

Approaches to a Covariant Hamiltonian Formulation of Classical Field Theory

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- 1 Introduction and motivation
 - On the need for a covariant hamiltonian formalism
 - Ingredients for a covariant hamiltonian formalism
- 2 Multisymplectic formalism
 - Analogies with the symplectic formalism
 - Polysymplectic forms: algebraic theory
 - Multisymplectic forms: algebraic theory
 - Polysymplectic fiber bundles and manifolds
 - Multisymplectic fiber bundles
- 3 Covariant functional formalism
 - Setup and the functional symplectic form
 - Qualitative description of results
 - The covariant functional Poisson bracket
- 4 Conclusions and open problems

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On the need for a covariant hamiltonian formalism

Central principles of relativistic field theory (classical and quantum)

- **Principle of covariance** (this means Lorentz covariance in the context of special relativity and general covariance in the context of general relativity): results of physical experiments do not depend on the coordinates or reference frames in terms of which we choose to describe them.
- **Principle of locality** (or microcausality): results of physical experiments performed in a certain space-time region do not depend on what happens in its spacelike complement.

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On the need for a covariant hamiltonian formalism

RCFT: defects of the usual hamiltonian formalism

- In relativistic classical field theories, the requirement of covariance is satisfied in the lagrangian formulation but violated in the usual hamiltonian formulation. This is so because the latter uses Cauchy data for its basic dynamical variables and hence presupposes the choice of a Cauchy surface in space-time. Moreover, the issue of locality is usually not even addressed.
- For non-relativistic field theories, these problems do not arise since in this case there exists a preferred family of Cauchy surfaces, namely the level sets of Newton's absolute time, and the postulate of locality is void.

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On the need for a covariant hamiltonian formalism

RQFT: role of covariance and locality, absence of Cauchy data

- In relativistic quantum field theories, the requirements of covariance and locality are incorporated from the start. (A recent development which extends the framework to quantum fields over curved space-times even unifies them into a common principle of functoriality under isometric embeddings.) On the other hand, in QFT there is no natural place for Cauchy data. In fact, restricting quantum fields to Cauchy surfaces will in general make no sense, due to their nature as distributions.

On the need for a covariant hamiltonian formalism

The issue of quantization in field theory

- The previous observations imply that quantization of relativistic classical field theories should not be based on their usual non-covariant hamiltonian formulation.
- What is needed is a manifestly covariant hamiltonian formalism where locality is built in right from the start.

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Ingredients for a covariant hamiltonian formalism

Multisymplectic and covariant functional formalism

- Among the many approaches to this problem that have been proposed, two have become particularly fruitful: the multisymplectic formalism and the covariant functional formalism.
- Each of them has its own strengths and drawbacks.
- Best results are obtained by adequately combining both approaches.

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Ingredients for a covariant hamiltonian formalism

Multisymplectic formalism

- The multisymplectic formalism is based on methods of differential geometry in a purely finite-dimensional setting and hence it is mathematically rigorous.
- Coordinate form (local), going back to the work of De Donder and Weyl in the 1930's: introduces the concept of n conjugate **multimomenta** associated with each generalized coordinate, where n is the dimension of the underlying space-time.
- Geometric form (global), appearing gradually since the 1970's: introduces the concept of a **multiphase space** which, as was clearly realized only later, comes in two variants that might be called **internal** and **extended** multiphase space.

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Covariant functional formalism

- The covariant functional formalism can be viewed as a covariant version of the usual hamiltonian formulation in which the “space” of Cauchy data is replaced by the “space” of solutions of the equations of motion – usually referred to as **covariant phase space**.
- Main advantage: it is closely similar to the symplectic formalism in mechanics.
- Main difficulty: it is inherently infinite-dimensional and hence most of its basic constructions either remain formal or else require enormous mathematical efforts to be made mathematically rigorous.

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Analogs with the symplectic formalism

Classical mechanics: dynamics governed by odes

- Configuration space Q (local coordinates q^i)
 - Velocity space – the tangent bundle TQ of Q
(local coordinates q^i, \dot{q}^i), Lagrangian $L : TQ \rightarrow \mathbb{R}$
 - Phase space – the cotangent bundle T^*Q of Q
(local coordinates q^i, p_i), Hamiltonian $H : T^*Q \rightarrow \mathbb{R}$
- Legendre transformation $\dot{q}^i \rightarrow p_i$ given by

$$p_i = \frac{\partial L}{\partial \dot{q}^i}$$

- Symplectic structure on phase space

$$\omega = dq^i \wedge dp_i$$

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Analogy with the symplectic formalism

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- To deal with non-autonomous systems, the time coordinate must be included explicitly. This amounts to replacing TQ by $\mathbb{R} \times TQ$ (local coordinates t, q^i, \dot{q}^i) as the domain of L and T^*Q by $\mathbb{R} \times T^*Q$ (local coordinates t, q^i, p_i) as the domain of H . Note that the “internal phase space” $\mathbb{R} \times T^*Q$ is not symplectic but is a contact manifold. A symplectic extension is obtained by adding yet another copy of the real line to accommodate the range of H , passing to the “extended phase space” $\mathbb{R} \times T^*Q \times \mathbb{R}$ (local coordinates t, q^i, p_i, p).

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$$p_i = \frac{\partial L}{\partial \dot{q}^i} \quad , \quad p = \frac{\partial L}{\partial \dot{q}^i} \dot{q}^i - L$$

- Symplectic structure on extended phase space

$$\omega = dq^i \wedge dp_i - dp \wedge dt$$

- Contact structure on internal phase space

$$\hat{\omega} = dq^i \wedge dp_i$$

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Classical mechanics: dynamics governed by odes

- Euler-Lagrange equations

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} = 0$$

- Hamilton equations

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}$$

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Analogy with the symplectic formalism

Classical field theory: dynamics governed by pdes

- Space-time manifold M (local coordinates x^μ), $\dim M = n$
- Configuration bundle E over M (local coordinates x^μ, q^i)
→ Velocity space – comes in two variants:
 - the first order jet bundle JE of E
(local coordinates x^μ, q^i, q^i_μ),
 - the associated first order tangent bundle T^1E
(local coordinates x^μ, q^i, \dot{q}^i).

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→ Multiphase space – comes in two variants:

- internal multiphase space = the twisted linear dual $\vec{J}^{\otimes} E$ of $\vec{J}E$ (local coordinates x^{μ}, q^i, p_i^{μ}),
- extended multiphase space = the twisted affine dual $J^{\otimes} E$ of JE (local coordinates $x^{\mu}, q^i, p_i^{\mu}, p$),
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- Remark: Over E , JE is an affine bundle and $\vec{J}E$ is a vector bundle, namely the corresponding difference vector bundle. Explicitly, $\vec{J}E \cong \pi^*(T^*M) \otimes VE$ where π is the bundle projection from E to M and VE is the vertical bundle of E .
- Remark: If the q^i accommodate the possible values of the basic fields, represented by sections of E , then the $q_{,\mu}^i$ accommodate the possible values of their first order partial derivatives and the $\vec{q}_{,\mu}^i$ accommodate the possible values of their first order covariant derivatives (with respect to any connection in E) while the multimomenta p_i^μ are conjugate (dual) to both of these and p is an additional energy variable.

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Analogy with the symplectic formalism

Underlying geometric structure

- Mechanics: cotangent bundles \longrightarrow symplectic manifolds (or even Poisson manifolds)
- Other models: classical description of half-integer spin and Souriau's sphere S^2
- Field theory: cojet bundles \longrightarrow multisymplectic fiber bundles
- Other models: ???

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Some historical remarks

- The local, coordinate dependent form of the multisymplectic formalism can, as mentioned before, be traced back to the work of De Donder and Weyl in the 1930's. The so-called De Donder - Weyl equations appear even before that, namely in a paper by Volterra in the 19th century, but unfortunately the term "Volterra equation" is already occupied.
- The global, geometric form of the multisymplectic formalism is more recent, beginning in the 1970's with the work of the "Polish group" around Tulczyjew's seminar on mathematical physics in Warsaw during the late 1960's and early 1970's, where the term "multisymplectic" was coined.

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Some historical remarks

- Nevertheless, the area has meanwhile received contributions from many authors, and I can name just a few, apologizing in advance for the omissions.
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 - 1980's: Günther; Martin
 - 1990's: Cariñena, Crampin & Ibort; Awane; Gotay, Isenberg, Marsden & Montgomery (et al.); Cantrijn, Ibort & de León
 - after 2000: many other authors (including myself) ...
- For many years, progress in the area has been hampered by several annoying obstacles.

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- Nevertheless, the area has meanwhile received contributions from many authors, and I can name just a few, apologizing in advance for the omissions.
 - 1970's: Kijowski & Szczyrba; Goldschmidt & Sternberg; Garcia
 - 1980's: Günther; Martin
 - 1990's: Cariñena, Crampin & Ibort; Awane; Gotay, Isenberg, Marsden & Montgomery (et al.); Cantrijn, Ibort & de León
 - after 2000: many other authors (including myself) ...
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- One main obstacle was a long-lasting confusion between “internal” and “extended” multiphase space (see above). This was only overcome around 1990, when it was finally realized and clearly stated that both are needed in order for the theory to work.
- No less confusing was the proliferation of proposals in the literature as to what should be the relevant mathematical structure, among them that of a “polysymplectic structure” (Günther) or “ k -symplectic structure” (Awane), which are really the same thing, and that of a “multisymplectic structure” (Martin).

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- Another important obstacle has been the absence of a fully satisfactory definition of the concept of a multisymplectic structure, given that, unfortunately, the one proposed by Martin is inadequate, since it fails to cover the physically relevant case. (For the polysymplectic case, the situation is much better.)
- Fortunately, we have recently been able to overcome these problems and to finally come up with what we think is the “right” definition (Leandro Gomes, PhD thesis; MF & LG, submitted for publication). As a result, we would like to propose to the scientific community a “streamlining” of the terminology in the area. (Comments are most welcome!)

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- To be fair, I want to state in advance that the property of existence of a special type of lagrangian subspace/subbundle that emerges as being the cornerstone of the entire theory is already present in the work of Martin, though in a somewhat hidden form and combined with other hypotheses that exclude the application of physical interest, namely the hamiltonian formulation of classical field theory. (A physician would say: surgery successful, patient dead.)
- In our approach, we stress the main structural feature of this idea, which generalizes to other situations, rather than a mere dimension formula, which does not.

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Polysymplectic forms: algebraic theory

General vector-valued form $\hat{\omega} \in (\wedge^{k+1} V^*) \otimes \hat{T}$

- V vector space, \hat{T} auxiliary vector space, $\hat{n} = \dim \hat{T}$
- Consider contraction map $\hat{\omega}^b : V \longrightarrow \wedge^k V^* \otimes \hat{T}$
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Definition of polysymplectic and polylagrangian forms

- Facts: For $\hat{n} \geq 2$, a polylagrangian subspace L need not exist, but if it does, it is unique, is given by the (non-direct) sum of the kernels of the component forms of $\hat{\omega}$ and has dimension $\dim L = \dim \ker \hat{\omega} + \hat{n} \binom{N}{k}$. Conversely, if L is an isotropic subspace of V of this dimension, then L is polylagrangian.
- Definition: $\hat{\omega}$ is **polylagrangian** (**polypresymplectic** if $k = 1$) of **rank** N if V admits a polylagrangian subspace L of codimension N .

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$$\{e_i, e_a^{i_1 \dots i_k} \mid 1 \leq a \leq \hat{n}, 1 \leq i \leq N, 1 \leq i_1 < \dots < i_k \leq N\}$$

of a subspace of V complementary to $\ker \hat{\omega}$, where the vectors $e_a^{i_1 \dots i_k}$ generate $L / \ker \hat{\omega}$, such that in terms of the dual basis of $\text{supp } \hat{\omega}$

$$\hat{\omega} = \frac{1}{k!} (e_{i_1 \dots i_k}^a \wedge e^{i_1} \wedge \dots \wedge e^{i_k}) \otimes \hat{e}_a$$

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$$0 \longrightarrow V \longrightarrow W \xrightarrow{\pi} T \longrightarrow 0$$

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Definition of multisymplectic and multilagrangian forms

- Facts: For $k - n + 2 \leq r \leq k$, a multilagrangian subspace L need not exist, but if it does, it is unique and has dimension $\dim L = \dim \ker \omega + \sum_{s=0}^{r-1} \binom{N}{s} \binom{n}{k-s}$. Conversely, if L is an isotropic subspace of V of this dimension, then L is multilagrangian.
- Definition: ω is **multilagrangian** (multipresymplectic if $k = n$ and $r = 2$) of **rank** N if the vertical space V admits a multilagrangian subspace L of codimension N .

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$$\hat{\omega}(v_1, \dots, v_r) = i_{v_1} \dots i_{v_r} \omega \quad \text{for } v_1, \dots, v_r \in V$$

(Use the identification $\Lambda^{k+1-r} T^* \cong \Lambda_0^{k+1-r} W^*$.)

- Theorem: If ω is multilagrangian (multipresymplectic), then $\hat{\omega}$ is polylagrangian (polypresymplectic), with respect to the same multi/polylagrangian subspace L . Moreover, $\ker \omega$ is contained in $\ker \hat{\omega}$ (with codimension ≤ 1).

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Multisymplectic forms: algebraic theory

Theorem (algebraic Darboux theorem for multilagrangian forms)

Let $\omega \in \bigwedge_r^{k+1} W^*$ be a multilagrangian form of rank N with multilagrangian subspace L . Then there exists a basis

$$\left\{ e_i, e_\mu, e^{i_1 \dots i_s; \mu_1 \dots \mu_{k-s}} \mid 0 \leq s \leq r-1, \begin{array}{l} 1 \leq i \leq N, 1 \leq i_1 < \dots < i_s \leq N \\ 1 \leq \mu \leq n, 1 \leq \mu_1 < \dots < \mu_{k-s} \leq n \end{array} \right\}$$

of a subspace of W complementary to $\ker \omega$, where the vectors $e^{i_1 \dots i_s; \mu_1 \dots \mu_{k-s}}$ generate $L / \ker \omega$ and the vectors e_i generate V / L , such that in terms of the dual basis of $\text{supp } \omega$

$$\omega = \sum_{s=0}^{r-1} \frac{1}{s!} \frac{1}{(k-s)!} e_{i_1 \dots i_s; \mu_1 \dots \mu_{k-s}} \wedge e^{i_1} \wedge \dots \wedge e^{i_s} \wedge e^{\mu_1} \wedge \dots \wedge e^{\mu_{k-s}}$$

$$\hat{\omega} = \frac{1}{(r-1)!} \frac{1}{(k+1-r)!} e_{i_1 \dots i_r; \mu_1 \dots \mu_{k+1-r}} \wedge e^{i_1} \wedge \dots \wedge e^{i_r} \otimes e^{\mu_1} \wedge \dots \wedge e^{\mu_{k+1-r}}$$

Polysymplectic fiber bundles and manifolds

Basic tool: Cartan calculus for vertical forms

- Given a fiber bundle P over M with projection π together with an auxiliary vector bundle \hat{T} over M , let VP be the vertical bundle of P and $\pi^*\hat{T}$ be the pull-back of \hat{T} to P ; then by abuse of language, elements of the space

$$\Omega_V^r(P; \pi^*\hat{T}) = \Gamma(\wedge^r V^*P \otimes \pi^*\hat{T})$$

of sections of the vector bundle $\wedge^r V^*P \otimes \pi^*\hat{T}$ over P are called **vertical r -forms** on P with values in $\pi^*\hat{T}$.

Polysymplectic fiber bundles and manifolds

Basic tool: Cartan calculus for vertical forms

- Strictly speaking, vertical forms are not differential forms but rather equivalence classes of differential forms on the total space of the bundle. Intuitively, they represent families of differential forms along its fibers, smoothly parametrized by the points of its base manifold.
- More precisely, a vertical r -form $\hat{\alpha} \in \Omega_V^r(P; \pi^* \hat{T})$ on P with values in $\pi^* \hat{T}$ defines, for every $x \in M$, an ordinary r -form $\hat{\alpha}_x \in \Omega^r(P_x; \hat{T}_x)$ on P_x with values in \hat{T}_x : for $p \in P_x$ and $v_1, \dots, v_r \in V_p P = T_p P_x$,

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- Remarkably, one can develop a Cartan calculus for vertical forms which has all the properties of the ordinary Cartan calculus for differential forms on manifolds, defining the operations i_X of contracting with a vertical vector field X , L_X of taking the Lie derivative along a vertical vector field X and d_V of taking the vertical exterior derivative, subject to all the usual rules; in particular, $d_V^2 = 0$ and $(d_V \hat{\alpha})_x = d \hat{\alpha}_x$.

Polysymplectic fiber bundles and manifolds

Definition of a polysymplectic fiber bundle

Definition: A **polysymplectic fiber bundle** is a fiber bundle P over a base manifold M equipped with a non-degenerate vertical 2-form $\hat{\omega}$ on the total space P taking values in $\pi^*\hat{T}$, where \hat{T} is a given auxiliary vector bundle over M and π is the bundle projection from P to M , called the **polysymplectic form**, such that the vertical bundle VP of P admits a **polylagrangian subbundle** L and such that $\hat{\omega}$ satisfies the integrability condition of being vertically closed:

$$\hat{\omega} \in \Omega_V^2(P; \pi^*\hat{T}) \quad , \quad d_V \hat{\omega} = 0 .$$

(The definition of a polylagrangian fiber bundle is analogous.)

Polysymplectic fiber bundles and manifolds

Definition of a polysymplectic manifold

Definition: A **polysymplectic manifold** is a polysymplectic fiber bundle whose base manifold reduces to a point (and hence \hat{T} is simply a given auxiliary vector space).

Polysymplectic fiber bundles and manifolds

Integrability of the polylagrangian distribution

- Theorem: If $\hat{n} \geq 3$, then L is integrable.
- Remark: The theorem does not hold if $\hat{n} = 2$, and interesting counterexamples can be constructed.
- Remark: Whenever L is integrable, it defines a foliation of P which we shall call the **polylagrangian foliation**.

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Polysymplectic fiber bundles and manifolds

Theorem (polysymplectic Darboux theorem)

Let $\hat{\omega} \in \Omega_{\mathbb{V}}^2(P; \pi^* \hat{T})$ be a polysymplectic form of rank N with integrable polylagrangian subbundle L . Given any basis $\{\hat{e}_a \mid 1 \leq a \leq \hat{n}\}$ of local sections of \hat{T} , there exists a system of local coordinates for P , consisting of

- local coordinates x^μ for M
- local coordinates q^i transversal to L
- local coordinates p_i^a along L

such that

$$\hat{\omega} = dp_i^a \wedge dq^i \otimes \hat{e}_a$$

Remark: An analogous theorem holds for polylagrangian forms.

Multisymplectic fiber bundles

Definition of a multisymplectic fiber bundle

Definition: A **multisymplectic fiber bundle** is a fiber bundle P over an n -dimensional base manifold M equipped with a non-degenerate $(n-1)$ -horizontal $(n+1)$ -form ω on the total space P , called the **multisymplectic form**, such that the vertical bundle VP of P admits a **multilagrangian subbundle** L and such that ω satisfies the integrability condition of being closed:

$$\omega \in \Omega_2^{n+1}(P) \quad , \quad d\omega = 0 .$$

(The definition of a multilagrangian fiber bundle is analogous.)

Multisymplectic fiber bundles

The symbol: $\omega \in \Omega_{\frac{1}{2}}^{n+1}(P) \longrightarrow \hat{\omega} \in \Omega_V^2(P, \pi^*(\Lambda^{n-1} T^*M))$

- Given a partially horizontal form $\omega \in \Omega_{\frac{1}{2}}^{n+1}(P)$, its **symbol** is the bundle-valued form $\hat{\omega} \in \Omega_V^2(P, \pi^*(\Lambda^{n-1} T^*M))$ given by
$$\hat{\omega}(X_1, \dots, X_r) = i_{X_1} \dots i_{X_r} \omega \quad \text{for } X_1, \dots, X_r \in \mathfrak{X}_V(P)$$
(Use the identification $\pi^*(\Lambda^{n-1} T^*M) \cong \Lambda_0^{n-1} T^*P$.)
- Theorem: If ω is multipresymplectic, then $\hat{\omega}$ is polypresymplectic, with respect to the same multi/polylagrangian subbundle L . Moreover, $\ker \omega$ is contained in $\ker \hat{\omega}$ with codimension ≤ 1 .
- Remark: Except for the last statement, an analogous relation holds between multilagrangian and polylagrangian forms.

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- Theorem: If ω is multipresymplectic, then $\hat{\omega}$ is polypresymplectic, with respect to the same multi/polylagrangian subbundle L . Moreover, $\ker \omega$ is contained in $\ker \hat{\omega}$ with codimension ≤ 1 .
- Remark: Except for the last statement, an analogous relation holds between multilagrangian and polylagrangian forms.

Multisymplectic fiber bundles

The symbol: $\omega \in \Omega_{\frac{1}{2}}^{n+1}(P) \longrightarrow \hat{\omega} \in \Omega_V^2(P, \pi^*(\Lambda^{n-1} T^*M))$

- Given a partially horizontal form $\omega \in \Omega_{\frac{1}{2}}^{n+1}(P)$, its **symbol** is the bundle-valued form $\hat{\omega} \in \Omega_V^2(P, \pi^*(\Lambda^{n-1} T^*M))$ given by
$$\hat{\omega}(X_1, \dots, X_r) = i_{X_1} \dots i_{X_r} \omega \quad \text{for } X_1, \dots, X_r \in \mathfrak{X}_V(P)$$
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Multisymplectic fiber bundles

Theorem (multisymplectic Darboux theorem)

Let $\omega \in \Omega_2^{n+1}(P)$ be a multisymplectic form of rank N with integrable multilagrangian subbundle L . Then there exists a system of local coordinates for P , consisting of

- local coordinates x^μ for M
- local coordinates q^i transversal to L
- local coordinates p_i^μ along L

such that

$$\begin{aligned}\omega &= dp_i^\mu \wedge dq^i \wedge d^n x_\mu - dp \wedge d^n x \\ \hat{\omega} &= dp_i^\mu \wedge dq^i \otimes d^n x_\mu\end{aligned}$$

Remark: An analogous theorem holds for multilagrangian forms.

Multisymplectic fiber bundles

A note on terminology

- In the terminology adopted here, there is no such thing as a multisymplectic manifold (the values of the parameters are such that one cannot allow the base manifold M to reduce to a point), and what Martin called a multisymplectic manifold is a degenerate case: it can be viewed as a polylagrangian manifold with $\hat{n} = 1$ (so it is not really “poly” since the form $\hat{\omega}$ is not vector-valued) and also as a multilagrangian manifold with $r = k + 1$ (so it is not really “multi” since the condition of horizontality becomes void). We believe it would be convenient to change the terminology accordingly.

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Setup and the functional symplectic form

Space of field configurations and space of solutions

- In the covariant functional approach to classical field theory, one considers the “space” \mathcal{C} of all possible field configurations of a given model, usually realized as the space of all smooth sections ϕ of a given fiber bundle F over space-time M with certain prescribed support properties or asymptotic properties at infinity, and within it, the “space” \mathcal{S} of all solutions of the equations of motion of the model at hand.
- Formally, \mathcal{S} can be regarded as a submanifold of \mathcal{C} , even though it may have singularities and even though both \mathcal{C} and \mathcal{S} are infinite-dimensional.

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Setup and the functional symplectic form

Formal tangent and cotangent spaces

- Next, given a specific field configuration, that is, a point ϕ in \mathcal{C} , one introduces the formal tangent space $T_\phi\mathcal{C}$ to \mathcal{C} at ϕ , realized as the space of all smooth sections $\delta\phi$ of the pull-back ϕ^*VF of the vertical bundle of F to M via ϕ with corresponding support properties or asymptotic properties at infinity, which are usually referred to as variations of ϕ .

Setup and the functional symplectic form

Formal tangent and cotangent spaces

- Applying the standard rules of dualization leads to the formal cotangent space $T_\phi^*\mathcal{C}$ to \mathcal{C} at ϕ , realized as the space of all distributional sections of the twisted dual bundle $\phi^*V^{\otimes}F$ of ϕ^*VF with corresponding dual support properties or asymptotic properties at infinity: these appear when one considers variational derivatives of functionals on \mathcal{C} .
- One of these functionals is the action functional S , whose stationary points are precisely the points of \mathcal{S} .

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Setup and the functional symplectic form

Formal tangent and cotangent spaces

- Similarly, given a specific solution, that is, a point ϕ in \mathcal{S} , one also introduces the formal tangent space $T_\phi\mathcal{S}$ to \mathcal{S} at ϕ ; of course, $T_\phi\mathcal{S}$ is a subspace of $T_\phi\mathcal{C}$, and it consists of those elements of $T_\phi\mathcal{C}$ that are solutions of the linearized equations of motion (where linearization is to be performed around ϕ).

Setup and the functional symplectic form

Covariant phase space

- The space \mathcal{S} is generally known under the name **covariant phase space** since it carries a naturally defined symplectic form Ω which, according to the approach advocated by Crnković & Witten and by Zuckerman in the 1980's, can be written as the integral of a conserved so-called symplectic current over any spacelike hypersurface Σ in space-time:

$$\Omega_\phi(\delta\phi_1, \delta\phi_2) = \int_\Sigma d\sigma_\mu J_\phi^\mu(\delta\phi_1, \delta\phi_2)$$

Qualitative description of results

Previous state of the art

- In the work of Crnković & Witten and of Zuckerman, explicit expressions for the symplectic current of various important models of field theory such as gauge theories and general relativity are worked out, but a general prescription in terms of the standard lagrangian or hamiltonian formalism is not given. Moreover, the question as to what is the covariant functional Poisson bracket, that is, the Poisson bracket associated with this symplectic form according to the usual rules of symplectic geometry, is not addressed.

Qualitative description of results

Peierls - De Witt bracket as covariant functional Poisson bracket

- Within the context of multiphase space(s) defined from a given configuration bundle E over M by taking appropriate twisted cojet bundles, one can establish a direct connection between the multisymplectic formalism and the covariant functional formalism which allows to give a simple definition of the symplectic form on covariant phase space and of the covariant functional Poisson bracket, which turns out to be identical with the Peierls - De Witt bracket familiar from field theory (Sandro Romero, PhD thesis; MF & SR, Commun. Math. Phys. 256 (2005) 375).

Qualitative description of results

Multisymplectic brackets from the Peierls - De Witt bracket

- Moreover, it can be shown that restricting to a special class of functionals, obtained by integrating so-called hamiltonian $(n - 1)$ -forms on multiphase space, pulled back to space-time with a solution of the field equations, over compact regions in some spacelike hypersurface, the Peierls - De Witt bracket reproduces the multisymplectic Poisson bracket already studied in the 1970's: this is therefore a derived object and not a fundamental one (Mário Salles, PhD thesis; MF & MS, to be submitted for publication). What is still missing is the extension to forms of other degree and the correct treatment of boundary terms.

The covariant functional Poisson bracket

The role of the causal Green function of the Jacobi operator

- The basic issue in the transition from the symplectic form to the Poisson bracket is, as in the finite-dimensional case, the task of inverting the linear operator $\Omega_\phi^b : T_\phi\mathcal{S} \rightarrow T_\phi^*\mathcal{S}$. The basic theorem in the paper by FR is that this linear operator is just the Jacobi operator of the theory at ϕ , i.e., the linear differential operator obtained by linearizing its full equations of motion around ϕ , and that the desired inversion is achieved by taking the convolution with the causal Green function of this operator.

The covariant functional Poisson bracket

Explicit formula for functional hamiltonian vector fields

- This fact expresses itself in the following explicit formula for the functional hamiltonian vector field X_F generated by a functional F on \mathcal{S} which is “local in time”:

$$X_F[\phi]^i(x) = \int_M d^n y G_{\phi}^{ij}(x, y) \frac{\delta F}{\delta \phi^j}[\phi](y)$$

The covariant functional Poisson bracket

Explicit formula for the covariant functional Poisson bracket

- As a corollary, one arrives at the following explicit formula for the covariant functional Poisson bracket between two functionals F and G on \mathcal{S} which are, e.g., regular and local:

$$\{F, G\}[\phi] = \int_M d^n x \int_M d^n y \frac{\delta F}{\delta \phi^i}[\phi](x) G_{\phi}^{ij}(x, y) \frac{\delta G}{\delta \phi^j}[\phi](y)$$

The covariant functional Poisson bracket

Basic properties of the covariant functional Poisson bracket

- It satisfies all structural properties to be expected from a decent bracket: it is \mathbb{R} -bilinear, antisymmetric and satisfies the Jacobi identity as well as the Leibniz rule with respect to the ordinary product of functionals.
- It satisfies the axiom of locality: functionals localized in spacelike separated regions of space-time commute.

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- Therefore, it is the natural candidate for the classical limit of the commutator of relativistic quantum field theory and must enter naturally into any attempt of quantization of classical field theories by deformation.
- In contrast to the more familiar equal-time Poisson brackets, it depends nontrivially on the dynamics of the theory.

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Conclusions and open problems

Some open problems

The new area of multisymplectic geometry is still in its infancy, and there are many open problems. Here are a few:

- To what extent can general multisymplectic fiber bundles deviate from the familiar class of affine duals of first order jet bundles?
- Are there interesting and, more importantly, physically relevant examples of multisymplectic fiber bundles beyond the familiar class of affine duals of first order jet bundles? (We no longer believe that there is a full multisymplectic analogue of the coadjoint orbit construction.)

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Conclusions and open problems

More open problems

Some more:

- How do the functional constructions discussed here generalize to general multisymplectic fiber bundles?
- Multisymplectic structures are obviously G -structures, but what is the underlying Lie group G ?
- How does one deal with symmetries? It is almost certain that this requires replacing Lie groups/algebras by Lie groupoids/algebroids and adapting concepts such as actions, momentum maps and reduction to this new notion of symmetry. This will also provide new methods for treating gauge theories.

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The end

THANK YOU
GRACIAS
OBRIGADO