Generalized nonholonomic mechanics on Lie Algebroids

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Motivation

The constraint distribution play a double role in nonholonomic mechanics.

- Kinematic contraints. They restrict the phase space velocities.
- Variational contraints. They restrict the variational space by means of the **D'Alembert Priciple**.

Generalized nonholonomic systems:
Examples:

- control of mechanical systems,
- stabilization of an unstable equilibrium,
- impose constants of motion to controlled systems,
- ...

Motivation

The constraint distribution play a double role in nonholonomic mechanics.

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Generalized nonholonomic systems:
Examples:

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Nonholonomic mechanical systems

Let $g : E \times_Q E \to \mathbb{R}$ be a fiber metric in the Lie algebroid e $(E, [\cdot\,,\cdot], \rho)$.
The systems considered are the *nonholonomic mechanical systems* determined by:

- A Lagrangian function L:

$$L(e) = \frac{1}{2} g(e, e) - V(\tau(e)), \quad e \in E,$$

  with V a potential function on M.
- Nonholonomic (linear) constraints determined by a subbundle D of E.

Let us consider the orthogonal decomposition $\mathsf{E} = \mathsf{D} \oplus \mathsf{D}^{\perp}$, and the associated orthogonal projectors

$$\mathsf{P} : \mathsf{E} \longrightarrow \mathsf{D} \qquad \mathsf{Q} : \mathsf{E} \longrightarrow \mathsf{D}^{\perp}$$

We choose a local basis of sections of E, $\{e_A\}$, adapted to (L, D), that is:

(i) $\{e_A\}$ is an orthogonal basis with respect to g (i.e., $g(e_A, e_B) = \delta_{AB}$)

(ii) $\{e_A\} = \{e_a, e_\alpha\}$ where $\mathsf{D} = \text{span}\{e_a\}$, $\mathsf{D}^{\perp} = \text{span}\{e_\alpha\}$,

(iii) and denote $(x^\mu, y^A) = (x^\mu, y^a, y^\alpha)$ the induced coordinates on E

Then we have the local *structure functions*:

$$(\rho^{\mu}_{A}, \mathcal{C}^{C}_{AB})$$

induced by the adapted basis. That is,

$$\rho(e_A) = \rho^{\mu}_{A} \frac{\partial}{\partial x^{\mu}}, \quad [\![e_A, e_B]\!] = \mathcal{C}^{C}_{AB} e_C$$

and,

- $L(x^{\mu}, y^{A}) = \sum_{A} (y^{A})^2 - V(x^{\mu})$,
- $y^{\alpha} = 0$ are the local equations determining the vector subbundle D of E.

Motion equations of the nonholonomic mechanical system

$\gamma(t) = (x^\mu(t), y^A(t))$ a curve in E

γ is a solution

$$\Updownarrow$$

$$\dot{x}^\mu = \rho^\mu_a y^a$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial y^a}\right) + \frac{\partial L}{\partial y^C}\mathcal{C}^C_{ab} y^b - \rho^\mu_a \frac{\partial L}{\partial x^\mu} = 0$$

or equivalently

$$\dot{x}^\mu = \rho^\mu_a y^a$$

$$\dot{y}^a + \mathcal{C}^c_{ab} y^b y^c + \rho^\mu_a \frac{\partial V}{\partial x^\mu} = 0$$

The structure of an *almost-Lie algebroid*

Definition

An *almost-Lie algebroid structure* in the vector fiber bundle $\tau_D : D \to Q$ is a $\mathbb{R}$-linear bracket $[\![\cdot,\cdot]\!]_D : \Gamma(\tau_D) \times \Gamma(\tau_D) \to \Gamma(\tau_D)$ and a morphism of vectorial fiber bundles $\rho_D : D \to TQ$, the *anchor map*, such that

1. $[\![\cdot,\cdot]\!]_D$ is skew-symmetric,

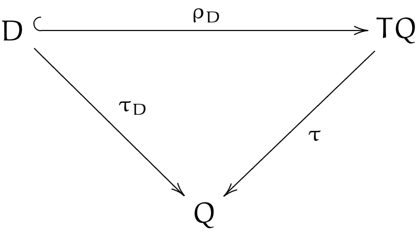
$$[\![X,Y]\!]_D = -[\![Y,X]\!]_D, \quad \text{for } X, Y \in \Gamma(\tau_D).$$

2. If we denote $\rho_D : \Gamma(\tau_D) \to \mathfrak{X}(Q)$ the morphism of $C^\infty(Q)$-modulo induced by the anchor map, the

$$[\![X, fY]\!]_D = f[\![X,Y]\!]_D + \rho_D(X)(f)Y,$$

for $X, Y \in \Gamma(D)$ and $f \in C^\infty(Q)$.

Nonholonomic mechanics & Mechanics in almost-Lie algebroids



For all $X, Y \in \Gamma(\tau_D)$ we define

- $[\![X, Y]\!]_D = P[\![i_D \circ X, i_D \circ Y]\!]$
- $\rho_D(X) = \rho(i_D \circ X)$

Then the local structure functions are

- $[\![e_a, e_b]\!]_D = P[\![e_a, e_b]\!] = P\left(\mathcal{C}^c_{ab} e_c + \mathcal{C}^\alpha_{ab} e_\alpha\right) = \mathcal{C}^c_{ab} e_c,$
- $\rho_D(e_a) = \rho^\mu_a \frac{\partial}{\partial x^\mu}$

Almost-Lie algebroid$(D, [\![\ ,\ ]\!]_D, \rho_D)$

$\Updownarrow$

Linear bivector Λ_{D^*} in D^*

$$\Lambda_{D^*} = \rho_a^\mu \frac{\partial}{\partial x^\mu} \wedge \frac{\partial}{\partial p_a} - \frac{1}{2} \mathcal{C}^c_{ab} p_c \frac{\partial}{\partial p_a} \wedge \frac{\partial}{\partial p_b}.$$

where (x^μ, p_a) are the induced coordinates on D^*.

Linear almost Poisson structure $\{\ ,\}_{D^*}$
$$\begin{cases} \{x^\mu, x^\gamma\}_{D^*} = 0 \\ \{x^\mu, p_a\}_{D^*} = \rho_a^\mu \\ \{p_a, p_b\}_{D^*} = -\mathcal{C}^c_{ab} p_c \end{cases}$$

- $(D, \{\ ,\ \}_{D^*}, \rho_D, h)$ is a hamiltonian system,
- *The Hamiltonian vector field* on D^*: $X_h = -i(dh)\Lambda_{D^*}$

- Hamilton equations:

$$
\begin{aligned}
\dot{q}^i &= \rho^i_a \frac{\partial h}{\partial p_a} \\
\dot{p}_a &= -\left(\rho^i_a \frac{\partial h}{\partial q^i} + C^c_{ab} p_c \frac{\partial h}{\partial p_b} \right)
\end{aligned}
$$

With the Legendre tranformation we obtain an *almost Poisson bracket* on D: $\{\ ,\ \}_{l,D}$

In this case,

- $l : D \longrightarrow \mathbb{R}$ is $l(x^\mu, y^a) = \sum_a (y^a)^2 - V(x^\mu)$.
- The Euler-Lagrange equation are:

$$\begin{aligned}
\frac{dx^\mu}{dt} &= \{x^\mu, E_l\}_{l,D} = \rho^\mu_a y^a, \\
\frac{dy^a}{dt} &= \{y^a, E_l\}_{l,D} = -\mathcal{C}^c_{ab} y^c y^b - \rho^\mu_a \frac{\partial V}{\partial x^\mu}.
\end{aligned}$$

where E_l is the energy defined as $E_l(e) = \langle Leg_l(e), e\rangle - l(e)$

Geometric formulation of the generalized nonholonomic mechanics

Let $g : E \times_Q E \to \mathbb{R}$ a fiber metric in the Lie algebroid $(E, [\cdot\,,\cdot], \rho)$.

Here we will consider the *generalized nonholonomic mechanical systems* which are determined by:

- A lagrangian function L:

$$L(e) = \frac{1}{2} g(e, e) - V(\tau(e)), \quad e \in E,$$

  with V a potential function on M.

- **Kinematic constraints** determined by the distribution D of E,
- **Variational constraints** determined by the distribution $\widetilde{D}$.

We assume the **compatibility condition** $E = D \oplus \widetilde{D}^{\perp}$.

Given local coordinates (x^i) in Q and a local basis of sections in E, $\{e_A\}$, adapted to the generalized nonholonomic problem $(L, D, \widetilde{D})$:

(i) $\{e_A\} = \{e_a, e_\alpha\}$ where $D = \text{span}\{e_a\}$, $\widetilde{D}^{\perp} = \text{span}\{e_\alpha\}$.
(ii) $\{e_a\}$ y $\{e_\alpha\}$ are orthogonal with respect to g.
(iii) and denote $(x^\mu, y^\alpha) = (x^\mu, y^a, y^\alpha)$ the induced coordinates on E.

Then $(\rho^{\mu}_{A}, \mathcal{C}^{C}_{AB})$ are the local structure functions, induced by the orthonormal basis adapted to D and to $\widetilde{D}^{\perp}$.

$$\rho(e_A) = \rho^{\mu}_{A} \frac{\partial}{\partial x^{\mu}}, \quad [\![e_A, e_B]\!] = \mathcal{C}^{C}_{AB} e_C$$

In this basis,

- $L(x^{\mu}, y^{a}, y^{\alpha}) = \sum_A (y^A)^2 + g_{a\alpha} y^a y^{\alpha} - V(x^{\mu})$,
- $y^{\alpha} = 0$ are the local equations defining the vector subbundle D on E.
- $\widetilde{D}^{o} = \mathrm{span}\{g_{a\alpha} e^{a} + e^{\alpha}\}$

Motion equations of the generalized nonholonomic system

$\gamma(t) = (x^\mu(t), y^A(t))$ curve in E

γ is a solution

$$\Updownarrow$$

$$\dot{x}^\mu = \rho^\mu_a y^a$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial y^a}\right) + \frac{\partial L}{\partial y^C}\mathcal{C}^C_{ab} y^b - \rho^\mu_a \frac{\partial L}{\partial x^\mu} = \lambda^\alpha g_{a\alpha}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial y^\alpha}\right) + \frac{\partial L}{\partial y^C}\mathcal{C}^C_{\alpha b} y^b - \rho^\mu_\alpha \frac{\partial L}{\partial x^\mu} = \lambda^\alpha$$

After some computations we get

$$\begin{aligned}\dot{y}^a &= -\Bigg[[W^{-1}]^{ad}\Big(\mathcal{C}^c_{db} + g_{c\beta}\mathcal{C}^\beta_{db} - g_{d\alpha}\mathcal{C}^c_{\alpha b} \\ &\quad -g_{c\alpha}g_{d\gamma}\mathcal{C}^\gamma_{\alpha b} - g_{d\alpha}\frac{\partial g_{c\alpha}}{\partial x^\mu}\rho^\mu_b\Big)\Bigg] y^c y^b \\ &\quad -[W^{-1}]^{ad}(\rho^\mu_d - g_{d\alpha}\rho^\mu_\alpha)\frac{\partial V}{\partial x^\mu} \\ \dot{x}^\mu &= \rho^\mu_a y^a\end{aligned}$$

where W is the *hessian matrix* of $L : E \longrightarrow \mathbb{R}$.

$W = \begin{pmatrix} I_m & G \\ G^T & I_{n-m} \end{pmatrix}$ with $G = (g_{a\alpha})$.

A new geometric structure

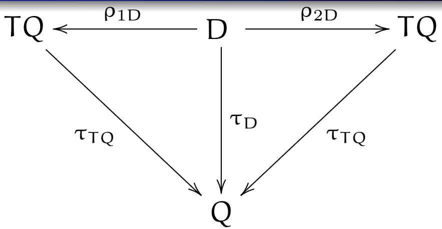
Let us denote

$$
\begin{aligned}
\widetilde{\mathcal{C}}^c_{ab} &= [W^{-1}]^{ad}\Big(\mathcal{C}^c_{db} + g_{c\beta}\mathcal{C}^\beta_{db} - g_{d\alpha}\mathcal{C}^c_{\alpha b} \\
&\quad - g_{c\alpha}g_{d\gamma}\mathcal{C}^\gamma_{\alpha b} - g_{d\alpha}\frac{\partial g_{c\alpha}}{\partial x^\mu}\rho^\mu_b\Big) \\
(\rho_{1D})^\mu_a &= [W^{-1}]^{ad}(\rho^\mu_d - g_{d\alpha}\rho^\mu_\alpha) \\
(\rho_{2D})^\mu_a &= \rho^\mu_a
\end{aligned}
$$

Then we rewrite the motion equations as:

$$
\begin{aligned}
\dot{x}^i &= (\rho_{2D})^i_a y^a \\
\dot{y}^a &= -\widetilde{\mathcal{C}}^c_{ab} y^b y^c - (\rho_{1D})^i_a \frac{\partial V}{\partial x^i}
\end{aligned}
$$

Almost-Leibniz algebroid structure



Definition

An *almost-Leibniz algebroid structure* in the vector bundle $\tau_D : D \to Q$ is a $\mathbb{R}$-linear bracket $[\![\cdot,\cdot]\!]_D : \Gamma(\tau_D) \times \Gamma(\tau_D) \to \Gamma(\tau_D)$ and two vector bundle morphism $\rho_{1D} : D \to TQ$, and $\rho_{2D} : D \to TQ$ (*anchor maps*), such that

$$[\![fX, gY]\!]_D = f\rho_{1D}(X)(g)Y - g\rho_{2D}(Y)(f)X + fg[\![X, Y]\!]_D$$

for $X, Y \in \Gamma(D)$ and $f, g \in C^\infty(Q)$.

Generalized nonholonomic mechanics and mechanics on almost-Leibniz algebroids

In a generalized nonholonomic system we have a metric g and two distributions D and $\tilde{\mathsf{D}}$.
Associated to the decomposition $\mathsf{D} \oplus \mathsf{D}^{\perp} = \mathsf{E}$ we have the projector

$$\mathsf{P} : \quad \mathsf{E} \quad \longrightarrow \quad \mathsf{D}$$

and associated to $\widetilde{\mathsf{D}} \oplus \mathsf{D}^{\perp} = \mathsf{E}$ the projector

$$\widetilde{\mathsf{P}} : \quad \mathsf{E} \quad \longrightarrow \quad \widetilde{\mathsf{D}}$$

For all $X, Y \in \Gamma(\tau_D)$ we define

- $[\![X, Y]\!]_{(D,\widetilde{D})} = P[\![i_{\widetilde{D}} \circ \widetilde{P}X, i_D \circ Y]\!]$
- $\rho_{1D}(X) = \rho(i_{\widetilde{D}} \circ \widetilde{P}X)$
- $\rho_{2D}(X) = \rho(i_D X)$

Then it is verified:

$$[\![fX, gY]\!]_{(D,\widetilde{D})} = f\rho_{1D}(X)(g)Y - g\rho_{2D}(Y)(f)X + fg[\![X, Y]\!]_{D,\widetilde{D}}$$

and

$$\begin{aligned} [\![e_a, e_b]\!]_{(D,\widetilde{D})} &= \widetilde{\mathcal{C}}^c_{ab} e_c, \\ \rho_{1D}(e_a) &= [W^{-1}]^{ad}(\rho^\mu_d - g_{d\alpha}\rho^\mu_\alpha)\frac{\partial}{\partial x^\mu}, \\ \rho_{2D}(e_a) &= \rho^\mu_a \frac{\partial}{\partial x^\mu}. \end{aligned}$$

For all $X, Y \in \Gamma(\tau_D)$ we define

- $[\![X, Y]\!]_{(D,\tilde{D})} = P[\![i_{\tilde{D}} \circ \tilde{P}X, i_D \circ Y]\!]$
- $\rho_{1D}(X) = \rho(i_{\tilde{D}} \circ \tilde{P}X)$
- $\rho_{2D}(X) = \rho(i_D X)$

Then it is verified:

$$[\![fX, gY]\!]_{(D,\tilde{D})} = f\rho_{1D}(X)(g)Y - g\rho_{2D}(Y)(f)X + fg[\![X, Y]\!]_{D,\tilde{D}}$$

and

$$\begin{aligned} [\![e_a, e_b]\!]_{(D,\tilde{D})} &= \tilde{\mathcal{C}}^c_{ab} e_c, \\ \rho_{1D}(e_a) &= [W^{-1}]^{ad}(\rho^\mu_d - g_{d\alpha}\rho^\mu_\alpha)\frac{\partial}{\partial x^\mu}, \\ \rho_{2D}(e_a) &= \rho^\mu_a \frac{\partial}{\partial x^\mu}. \end{aligned}$$

Almost-Leibniz algebroid structure $(D, [\![\;,\;]\!]_{(D,\widetilde{D})}, \rho_{1D}, \rho_{2D})$

$$\Updownarrow$$

2-contravariant linear tensor $\Lambda_{(D^*,\widetilde{D})}$ in D^*

$$\Lambda_{(D^*,\widetilde{D})} = (\rho_{2D})^{\mu}_{a} \frac{\partial}{\partial x^{\mu}} \otimes \frac{\partial}{\partial p_a} - (\rho_{1D})^{\mu}_{a} \frac{\partial}{\partial p_a} \otimes \frac{\partial}{\partial x^{\mu}} - \widetilde{\mathcal{C}}^{c}_{ab} p_c \frac{\partial}{\partial p_a} \otimes \frac{\partial}{\partial p_b}.$$

where (x^{μ}, p_a) are the induced coordinates on D^*.

Linear *almost* Poisson structure $\{\;,\}_{(D^*,\widetilde{D})}$

$$\begin{cases} \{x^{\mu}, x^{\gamma}\}_{(D^*,\widetilde{D})} = 0 \\ \{x^{\mu}, p_a\}_{(D^*,\widetilde{D})} = (\rho_{2D})^{\mu}_{a} \\ \{p_a, x^{\mu}\}_{(D^*,\widetilde{D})} = -(\rho_{1D})^{\mu}_{a} \\ \{p_a, p_b\}_{(D^*,\widetilde{D})} = -\widetilde{\mathcal{C}}^{c}_{ab} p_c \end{cases}$$

- $h : D^* \to \mathbb{R}$ a hamiltonian function on D^*.
- $(D, \widetilde{D}, \{\ ,\}_{D^*}, \rho_{1D}, \rho_{2D}, h)$ is a hamiltonian system,
- *The Hamiltonian vector field* on D^*: $X_h = -i(dh)\Lambda_{D^*}$

- Hamilton equations:

$$
\begin{aligned}
\dot{q}^i &= (\rho_{2D})^i_a \frac{\partial h}{\partial p_a} \\
\dot{p}_a &= -\left((\rho_{1D})^i_a \frac{\partial h}{\partial q^i} + \widetilde{C}^c_{ab} p_c \frac{\partial h}{\partial p_b} \right)
\end{aligned}
$$

Mechanics in almost-Leibniz algebroids

If $\mathfrak{l} : D \longrightarrow \mathbb{R}$ is a regular Lagragian, with the *pull back* of the Legendre transformation we obtain an *almost Poisson bracket* in D: $\{\ ,\ \}_{(\mathfrak{l},D,\widetilde{D})}$

In this case, $\mathfrak{l} : D \longrightarrow \mathbb{R}$ is $\mathfrak{l}(x^\mu, y^a) = \sum_a (y^a)^2 - V(x^\mu)$, then the Euler-Lagrange equations are:

$$\begin{aligned} \frac{dx^\mu}{dt} &= \{x^\mu, E_{\mathfrak{l}}\}_{(\mathfrak{l},D,\widetilde{D})} = (\rho_{2D})^\mu_a y^a, \\ \frac{dy^a}{dt} &= \{y^a, E_{\mathfrak{l}}\}_{(\mathfrak{l},D,\widetilde{D})} = -\widetilde{C}^c_{ab} y^c y^b - (\rho_{1D})^\mu_a \frac{\partial V}{\partial x^\mu}. \end{aligned}$$

with the *energy* defined as $E_{\mathfrak{l}}(e) = \langle \mathrm{Leg}_{\mathfrak{l}}(e), e\rangle - \mathfrak{l}(e)$.

If $l : D \longrightarrow \mathbb{R}$ is a regular Lagragian, with the *pull back* of the Legendre transformation we obtain an *almost Poisson bracket* in D: $\{\ ,\ \}_{(l,D,\widetilde{D})}$

In this case, $l : D \longrightarrow \mathbb{R}$ is $l(x^\mu, y^a) = \sum_a (y^a)^2 - V(x^\mu)$, then the Euler-Lagrange equations are:

$$
\begin{aligned}
\frac{dx^\mu}{dt} &= \{x^\mu, E_l\}_{(l,D,\widetilde{D})} = (\rho_{2D})^\mu_a y^a, \\
\frac{dy^a}{dt} &= \{y^a, E_l\}_{(l,D,\widetilde{D})} = -\widetilde{C}^c_{ab} y^c y^b - (\rho_{1D})^\mu_a \frac{\partial V}{\partial x^\mu}.
\end{aligned}
$$

with the *energy* defined as $E_l(e) = \langle Leg_l(e), e\rangle - l(e)$.

$$\text{Almost-Lie algebroid } (D, [\![\ ,\ ]\!]_D, \rho_D)$$

$$\Updownarrow$$

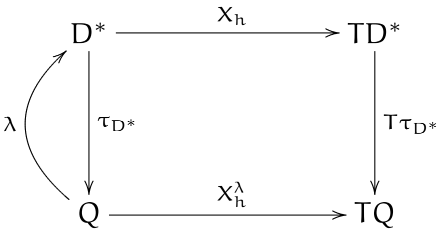
$$\text{An almost differential } d^D : \Gamma(\Lambda^k D^*) \longrightarrow \Gamma(\Lambda^{k+1} D^*)$$

$$\begin{aligned}
(1)\ (d^D f)(\xi) &= \rho_D(\xi)(f), \quad f \in C^\infty(Q), \quad \xi \in \Gamma(\tau_D) \\
(2)\ d^D \sigma(\xi_1, \xi_2) &= \rho_D(\xi_1)(\sigma(\xi_2)) - \rho_D(\xi_2)(\sigma(\xi_1)) \\
&\quad -\sigma[\![\xi_1,\ \xi_2]\!]_D,\ \sigma \in \Gamma(\tau_{D^*}),\ \xi_1, \xi_2 \in \Gamma(\tau_D)
\end{aligned}$$

In general $\boxed{(d^D)^2 \neq 0}$

Hamilton-Jacobi Theorem on an almost-Lie algebroid

Let $\lambda : Q \longrightarrow D^*$ be a section of $\tau_{D^*} : D^* \longrightarrow Q$.



Let us define $X_h^\lambda = T\tau_{D^*} \circ X_h \circ \lambda$.

Theorem (Hamilton-Jacobi)

Let $d^D\lambda = 0$.

$$(i)\ \sigma : I \to Q \text{ integral curve of } X_h^\lambda \Rightarrow \lambda \circ \sigma \text{ integral curve of } X_h$$

$$\Updownarrow$$

$$(ii) \quad d^D(h \circ \lambda) = 0$$

The differential map on an almost-Leibniz algebroid

Let $(D, [\![\ ,\ ]\!]_{(D,\tilde{D})}, \rho_{1D}, \rho_{2D})$ be an almost-Leibniz algebroid.

$$\text{Almost-Leibniz algebroid}(D, [\![\ ,\ ]\!]_{(D,\tilde{D})}, \rho_{1D}, \rho_{2D})$$

$$\Downarrow$$

$$\text{An almost differential } d^{(D,\tilde{D})} : \Gamma(\mathcal{T}^k D^*) \longrightarrow \Gamma(\mathcal{T}^{k+1} D^*), k = 0, 1$$

$$(1)\ (d^{(D,\tilde{D})} f)(\xi) = \rho_{2D}(\xi)(f), \quad f \in C^\infty(Q), \quad \xi \in \Gamma(\tau_D)$$

$$\begin{aligned}(2)\ d^{(D,\tilde{D})}\sigma(\xi_1, \xi_2) &= \rho_{2D}(\xi_1)(\sigma(\xi_2)) - \rho_{1D}(\xi_2)(\sigma(\xi_1)) \\ &- \sigma\left([\![\xi_1, \xi_2]\!]_{D,\tilde{D}}\right), \quad \sigma \in \Gamma(\tau_{D^*}), \xi_1, \xi_2 \in \Gamma(\tau_D)\end{aligned}$$

In general $\boxed{(d^{(D,\tilde{D})})^2 \neq 0}$

The differential map on an almost-Leibniz algebroid

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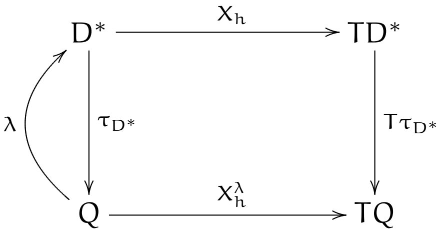
In general $\boxed{(d^{(D,\widetilde{D})})^2 \neq 0}$

Hamilton-Jacobi Theorem for generalized nonholonomic systems

Consider $\lambda : Q \longrightarrow D^*$ a section of $\tau_{D^*} : D^* \longrightarrow Q$.



Then, let us define $X_h^\lambda = T\tau_{D^*} \circ X_h \circ \lambda$.

Hamilton-Jacobi Theorem for generalized nonholonomic systems

Theorem (Hamilton-Jacobi)

Sea $d^{(D,\widetilde{D})}\lambda = 0$.

$$(i)\ \sigma : I \to Q \text{ integral curve of } X_h^\lambda \Rightarrow \lambda \circ \sigma \text{ integral curve of } X_h$$

$$\Updownarrow$$

$$(ii) \quad d^{(D,\widetilde{D})}(h \circ \lambda) = 0$$