Stability of Relative Equilibria in Simple Mechanical Systems with Symmetry

Miguel Rodríguez-Olmos
EPFL, Switzerland

I Iberoamerican Meeting on Geometry, Mechanics and Control
Santiago de Compostela

Outline

Outline

1. Hamiltonian Systems and Relative Equilibria

2. Simple Mechanical Systems and Cotangent Bundle Geometry

Outline

Symmetric Hamiltonian Systems

Let $(\mathcal{P}, \omega, G, \mathbf{J}, h)$ be a symmetric Hamiltonian system, where

Symmetric Hamiltonian Systems

Let $(\mathcal{P}, \omega, G, \mathbf{J}, h)$ be a symmetric Hamiltonian system, where

- $(\mathcal{P}, \omega)$ is a symplectic manifold,

$$\omega \in \Omega^2(\mathcal{P}), \quad \mathbf{d}\omega = 0, \quad \omega \text{ non-degenerate}$$

Symmetric Hamiltonian Systems

Let $(\mathcal{P}, \omega, G, \mathbf{J}, h)$ be a symmetric Hamiltonian system, where

- $(\mathcal{P}, \omega)$ is a symplectic manifold,

$$\omega \in \Omega^2(\mathcal{P}), \quad \mathbf{d}\omega = 0, \quad \omega \text{ non-degenerate}$$

- $\varphi : G \times \mathcal{P} \to \mathcal{P}$ is a proper smooth Hamiltonian action of a Lie group G,

$$\varphi_g^* \omega = \omega \quad \forall g \in G$$

Symmetric Hamiltonian Systems

Let $(\mathcal{P}, \omega, G, \mathbf{J}, h)$ be a symmetric Hamiltonian system, where

- $(\mathcal{P}, \omega)$ is a symplectic manifold,

$$\omega \in \Omega^2(\mathcal{P}), \quad \mathbf{d}\omega = 0, \quad \omega \text{ non-degenerate}$$

- $\varphi : G \times \mathcal{P} \to \mathcal{P}$ is a proper smooth Hamiltonian action of a Lie group G,

$$\varphi_g^* \omega = \omega \quad \forall g \in G$$

- $\mathbf{J} : \mathcal{P} \to \mathfrak{g}^*$ is the associated equivariant momentum map,

$$\begin{aligned} \omega(\xi_{\mathcal{P}}, \cdot) &= \langle \mathbf{dJ}(\cdot), \xi \rangle \quad \forall \xi \in \mathfrak{g} \\ \mathbf{J}(g \cdot z) &= \mathrm{Ad}^*_{g^{-1}}(\mathbf{J}(z)) \end{aligned}$$

Symmetric Hamiltonian Systems

Let $(\mathcal{P}, \omega, G, \mathbf{J}, h)$ be a symmetric Hamiltonian system, where

- $(\mathcal{P}, \omega)$ is a symplectic manifold,

$$\omega \in \Omega^2(\mathcal{P}), \quad \mathbf{d}\omega = 0, \quad \omega \text{ non-degenerate}$$

- $\varphi : G \times \mathcal{P} \to \mathcal{P}$ is a proper smooth Hamiltonian action of a Lie group G,

$$\varphi_g^* \omega = \omega \quad \forall g \in G$$

- $\mathbf{J} : \mathcal{P} \to \mathfrak{g}^*$ is the associated equivariant momentum map,

$$\begin{aligned} \omega(\xi_{\mathcal{P}}, \cdot) &= \langle \mathbf{dJ}(\cdot), \xi \rangle \quad \forall \xi \in \mathfrak{g} \\ \mathbf{J}(g \cdot z) &= \mathrm{Ad}^*_{g^{-1}}(\mathbf{J}(z)) \end{aligned}$$

- $h \in C^\infty(\mathcal{P})$ is a G-invariant function (the Hamiltonian).

$$h(g \cdot z) = h(z)$$

- Hamilton's equations:

$$\omega(X_h, \cdot) = \mathbf{d}h$$

- Hamilton's equations:

$$\omega(X_h, \cdot) = \mathbf{d}h$$

- Hamiltonian flow: $X_h \Rightarrow \phi^t_{X_h} : \mathcal{P} \to \mathcal{P}$ is G-equivariant

$$\phi^t_{X_h}(g \cdot z) = g \cdot \phi^t_{X_h}(z), \quad \forall\, \mathfrak{g} \in G,\, z \in \mathcal{P}$$

- Hamilton's equations:

$$\omega(X_h, \cdot) = \mathbf{d}h$$

- Hamiltonian flow: $X_h \Rightarrow \phi^t_{X_h} : \mathcal{P} \to \mathcal{P}$ is G-equivariant

$$\phi^t_{X_h}(g \cdot z) = g \cdot \phi^t_{X_h}(z), \quad \forall\, \mathfrak{g} \in G,\, z \in \mathcal{P}$$

- Conservation of Energy: (antisymmetry of ω)

$$h(\phi^t_{X_h}(z)) = h(z)$$

- Hamilton's equations:

$$\omega(X_h, \cdot) = \mathbf{d}h$$

- Hamiltonian flow: $X_h \Rightarrow \phi^t_{X_h} : \mathcal{P} \to \mathcal{P}$ is G-equivariant

$$\phi^t_{X_h}(g \cdot z) = g \cdot \phi^t_{X_h}(z), \quad \forall\, \mathfrak{g} \in G,\, z \in \mathcal{P}$$

- Conservation of Energy: (antisymmetry of ω)

$$h(\phi^t_{X_h}(z)) = h(z)$$

- Nöether's Theorem: (G-invariance of h)

$$\mathbf{J}(\phi^t_{X_h}(z)) = \mathbf{J}(z)$$

Relative Equilibria

Definition

A point $z \in \mathcal{P}$ is a relative equilibrium if

$$\phi^t_{X_h}(z) \subseteq G \cdot z$$

Relative Equilibria

Definition

A point $z \in \mathcal{P}$ is a relative equilibrium if

$$\phi^t_{X_h}(z) \subseteq G \cdot z$$

Important Properties

- If z is a RE with $\mathbf{J}(z) = \mu$ then $\exists \xi \in \mathfrak{g}_\mu$ s.t.

$$\phi^t_{X_h}(z) = \exp(t\xi) \cdot z$$

$$G_\mu = \{g \in G \,:\, \mathrm{Ad}^*_g \mu = \mu\} \quad \mathfrak{g}_\mu = \mathrm{Lie}\, G_\mu$$

Relative Equilibria

Definition

A point $z \in \mathcal{P}$ is a relative equilibrium if

$$\phi^t_{X_h}(z) \subseteq G \cdot z$$

Important Properties

- If z is a RE with $\mathbf{J}(z) = \mu$ then $\exists \xi \in \mathfrak{g}_\mu$ s.t.

$$\phi^t_{X_h}(z) = \exp(t\xi) \cdot z$$

$$G_\mu = \{g \in G \,:\, \mathrm{Ad}^*_g \mu = \mu\} \quad \mathfrak{g}_\mu = \mathrm{Lie}\, G_\mu$$

- If z is a relative equilibrium then all points in $G \cdot z$ are also RE

Relative Equilibria

Definition

A point $z \in \mathcal{P}$ is a relative equilibrium if

$$\phi^t_{X_h}(z) \subseteq G \cdot z$$

Important Properties

- If z is a RE with $\mathbf{J}(z) = \mu$ then $\exists \xi \in \mathfrak{g}_\mu$ s.t.

$$\phi^t_{X_h}(z) = \exp(t\xi) \cdot z$$

$$G_\mu = \{g \in G \, : \, \mathrm{Ad}^*_g \mu = \mu\} \quad \mathfrak{g}_\mu = \mathrm{Lie}\, G_\mu$$

- If z is a relative equilibrium then all points in $G \cdot z$ are also RE
- A point $z \in \mathcal{P}$ with $\mathbf{J}(z) = \mu$ is a RE iff $[z] \in \mathbf{J}^{-1}(\mu)/G_\mu$ is a fixed point for the Hamiltonian continuous flow induced by $\phi^t_{X_h}$.

Isotropy

- Every $\xi \in \mathfrak{g}_\mu$ for which

$$\phi^t_{X_h}(z) = \exp(t\xi) \cdot z$$

is called a velocity of the RE.

Isotropy

- Every $\xi \in \mathfrak{g}_\mu$ for which

$$\phi^t_{X_h}(z) = \exp(t\xi) \cdot z$$

is called a velocity of the RE.

- If the G-action is not free, the velocity is not unique. If $\xi \in \mathfrak{g}$ is a velocity for a relative equilibrium z, then

$$\xi + \xi'$$

is another velocity for z, where $\xi' \in \mathfrak{g}_z = \mathrm{Lie}\, G_z$.

Isotropy

- Every $\xi \in \mathfrak{g}_\mu$ for which

$$\phi^t_{X_h}(z) = \exp(t\xi) \cdot z$$

is called a velocity of the RE.

- If the G-action is not free, the velocity is not unique. If $\xi \in \mathfrak{g}$ is a velocity for a relative equilibrium z, then

$$\xi + \xi'$$

is another velocity for z, where $\xi' \in \mathfrak{g}_z = \mathrm{Lie}\, G_z$.

- By equivariance of $\mathbf{J} : \mathcal{P} \to \mathfrak{g}^*$, $G_z \subset G_\mu$.

Isotropy

- Every $\xi \in \mathfrak{g}_\mu$ for which

$$\phi^t_{X_h}(z) = \exp(t\xi) \cdot z$$

is called a velocity of the RE.

- If the G-action is not free, the velocity is not unique. If $\xi \in \mathfrak{g}$ is a velocity for a relative equilibrium z, then

$$\xi + \xi'$$

is another velocity for z, where $\xi' \in \mathfrak{g}_z = \mathrm{Lie}\, G_z$.

- By equivariance of $\mathbf{J} : \mathcal{P} \to \mathfrak{g}^*$, $G_z \subset G_\mu$.

- Therefore in the case of singular Hamiltonian actions the same relative equilibrium can be assigned several velocities at a given point.

Examples of Relative Equilibria

- Kepler's Problem:

$$\mathcal{P} = T^*\mathbb{R}^3,\ G = \mathrm{SO}(3),\ h = \frac{|\mathbf{p}|^2}{2m} + V(|\mathbf{x}|).$$

RE correspond to circular orbits.

Examples of Relative Equilibria

- Kepler's Problem:

$$\mathcal{P} = T^*\mathbb{R}^3,\ G = \mathrm{SO}(3),\ h = \frac{|\mathbf{p}|^2}{2m} + V(|\mathbf{x}|).$$

RE correspond to circular orbits.

- Spherical Pendulum:

$$\mathcal{P} = T^*S^2,\ G = S^1, h = \frac{|\mathbf{p}|^2}{2m} - mgl\mathbf{x}\cdot\mathbf{e_3}.$$

RE correspond to circular motions with constant θ.

Examples of Relative Equilibria

- Kepler's Problem:

$$\mathcal{P} = T^*\mathbb{R}^3,\ G = \mathrm{SO}(3),\ h = \frac{|\mathbf{p}|^2}{2m} + V(|\mathbf{x}|).$$

RE correspond to circular orbits.

- Spherical Pendulum:

$$\mathcal{P} = T^*S^2,\ G = S^1, h = \frac{|\mathbf{p}|^2}{2m} - mgl\mathbf{x}\cdot\mathbf{e_3}.$$

RE correspond to circular motions with constant θ.

- Riemann Ellipsoids (self-gravitating figures of equilibrium)

Stability of Relative Equilibria

Definition (G. Patrick)

Let $z \in \mathcal{P}$ be a relative equilibrium of the symmetric Hamiltonian system $(\mathcal{P}, \omega, G, \mathbf{J}, h)$ with momentum $\mu = \mathbf{J}(z)$.

The relative equilibrium z is (nonlinearly) stable modulo G_μ if for every G_μ-invariant neighborhood $G_\mu \cdot z \subset V$ there is a neighborhood $z \in U$ such that

$$\phi^t_{X_h}(U) \subset V \quad \forall\, t \in I.$$

Stability of Relative Equilibria

Definition (G. Patrick)

Let $z \in \mathcal{P}$ *be a relative equilibrium of the symmetric Hamiltonian system* $(\mathcal{P}, \omega, G, \mathbf{J}, h)$ *with momentum* $\mu = \mathbf{J}(z)$.

The relative equilibrium z *is (nonlinearly)* ***stable modulo*** G_μ *if for every* G_μ*-invariant neighborhood* $G_\mu \cdot z \subset V$ *there is a neighborhood* $z \in U$ *such that*

$$\phi^t_{X_h}(U) \subset V \quad \forall\, t \in I.$$

- Nonlinear stability $\Rightarrow$ ($\not\Leftarrow$) Spectral (or linear) stability.
- Notion related to the Lyapunov stability of the corresponding fixed point in the reduced space.

Existence Conditions

Let $z \in \mathcal{P}$ with $\mathbf{J}(z) = \mu$ and fix an element $\xi \in \mathfrak{g}_\mu$. Assume in the following that G_μ is compact.

Definition

The augmented Hamiltonian is the function defined by

$$h_\xi(z) = h(z) - \langle \mathbf{J}(z), \xi \rangle.$$

Existence Conditions

Let $z \in \mathcal{P}$ with $\mathbf{J}(z) = \mu$ and fix an element $\xi \in \mathfrak{g}_\mu$. Assume in the following that G_μ is compact.

Definition

The augmented Hamiltonian is the function defined by

$$h_\xi(z) = h(z) - \langle \mathbf{J}(z), \xi \rangle.$$

Theorem

Let $z \in \mathcal{P}$ with $\mathbf{J}(z) = \mu$ and let $\xi \in \mathfrak{g}_\mu$. Then z is a RE with velocity ξ if and only if

$$\mathbf{d}h_\xi(z) = 0.$$

Stability Conditions

Construct the following G_z-invariant complements:

$$\ker T_x\mathbf{J} = \mathfrak{g}_\mu \cdot z \oplus N, \quad \text{and} \quad \mathfrak{g}_\mu = \mathfrak{g}_z^\perp \oplus \mathfrak{g}_z.$$

Stability Conditions

Construct the following G_z-invariant complements:

$$\ker T_x\mathbf{J} = \mathfrak{g}_\mu \cdot z \oplus N, \quad \text{and} \quad \mathfrak{g}_\mu = \mathfrak{g}_z^\perp \oplus \mathfrak{g}_z.$$

Theorem (Energy-Momentum Method. Lerman, Singer '98; Ortega, Ratiu 99)

Let $z \in \mathcal{P}$ be a R.E. with $\mathbf{J}(z) = \mu$ and velocity $\xi \in \mathfrak{g}_\mu$. Let $\xi^\perp$ be the projection of ξ onto $\mathfrak{g}_z^\perp$. Then $\xi^\perp$ is also a velocity for the z. If the bilinear form

$$\mathbf{d}_z^2 h_{\xi^\perp}\big|_N$$

is definite, then z is stable modulo G_μ.

Note: N is called the symplectic normal space, and $\xi^\perp$ an orthogonal velocity.

Outline

Symmetric Mechanical Systems

Most symmetric Hamiltonian systems can be constructed in the following way:

Symmetric Mechanical Systems

Most symmetric Hamiltonian systems can be constructed in the following way:

- Let G act by isometries on a smooth Riemannian manifold $(M, \ll \cdot, \cdot \gg)$ configuration space.

Symmetric Mechanical Systems

Most symmetric Hamiltonian systems can be constructed in the following way:

- Let G act by isometries on a smooth Riemannian manifold $(M, \ll \cdot, \cdot \gg)$ configuration space.
- Let V be a G-invariant function on M potential energy.

Symmetric Mechanical Systems

Most symmetric Hamiltonian systems can be constructed in the following way:

- Let G act by isometries on a smooth Riemannian manifold $(M, \ll \cdot, \cdot \gg)$ configuration space.
- Let V be a G-invariant function on M potential energy.
- The phase space $\mathcal{P} = T^*M$ has a canonical symplectic structure and the lifted action of G to T^*M is Hamiltonian with momentum map

$$\langle \mathbf{J}(p_x), \xi \rangle = \langle p_x, \xi_M(x) \rangle.$$

Symmetric Mechanical Systems

Most symmetric Hamiltonian systems can be constructed in the following way:

- Let G act by isometries on a smooth Riemannian manifold $(M, \ll \cdot, \cdot \gg)$ configuration space.
- Let V be a G-invariant function on M potential energy.
- The phase space $\mathcal{P} = T^*M$ has a canonical symplectic structure and the lifted action of G to T^*M is Hamiltonian with momentum map

$$\langle \mathbf{J}(p_x), \xi \rangle = \langle p_x, \xi_M(x) \rangle.$$

- The hamiltonian function $h \in C^\infty(T^*M)$

$$h(p_x) = \frac{1}{2}|p_x|^2 + V(x)$$

is invariant by the lifted action of G on T^*M.

Definition

The symmetric Hamiltonian system constructed on the data $(M, \ll \cdot, \cdot \gg, V, G)$ is a (symmetric) simple mechanical system on M.

Definition

The symmetric Hamiltonian system constructed on the data $(M, \ll \cdot, \cdot \gg, V, G)$ is a (symmetric) simple mechanical system on M.

Objective: Particularize to simple mechanical systems the conditions for existence and stability of RE in terms of M instead of $\mathcal{P} = T^*M$. (computational advantage, since $\dim M = \frac{1}{2} \dim T^*M$).

Definition

The symmetric Hamiltonian system constructed on the data $(M, \ll \cdot, \cdot \gg, V, G)$ is a (symmetric) simple mechanical system on M.

Objective: Particularize to simple mechanical systems the conditions for existence and stability of RE in terms of M instead of $\mathcal{P} = T^*M$. (computational advantage, since $\dim M = \frac{1}{2} \dim T^*M$).

For that, we need to re-express

(i) The augmented hamiltonian h_ξ.

(ii) The symplectic normal space N.

(iii) The restricted Hessian $\mathbf{d}^2_z h_{\xi^\perp}\big|_N$.

in terms of the data $(M, \ll \cdot, \cdot \gg, V, G)$.

Existence of Relative Equilibria

Proposition (Abraham, Marsden '78)

Relative Equilibria for the SMS $(M, \ll \cdot, \cdot \gg, V, G)$ *are characterized by*

$$\mathbf{d}V_\xi(x) = 0, \qquad x \in M,\ \xi \in \mathfrak{g}$$

where

$$V_\xi(x) = V(x) - \frac{1}{2}\langle \xi, \mathbb{I}(x)\xi \rangle$$

is the augmented potential, and

$$\langle \xi, \mathbb{I}(x)\eta \rangle = \ll \xi_M(x), \eta_M(x) \gg \quad \forall x \in M,\ \xi, \eta \in \mathfrak{g}$$

is the locked inertia tensor.

Existence of Relative Equilibria

Proposition (Abraham, Marsden '78)

Relative Equilibria for the SMS $(M, \ll\cdot,\cdot\gg, V, G)$ are characterized by

$$\mathbf{d}V_\xi(x) = 0, \qquad x \in M,\ \xi \in \mathfrak{g}$$

where

$$V_\xi(x) = V(x) - \frac{1}{2}\langle \xi, \mathbb{I}(x)\xi\rangle$$

is the augmented potential, and

$$\langle \xi, \mathbb{I}(x)\eta\rangle = \ll \xi_M(x), \eta_M(x) \gg \quad \forall x \in M,\ \xi, \eta \in \mathfrak{g}$$

is the locked inertia tensor.

The solutions (x, ξ) define a relative equilibrium $p_x \in T^*_x M$ by

- $p_x = \ll \xi_M(x), \cdot \gg \in T^*_x M$.
- *velocity of the RE is ξ.*
- *momentum $\mu = \mathbb{I}(x)\xi \in \mathfrak{g}^*$.*

A normal form for N

We first need like a realization of the symplectic normal space N adapted to the geometric structure of simple mechanical systems.

Let $p_x \in T_x^*M$ be a relative equilibrium with configuration x, velocity ξ and momentum $\mu = \mathbb{I}(x)\xi$.

A normal form for N

We first need like a realization of the symplectic normal space N adapted to the geometric structure of simple mechanical systems.

Let $p_x \in T_x^*M$ be a relative equilibrium with configuration x, velocity ξ and momentum $\mu = \mathbb{I}(x)\xi$.

Step 1: Since $G_{p_x} = G_x \cap G_\mu$ is compact we can obtain a G_{p_x}-invariant splitting of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{g}_x \oplus \mathfrak{g}_{p_x}^\perp \oplus \mathfrak{t} \quad \text{with} \quad \mathfrak{g}_\mu = \mathfrak{g}_{p_x} \oplus \mathfrak{g}_{p_x}^\perp \quad \text{and} \quad \langle \mathfrak{g}_{p_x}^\perp, \mathbb{I}(x)\mathfrak{t} \rangle = 0.$$

A normal form for N

We first need like a realization of the symplectic normal space N adapted to the geometric structure of simple mechanical systems.

Let $p_x \in T_x^* M$ be a relative equilibrium with configuration x, velocity ξ and momentum $\mu = \mathbb{I}(x)\xi$.

Step 1: Since $G_{p_x} = G_x \cap G_\mu$ is compact we can obtain a G_{p_x}-invariant splitting of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{g}_x \oplus \mathfrak{g}_{p_x}^\perp \oplus \mathfrak{t} \quad \text{with} \quad \mathfrak{g}_\mu = \mathfrak{g}_{p_x} \oplus \mathfrak{g}_{p_x}^\perp \quad \text{and} \quad \langle \mathfrak{g}_{p_x}^\perp, \mathbb{I}(x)\mathfrak{t} \rangle = 0.$$

Step 2: Let $\xi^\perp = \mathbb{P}_{\mathfrak{g}_{p_x}^\perp}(\xi)$ the orthogonal velocity of the RE. Define

$$\begin{aligned} \mathbf{S} &:= (\mathfrak{g} \cdot x)^\perp \subset T_x M \\ \mathfrak{q}^\mu &:= \{\lambda \in \mathfrak{t} : \mathrm{ad}_\lambda^* \mu \in \mathfrak{g}_x^\circ\} \subset \mathfrak{g} \\ \Sigma_{\mathrm{int}} &:= \left\{ \lambda_M(x) + a : \lambda \in \mathfrak{q}^\mu, a \in \mathbf{S}, \left(\mathbf{D}\mathbb{I} \cdot (\lambda_M(x) + a)\right)(\xi^\perp) \in {\mathfrak{g}_{p_x}^\perp}^* \right\} \subset T_x M \end{aligned}$$

Step 3: Let $\hat{\mathbb{I}}_0 : (\mathfrak{g}_{p_x}^\perp \oplus \mathfrak{t}) \times (\mathfrak{g}_{p_x}^\perp \oplus \mathfrak{t}) \to \mathbb{R}$ be the restriction of $\mathbb{I}(x)$. It is non-degenerate. Define the singular Arnold form $\mathrm{Ar} : \mathfrak{q}^\mu \times \mathfrak{q}^\mu \to \mathbb{R}$ as

$$\langle \lambda_1, \mathrm{Ar}\, \lambda_2 \rangle = \langle \mathrm{ad}^*_{\lambda_1}\mu, \hat{\mathbb{I}}_0^{-1}(\mathrm{ad}^*_{\lambda_2}\mu) + \mathbb{P}_{\mathfrak{g}_{p_x}^\perp \oplus \mathfrak{t}} \left[\mathrm{ad}_\lambda \left(\hat{\mathbb{I}}_0^{-1} \mu \right) \right] \rangle \quad \forall \lambda_1, \lambda_2 \in \mathfrak{q}^\mu$$

Step 3: Let $\hat{\mathbb{I}}_0 : (\mathfrak{g}_{p_x}^{\perp} \oplus \mathfrak{t}) \times (\mathfrak{g}_{p_x}^{\perp} \oplus \mathfrak{t}) \to \mathbb{R}$ be the restriction of $\mathbb{I}(x)$. It is non-degenerate. Define the singular Arnold form $\mathrm{Ar} : \mathfrak{q}^{\mu} \times \mathfrak{q}^{\mu} \to \mathbb{R}$ as

$$\langle \lambda_1, \mathrm{Ar}\,\lambda_2 \rangle = \langle \mathrm{ad}^*_{\lambda_1}\mu, \hat{\mathbb{I}}_0^{-1}(\mathrm{ad}^*_{\lambda_2}\mu) + \mathbb{P}_{\mathfrak{g}_{p_x}^{\perp}\oplus\mathfrak{t}}\left[\mathrm{ad}_{\lambda}\left(\hat{\mathbb{I}}_0^{-1}\mu\right)\right]\rangle \quad \forall \lambda_1, \lambda_2 \in \mathfrak{q}^{\mu}$$

Proposition (M.R.-O., 2006)

At a relative equilibrium with configuration $x \in M$, $\xi \in \mathfrak{g}$ and momentum μ, if the singular Arnold form is non-degenerate, then there is a linear isomorphism

$$\varphi : \mathfrak{q}^{\mu} \oplus \Sigma_{\mathrm{int}} \oplus \mathbf{S}^* \to N \in T_{p_x}(T^*M).$$

Idea of the proof

The proof is based on the following identifications:

Idea of the proof

The proof is based on the following identifications:

1. The Sasaki metric on T^*M allows to write down an isomorphism

$$T_xM \oplus T_x^*M \to T_{p_x}(T^*M)$$

where

$$\begin{aligned}\mathrm{Ver}_{p_x} &= T_{p_x}(T_x^*M) \simeq T_x^*M \\ \mathrm{Hor}_{p_x} &= \mathrm{Ver}_{p_x}^{\perp_{\mathsf{Sasaki}}} \simeq T_xM\end{aligned}$$

Idea of the proof

The proof is based on the following identifications:

1. The Sasaki metric on T^*M allows to write down an isomorphism

$$T_xM \oplus T_x^*M \to T_{p_x}(T^*M)$$

where

$$\begin{aligned} \mathrm{Ver}_{p_x} &= T_{p_x}(T_x^*M) \simeq T_x^*M \\ \mathrm{Hor}_{p_x} &= \mathrm{Ver}_{p_x}^{\perp_{\mathsf{Sasaki}}} \simeq T_xM \end{aligned}$$

The horizontal space can also be defined as

$$\mathrm{Hor}_{p_x} = \ker K_{p_x},$$

where $K_{p_x} : T_{p_x}(T^*M) \to T_xM$ is the connection map of the Ehreshman connection on T^*M associated to the Levi-Civita connection on M.

The connection map is defined by

$$K(Y) = \left.\frac{D^{\nabla}\hat{c}(t)}{Dt}\right|_{t=0},$$

where

- $Y \in T_{p_x}(T^*M)$
- $\left.\frac{dc(t)}{dt}\right|_{t=0} = Y$
- $\hat{c}(t) = \tau_M(c(t)) \in M$
- ∇ is the Levi-Civita connection of $\ll \cdot, \cdot \gg$.

The connection map is defined by

$$K(Y) = \left.\frac{D^{\nabla}\hat{c}(t)}{Dt}\right|_{t=0},$$

where

- $Y \in T_{p_x}(T^*M)$
- $\left.\frac{dc(t)}{dt}\right|_{t=0} = Y$
- $\hat{c}(t) = \tau_M(c(t)) \in M$
- ∇ is the Levi-Civita connection of $\ll \cdot, \cdot \gg$.

2. The splitting $\mathfrak{g} = \mathfrak{g}_x \oplus \mathfrak{g}_{p_x}^{\perp} \oplus \mathfrak{t}$ and the metric $\ll \cdot, \cdot \gg$ allows us to write down the following isomorphism

$$\begin{array}{rcl} (\mathfrak{g}_{p_x}^{\perp} \oplus \mathfrak{t}) \oplus \mathbf{S} & \to & T_x M \\ ((\xi_1, \xi_2), a) & \mapsto & (\xi_1 + \xi_2)_M(x) + a \end{array}$$

3. Then we can identify

$$N \subset \quad (\mathfrak{g}_{p_x}^{\perp} \oplus \mathfrak{t}) \oplus \mathbf{S} \quad \oplus \quad (\mathfrak{g}_{p_x}^{\perp} \oplus \mathfrak{t})^* \oplus \mathbf{S}^* \quad \simeq T_{p_x}(T^*M)$$

3. Then we can identify

$$N \subset \quad (\mathfrak{g}_{p_x}^{\perp} \oplus \mathfrak{t}) \oplus \mathbf{S} \quad \oplus \quad (\mathfrak{g}_{p_x}^{\perp} \oplus \mathfrak{t})^* \oplus \mathbf{S}^* \quad \simeq T_{p_x}(T^*M)$$

T_xM

3. Then we can identify

$$N \subset \quad (\mathfrak{g}_{p_x}^{\perp} \oplus \mathfrak{t}) \oplus \mathbf{S} \quad \oplus \quad (\mathfrak{g}_{p_x}^{\perp} \oplus \mathfrak{t})^* \oplus \mathbf{S}^* \quad \simeq T_{p_x}(T^*M)$$

$$T_xM \quad \oplus \quad T_x^*M \quad \simeq T_{p_x}(T^*M)$$

3. Then we can identify

$$\begin{array}{lccccc} N \subset & (\mathfrak{g}_{p_x}^{\perp} \oplus \mathfrak{t}) \oplus \mathbf{S} & \oplus & (\mathfrak{g}_{p_x}^{\perp} \oplus \mathfrak{t})^* \oplus \mathbf{S}^* & \simeq T_{p_x}(T^*M) \\ & T_x M & \oplus & T_x^* M & \simeq T_{p_x}(T^*M) \end{array}$$

4. Let $\xi^{\perp} = \mathbb{P}_{\mathfrak{g}_{p_x}^{\perp} \oplus \mathfrak{t}} \xi$ (the orthogonal velocity). The explicit isomorphism of the Proposition is given by

$$\begin{array}{cccccccc} \mathfrak{q}^{\mu} \oplus \Sigma_{\mathrm{int}} \oplus \mathbf{S}^* & \longrightarrow & N \subset (\mathfrak{g}_{p_x}^{\perp} \oplus \mathfrak{t}) & \oplus & \mathbf{S} & \oplus & (\mathfrak{g}_{p_x}^{\perp} \oplus \mathfrak{t})^* & \oplus \mathbf{S}^* \\ (\lambda_1, ((\lambda_2)_M(x) + a), \gamma) & \longmapsto & (\lambda_1 + \lambda_2, & & a, & & N_3, & \gamma) \end{array}$$

with

$$N_3 = \frac{1}{2} \mathbb{P}_{\mathfrak{g}_{p_x}^{\perp}{}^* \oplus \mathfrak{t}^*} \left[(\mathbf{D}\mathbb{I} \cdot \xi_M^{\perp}(x))(\lambda_1 + \lambda_2) + \mathrm{ad}^*_{\lambda_1 + \lambda_2} \mu - (\mathbf{D}\mathbb{I} \cdot a)(\xi^{\perp}) \right]$$

Outline

Stability of Relative Equilibria

M. R.-O. "Stability of Relative Equilibria with Singular Momentum Values in Simple Mechanical Systems". Nonlinearity **19** *(2006) 853–877.*

Stability of Relative Equilibria

M. R.-O. "Stability of Relative Equilibria with Singular Momentum Values in Simple Mechanical Systems". Nonlinearity **19** *(2006) 853–877.*

In this adapted coordinates we can express the Hessian of the augmented Hamiltonian (stability test) as

$$\begin{array}{ccccc} & & \mathfrak{q}^{\mu} & \Sigma_{\text{int}} & \mathbf{S}^{*} \\ \mathbf{d}^2_{p_x} h_{\xi^\perp}\Big|_N & = & \left(\begin{array}{c} \mathrm{Ar} \\ 0 \\ 0 \end{array}\right. & \begin{array}{c} 0 \\ (\mathbf{d}^2_x V_{\xi^\perp} + \mathrm{corr}_{\,\xi^\perp})\Big|_{\Sigma_{\text{int}}} \\ 0 \end{array} & \left.\begin{array}{c} 0 \\ 0 \\ \ll \cdot, \cdot \gg \end{array}\right) \end{array}$$

with

$$\mathrm{corr}_{\,\xi^\perp}(v_1, v_2) = \langle \mathbb{P}_{\mathfrak{g}^\perp_{p_x} \oplus \mathfrak{t}} \left[(\mathbf{D}\mathbb{I} \cdot v_1)(\xi^\perp) \right], \hat{\mathbb{I}}_0^{-1} \left(\mathbb{P}_{\mathfrak{g}^\perp_{p_x} \oplus \mathfrak{t}} \left[(\mathbf{D}\mathbb{I} \cdot v_2)(\xi^\perp) \right] \right) \rangle$$

for every $v_1, v_2 \in T_x M$.

Stability of Relative Equilibria

M. R.-O. "Stability of Relative Equilibria with Singular Momentum Values in Simple Mechanical Systems". Nonlinearity ***19*** *(2006) 853–877.*

In this adapted coordinates we can express the Hessian of the augmented Hamiltonian (stability test) as

$$\begin{array}{cccc} & \mathfrak{q}^{\mu} & \Sigma_{\text{int}} & \mathbf{S}^{*} \\ \mathbf{d}^2_{p_x} h_{\xi^\perp}\big|_N = & \left(\begin{array}{c} \mathrm{Ar} \\ 0 \\ 0 \end{array} \right. & \begin{array}{c} 0 \\ (\mathbf{d}^2_x V_{\xi^\perp} + \mathrm{corr}_{\,\xi^\perp})\big|_{\Sigma_{\text{int}}} \\ 0 \end{array} & \left. \begin{array}{c} 0 \\ 0 \\ \ll \cdot, \cdot \gg \end{array} \right) \end{array}$$

with

$$\mathrm{corr}_{\,\xi^\perp}(v_1, v_2) = \langle \mathbb{P}_{\mathfrak{g}^\perp_{p_x} \oplus \mathfrak{t}} \left[(\mathbf{D}\mathbb{I} \cdot v_1)(\xi^\perp) \right], \hat{\mathbb{I}}_0^{-1} \left(\mathbb{P}_{\mathfrak{g}^\perp_{p_x} \oplus \mathfrak{t}} \left[(\mathbf{D}\mathbb{I} \cdot v_2)(\xi^\perp) \right] \right) \rangle$$

for every $v_1, v_2 \in T_x M$.

Since $\ll \cdot, \cdot \gg$ is positive-definite, $\mathbf{d}^2_{p_x} h_{\xi^\perp}\big|_N$ is definite if both Ar and $(\mathbf{d}^2_x V_{\lambda, \xi^\perp} + \mathrm{corr}_{\,\xi^\perp})\big|_{\Sigma_{\text{int}}}$ are positive definite.

Theorem (M.R.-O., 2006)

Let (x, ξ) *define a relative equilibrium with configuration* x*, velocity* ξ *and momentum* μ *for the simple mechanical system* $(M, \ll \cdot, \cdot \gg, V, G)$*.*

Theorem (M.R.-O., 2006)

Let (x, ξ) define a relative equilibrium with configuration x, velocity ξ and momentum μ for the simple mechanical system $(M, \ll \cdot, \cdot \gg, V, G)$.

Suppose that Ar is non-degenerate and let $\xi^\perp = \mathbb{P}_{\mathfrak{g}_{p_x}^\perp}(\xi)$ the orthogonal velocity. If

1. $\mathrm{Ar} > 0$, *and*
2. $\left(\mathbf{d}_x^2 V_{\xi^\perp} + \mathrm{corr}_{\xi^\perp}\right)\Big|_{\Sigma_{\mathrm{int}}} > 0$

then the relative equilibrium is (nonlinearly) G_μ-stable.

The case of a degenerate Arnold form

If the Arnold form is degenerate, the same techniques still produce a (less) optimal normal form for N

$$\varphi : \mathfrak{q}^{\mu} \oplus \mathbf{S} \oplus \mathbf{S}^{*} \longrightarrow N \subset T_{p_x}(T^{*}M),$$

In this case we have a block-diagonalization for the Hessian:

$$\begin{array}{ccc} & \Sigma & \mathbf{S}^{*} \\ \mathbf{d}^2_{p_x} h_{\xi^{\perp}}\big|_N = & \left(\begin{array}{c} (\mathbf{d}^2_x V_{\xi^{\perp}} + \mathrm{corr}_{\,\xi^{\perp}}(x))\big|_{\Sigma} \\ 0 \end{array} \right. & \left. \begin{array}{c} 0 \\ \ll \cdot, \cdot \gg \end{array} \right) \end{array}$$

where

$$\Sigma = \{\lambda_Q(x) + a \,:\, \forall\, \lambda \in \mathfrak{q}^{\mu},\, a \in \mathbf{S}\} \simeq \mathfrak{q}^{\mu} \times \mathbf{S}.$$

Theorem (M.R.-O., 2006)

Let (x, ξ) *define a relative equilibrium with configuration* x*, velocity* ξ *and momentum* μ *for the simple mechanical system* $(M, \ll \cdot, \cdot \gg, V, G)$*. If*

Theorem (M.R.-O., 2006)

Let (x, ξ) define a relative equilibrium with configuration x, velocity ξ and momentum μ for the simple mechanical system $(M, \ll \cdot, \cdot \gg, V, G)$. If

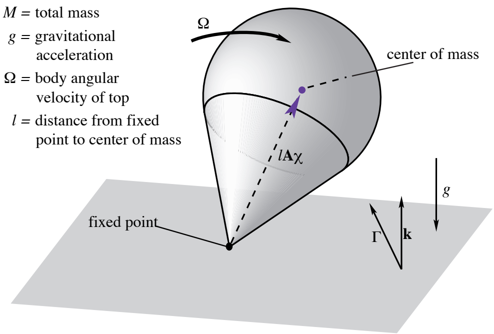
$$\left(\mathbf{d}_x^2 V_{\xi^\perp} + \mathrm{corr}_{\xi^\perp}\right)\Big|_{\Sigma} > 0$$

then the relative equilibrium is (nonlinearly) G_μ-stable.

Example: The Sleeping Lagrange Top

The Lagrange top (axisymmetric rigid body with fixed point in uniform gravity field).

- $M = \mathrm{SO}(3)$, $G = \mathbb{T}^2 = S^1 \times S^1$
- $h(\Lambda, \pi) = \frac{1}{2}\pi \cdot E_\Lambda^{-1}\pi + mgl\Lambda \mathbf{e_3} \cdot \mathbf{e_3}$, with $\Lambda \in \mathrm{SO}(3)$, $\pi \in \mathbb{R}^3 \simeq \mathfrak{so}(3)$
- $E = \mathrm{diag}\,(I_1, I_1, I_3)$ and $E_\Lambda = \Lambda E \Lambda^t$.



- $\Lambda = I, \xi = (\lambda, 0) \in \mathbb{R} \times \mathbb{R} = \mathfrak{t}^2$ is a R.E. (sleeping Lagrange top).
- $p_x = (I, \lambda\, I_3\, \mathbf{e_3}),\ G_{p_x} = S^1,\ G_\mu = \mathbb{T}^2$.
- Orthogonal velocity (choices parametrized by k):

$$\xi^\perp = \lambda \left(\frac{1}{1+k}, -\frac{k}{1+k} \right)$$

- $\Lambda = I, \xi = (\lambda, 0) \in \mathbb{R} \times \mathbb{R} = \mathfrak{t}^2$ is a R.E. (sleeping Lagrange top).
- $p_x = (I, \lambda I_3 \mathbf{e_3})$, $G_{p_x} = S^1$, $G_\mu = \mathbb{T}^2$.
- Orthogonal velocity (choices parametrized by k):

$$\xi^\perp = \lambda \left(\frac{1}{1+k}, -\frac{k}{1+k} \right)$$

Using the general Energy-Momentum Method:

Figure: Ortega, Ratiu, *Nonlinearity* '99

Now set $(\alpha_1, \alpha_2) = v_c(k)$ and combine the expressions (38) and (39) to obtain the following matrix of the Hessian $\boldsymbol{d}^2(h - \boldsymbol{J}^{v_c(k)})(z)$ restricted to W

$$\begin{pmatrix} -mgl - \lambda^2 I_3(\frac{1}{1+k} - \frac{I_3}{I_1}) & 0 & 0 & \lambda(\frac{I_3 - I_1}{I_1} + \frac{k}{1+k}) \\ 0 & -mgl - \lambda^2 I_3(\frac{1}{1+k} - \frac{I_3}{I_1}) & -\lambda(\frac{I_3 - I_1}{I_1} + \frac{k}{1+k}) & 0 \\ 0 & -\lambda(\frac{I_3 - I_1}{I_1} + \frac{k}{1+k}) & \frac{1}{I_1} & 0 \\ \lambda(\frac{I_3 - I_1}{I_1} + \frac{k}{1+k}) & 0 & 0 & \frac{1}{I_1} \end{pmatrix},$$

whose eigenvalues are

$$\sigma_\pm = A \pm \sqrt{-4I_1(1+k)^2 B + A^2},$$

with

$$A = (1+k)^2 - mglI_1(1+k)^2 + I_3\lambda^2(I_3(1+2k) - I_1(1+k))$$
$$B = \lambda^2(I_3 k + I_3 - I_1) - mgl(1+k)^2.$$

It is clear that $\boldsymbol{d}^2(h - \boldsymbol{J}^{v_c(k)})(z)$ is positive definite iff $B > 0$, that is

$$\lambda^2 > mgl \frac{(1+k)^2}{I_3 k + I_3 - I_1},$$

Using the method adapted to SMS

- $\mathfrak{t} = 0 \Rightarrow \mathfrak{q}^\mu = 0 \Rightarrow \Sigma = \mathbf{S}$.
- Augmented potential (trivial correction term):

$$V_{\xi^\perp}(\Lambda) = mgl\Lambda\mathbf{e_3}\cdot\mathbf{e_3} - \frac{\lambda^2}{2(1+k)^2}\left[\mathbf{e_3}\cdot E_\Lambda\mathbf{e_3} + 2kI_3\mathbf{e_3}\cdot\Lambda\mathbf{e_3} + k^2 I_3\right].$$

Using the method adapted to SMS

- $\mathfrak{t} = 0 \Rightarrow \mathfrak{q}^\mu = 0 \Rightarrow \Sigma = \mathbf{S}$.
- Augmented potential (trivial correction term):

$$V_{\xi^\perp}(\Lambda) = mgl\Lambda\mathbf{e_3}\cdot\mathbf{e_3} - \frac{\lambda^2}{2(1+k)^2}\left[\mathbf{e_3}\cdot E_\Lambda\mathbf{e_3} + 2kI_3\mathbf{e_3}\cdot\Lambda\mathbf{e_3} + k^2 I_3\right].$$

- $\left(\mathbf{d}_l^2 V_{\xi^\perp} + \mathrm{corr}_{\xi^\perp}\right)\Big|_\Sigma = \left(\frac{(kI_3+I_3-i_1)\lambda^2}{(1+k)^2} - mgl\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$
- Stability condition:

$$\lambda^2 > \frac{(1+k)^2 mgl}{kI_3 + I_3 - I_1}.$$

(same one)