constraints.
The first step toward the functional formalism is to identify $p_i = p_i^0$. The only way to define h from H is

$$h(t, x^a, q^i, p_i, q_a^i) = -p_i^a\, q_a^i + H(t, x^a, q^i, p_i, p_i^a)\ ,$$

where we assume that p_i^a is a function of t, x, q^i, p_i and q_a^i. This is possible if we can solve the equation

$$q_a^i = \frac{\partial H}{\partial p_i^a}$$

with respect to the variables t, x, q^i, p_i and q_a^i.

By the implicit function theorem, it is sufficient that the spatial Hessian matrix of H is non-singular, that is,

$$\det\left(\frac{\partial^2 H}{\partial p_i^a\,\partial p_j^b}\right) \neq 0 \,.$$

If we differentiate the equation that defines h with respect to q^i, p_i and q_a^i we get the following relations

$$\frac{\partial h}{\partial q^i} = -\,q_b^j\frac{\partial p_j^b}{\partial q^i} + \frac{\partial H}{\partial q^i} + \frac{\partial H}{\partial p_j^b}\frac{\partial\, p_j^b}{\partial q^i} = \frac{\partial H}{\partial q^i}$$

$$\frac{\partial h}{\partial p_i} = -\,q_b^j\frac{\partial p_j^b}{\partial p_i} + \frac{\partial H}{\partial p_i} + \frac{\partial H}{\partial p_j^b}\frac{\partial\, p_j^b}{\partial p_i} = \frac{\partial H}{\partial p_i}$$

$$\frac{\partial H}{\partial q_a^i} = -\,p_i^a - q_a^j\frac{\partial p_j^b}{\partial q_a^i} + \frac{\partial H}{\partial p_j^b}\frac{\partial\, p_j^b}{\partial q_a^i} = -\,p_i^a \,.$$

In particular we have

$$\frac{\partial h}{\partial \varphi^i} = \frac{\partial H}{\partial \varphi^i} \quad , \quad \frac{\partial h}{\partial \pi_i} = \frac{\partial H}{\partial \pi_i} \quad , \quad \frac{\partial h}{\partial\, \partial_a \varphi^i} = -\,\pi_i^a \,.$$

If we use the first and the second covariant Hamiltonian equations and the above relations we get

$$\partial_t \pi_i = -\left(\frac{\partial h}{\partial \varphi^i} - \partial_a\left(\frac{\partial h}{\partial\, \partial_a \varphi^i}\right)\right) \quad , \quad \partial_t \varphi^i = \frac{\partial h}{\partial \pi_i} \,,$$

which are the functional Hamilton equations.

Now, if we want to go back to the covariant Hamiltonian equations we start by identifying $p_i^0 = p_i$ and noting that the only way to define H from h is

$$H(t, x^a, q^i, p_i^0, p_i^a) = p_i^a q_a^i + h(t, x^a, q^i, p_i^0, q_a^i) ,$$

where we assume that q_a^i is a function of t, x, q^i, p_i and p_i^a. This is possible if we can solve the equation

$$p_i^a = -\frac{\partial h}{\partial q_a^i}$$

with respect to the variables t, x, q^i, p_i and p_i^a. By the implicit function theorem, it is sufficient that the spatial Hessian matrix of h is non-singular, that is,

$$\det\left(\frac{\partial^2 h}{\partial q_a^i\, \partial q_b^j}\right) \neq 0 .$$

If we differentiate the equation that defines H with respect to q^i and p_i^a we get

$$\frac{\partial H}{\partial q^i} = p_j^b \frac{\partial q_b^j}{\partial q^i} + \frac{\partial h}{\partial q^i} + \frac{\partial h}{\partial q_b^j}\frac{\partial\ q_b^j}{\partial q^i} = \frac{\partial h}{\partial q^i}$$

$$\frac{\partial H}{\partial p_i^0} = p_j^b \frac{\partial q_b^j}{\partial p_i^0} + \frac{\partial h}{\partial p_i^0} + \frac{\partial h}{\partial q_b^j}\frac{\partial\ q_b^j}{\partial p_i^0} = \frac{\partial h}{\partial p_i^0}$$

$$\frac{\partial H}{\partial p_i^a} = q_a^i + p_j^b \frac{\partial q_b^j}{\partial p_i^a} + \frac{\partial h}{\partial q_b^j}\frac{\partial\ q_b^j}{\partial p_i^a} = q_a^i .$$

So that

$$\frac{\partial H}{\partial \varphi^i} = \frac{\partial h}{\partial \varphi^i} \quad , \quad \frac{\partial H}{\partial \pi^0_i} = \frac{\partial h}{\partial \pi^0_i} \quad , \quad \frac{\partial H}{\partial \pi_i} = \partial_a \varphi^i \ .$$

Remembering the definition of p^a_i we also have

$$\pi^a_i = -\frac{\partial h}{\partial\, \partial_a \varphi^i} \ .$$

Now from the functional Hamiltonian equations we get

$$\partial_t \pi^0_i = -\partial_a \pi^a_i - \frac{\partial H}{\partial \varphi^i}$$

$$\partial_t \varphi^i = \frac{\partial H}{\partial \pi^0_i} \ ,$$

so that the calculus of the covariant Hamiltonian equations is finished.

7 Examples

The Hamiltonians of the scalar field theory and relativistic fluid mechanics satisfy the conditions the allows to pass from the covariant to the functional Hamilton equations and vice-versa.

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