# Examples of the Dirac Theory of Constraints for Dirac Manifolds and Further Generalizations

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## CONTENTS

# 1. Nonholonomic Systems

- $Q$: configuration space;
  $\Delta \subseteq TQ$: nonholonomic constraint;
  $L : TQ \to \mathbb{R}$ a Lagrangian.

- Equations of motion: Lagrange-d'Alembert's principle; an equivalent form of this principle:

$$\delta \int_{t_0}^{t_1} (p\dot{q} - \mathcal{E}(q, v, p))dt = 0,$$

  with $\mathcal{E} : TQ \oplus T^*Q \to \mathbb{R}$,
  $\mathcal{E}(q, v, p) = pv - L(q, v)$.

- Resulting equations:

$$\dot{p} - \frac{\partial L}{\partial q} \in \Delta^\circ \qquad (1)$$

$$\dot{q} = v \qquad (2)$$

$$p - \frac{\partial L}{\partial v} = 0 \qquad (3)$$

$$v \in \Delta \qquad (4)$$

PROBLEM: Show that (1) − (4), can be written as a Dirac equation

$$(x, \dot{x}) \oplus \alpha(x) \in D(x),$$

with $D \subseteq TM \oplus T^*M$ a given Dirac structure on some manifold $M$.
In order to unify the treatment of circuits and nonholonomic sytems we are going to consider a Dirac structure on $M = TQ \oplus T^*Q$. This makes a difference with some usual previous treatment by Yoshimura and Marsden [2006a], and it is more convenient. We consider a Dirac structure $\bar{D}_\Delta \subseteq TM \oplus T^*M$, with local expression,

$$\bar{D}_\Delta(q, v, p) = \{(q, v, p, \dot{q}, \dot{v}, \dot{p}, \alpha, \gamma, \beta) \mid \dot{q} \in \Delta(q), \\ \alpha + \dot{p} \in \Delta^\circ(q),\ \beta = \dot{q},\ \gamma = 0\}.$$

It is straightforward to check that the condition

$$(x, \dot{x}) \oplus d\mathcal{E}(x) \in \bar{D}_\Delta, \tag{5}$$

where $x = (q, v, p)$, is equivalent to

$$\dot{p} - \frac{\partial L}{\partial q} \in \Delta^\circ \tag{6}$$

$$\dot{q} = v \tag{7}$$

$$p = \frac{\partial L}{\partial v} \tag{8}$$

$$\dot{q} \in \Delta. \tag{9}$$

As we have written the equations as Dirac equations of motion, we can apply the Constraint Algorithm called CA algorithm.

The following formulas are easy to prove:

$$\begin{aligned} E_{\bar{D}_\Delta} &= \{(q,v,p,\dot{q},\dot{v},\dot{p}) \mid \dot{q} \in \Delta\}, \\ E^\flat_{\bar{D}_\Delta} &= \{(q,v,p,\alpha,\gamma,\beta) \mid \gamma = 0, \beta \in \Delta\}, \\ E^{\bar{D}_\Delta}_{\bar{D}_\Delta}(q,v,p) &= \{(q,v.p.\dot{q},\dot{v},\dot{p}) \mid \dot{q} = 0, \dot{p} \in \Delta^\circ\}. \end{aligned}$$

Then we have

$$\begin{aligned} M_1 &= \left\{(q,v,p) \;\middle|\; \left\langle d\mathcal{E}(q,v,p), E^{\bar{D}_\Delta}_{\bar{D}_\Delta}(q,v,p)\right\rangle = 0\right\} \\ &= \left\{(q,v,p) \;\middle|\; p - \frac{\partial L}{\partial v} = 0,\ v \in \Delta\right\}. \end{aligned}$$

We can continue applying the algorithm as it was previously explained in general, so we shall obtain formulas for $M_2$,....But we prefer to carry out this program in detail for the L-C circuits.

## 2. L-C Circuits

The description given in Yoshimura and Marsden [2006a]:

- $E$ is a vector space: *the charge space*;
- $TE$: *the current space*;
- $V = T^*E$ : *the flux linkage space*;
- $\Delta \subset TE$ : constant distribution that represents the *Kirchoff's Current Law (KCL)*;
- $\Delta^\circ \subset T^*E$ : represents the *Kirchoff's Voltage Law (KVL).*

There is a Dirac structure on the cotangent bundle $T^*E$, $D_\Delta \subset TV \oplus T^*V$ :

$$D_\Delta = \{(q, p, \dot{q}, \dot{p}) \oplus (q, p, \alpha_q, \alpha_p) \in TV \oplus T^*V \mid \dot{q} \in \Delta, , \alpha_p = \dot{q}, \ \alpha_q + \dot{p} \in \Delta^o\},$$

which does not depend on $(q, p)$ : $D_\Delta$ is a constant Dirac structure; for each point $(q, p)$ :

$$D_\Delta(q, p) = (q, p, \tilde{D}),$$

where

$$\tilde{D} = \{(\dot{q}, \dot{p}, \alpha, \beta) \in V \oplus V^* \mid \dot{q} \in \Delta, \ \beta = \dot{q}, \ \alpha + \dot{p} \in \Delta^o\}.$$

Dynamics:

$L : TE \longrightarrow \mathbb{R}$ : a Lagrangian given by a quadratic form on $E \times E$,

$$L(q,v) = \frac{1}{2}\sum_{i=1}^{n} L_i v_i^2 - \frac{1}{2}\sum_{i=1}^{n} \frac{1}{C_i} q_i^2.$$

The time evolution of the system is given by the equation

$$(q,p,\dot{q},\dot{p}) \oplus (q,p,\mathfrak{D}L(q,v)) \in D_\Delta(q,p),$$

with $(q,p) = \mathbb{F}L(q,v)$ and $(q,v) \in \Delta$, where $\mathfrak{D}L$ represents the Dirac differential of $L$.

Applying the CA Algorithm for L-C circuits:

We deal with L-C circuits exactly as we did with nonholonomic systems, by taking $Q = E$, $M = TE \oplus T^*E$, and $\bar{D}_\Delta$ and $\mathcal{E}$ as in the previous example, i.e.

$$\bar{D}_\Delta(q,v,p) = \{(q,v,p,\dot{q},\dot{v},\dot{p},\alpha,\gamma,\beta) \mid \dot{q} \in \Delta(q), \\ \alpha + \dot{p} \in \Delta^\circ(q),\ \beta = \dot{q},\ \gamma = 0\},$$

and $\mathcal{E} : TQ \oplus T^*Q \to \mathbb{R}$, $\mathcal{E}(q,v,p) = pv - L(q,v)$.

We define $\varphi : E \to E^*$ and $\psi : E \to E^*$ by

$$\varphi(v) = \frac{\partial L}{\partial v},$$

$$\psi(q) = \frac{\partial L}{\partial q},$$

which are linear maps given by,

$$\varphi(v) = (L_1 v_1, ..., L_n v_n)$$

and

$$\psi(q) = -\left(q_1/C_1, ..., q_n/C_n\right).$$

The evolution equations for a general nonholonomic system become in the case of L-C circuits:

$$\dot{p} + q/C \in \Delta^\circ \tag{10}$$

$$\dot{q} = v \tag{11}$$

$$p = Lv \tag{12}$$

$$\dot{q} \in \Delta \tag{13}$$

CA algorithm:

We calculate the first constraint submanifold, taking into account the general expressions stated before:

$$M_1 = \left\{(q,v,p) \in M \;\middle|\; \left\langle d\mathcal{E}, E_{\bar{D}_\Delta}^{\bar{D}_\Delta}\right\rangle = 0\right\}$$
$$= \{(q,v,p) \mid p = \varphi v,\, v \in \Delta\}.$$

We continue applying the algorithm. Let

$$W_1 = TM_1 \cap E_{\bar{D}_\Delta} = \{(q,v,p,\dot{q},\dot{v},\dot{p}) \mid (q,v,p) \in M_1,$$
$$\dot{q} \in \Delta,\, \dot{p} = \varphi \dot{v},\, \dot{v} \in \Delta\}.$$

We are going to calculate $W_1^{\bar{D}_\Delta} = (W_1^\flat)^\circ$.

$$W_1^\flat = \{(q,v,p,\alpha,\gamma,\beta) \mid (q,v,p) \in M_1,$$
$$\alpha \in \varphi(\Delta) + \Delta^\circ,\, \gamma = 0,\, \beta \in \Delta\}.$$

$$W_1^{\bar{D}_\Delta} = \{(q,v,p,\dot{q},\dot{v},\dot{p}) \mid (q,v,p) \in M_1,$$
$$\dot{q} \in (\varphi(\Delta) + \Delta^\circ)^\circ,\, \dot{p} \in \Delta^\circ\}.$$

Then,

$$M_2 = \{(q,v,p) \in M_1 \mid \left\langle d\mathcal{E}, W_1^{\bar{D}_\Delta}\right\rangle = 0\}$$
$$= \{(q,v,p) \mid q \in \psi^{-1}(\varphi(\Delta) + \Delta^\circ),\, p = \varphi v,\, v \in \Delta\}.$$

In the same way we can calculate $M_3$,

$$W_2 = TM_2 \cap E_{\bar{D}_\Delta} = \{(q,v,p,\dot{q},\dot{v},\dot{p}) \mid (q,v,p) \in M_2,$$
$$\dot{q} \in \Delta, \dot{q} \in \psi^{-1}(\varphi(\Delta) + \Delta^\circ),\, \dot{p} = \varphi\dot{v},\, \dot{v} \in \Delta\},$$

$$W_2^\flat = \{(q,v,p,\alpha,\gamma,\beta) \mid (q,v,p) \in M_2,\; \alpha \in \varphi(\Delta) + \Delta^\circ,$$
$$\gamma = 0,\, \beta \in \Delta \cap \psi^{-1}(\varphi(\Delta) + \Delta^\circ)\},$$

$$W_2^{\bar{D}_\Delta} = \{(q,v,p,\dot{q},\dot{v},\dot{p}) \mid (q,v,p) \in M_2,\; \dot{q} \in (\varphi(\Delta) + \Delta^\circ)^\circ,$$
$$\dot{p} \in (\Delta \cap \psi^{-1}(\varphi(\Delta) + \Delta^\circ))^\circ\}.$$

Then,

$$\begin{aligned} M_3 &= \left\{(q,v,p) \in M_2 \;\middle|\; \left\langle d\mathcal{E}, W_2^{\bar{D}_\Delta} \right\rangle = 0\right\} \\ &= \{(q,v,p) \mid q \in \psi^{-1}(\varphi(\Delta) + \Delta^\circ),\, p = \varphi v, \\ &\quad v \in \Delta \cap \psi^{-1}(\varphi(\Delta) + \Delta^\circ)\}. \end{aligned}$$

We can recursively define $M_k$. For all $k \geq 1$ define,

$$\Delta_k = \Delta \cap \psi^{-1}(\varphi(\Delta_{k-1}) + \Delta^\circ),$$

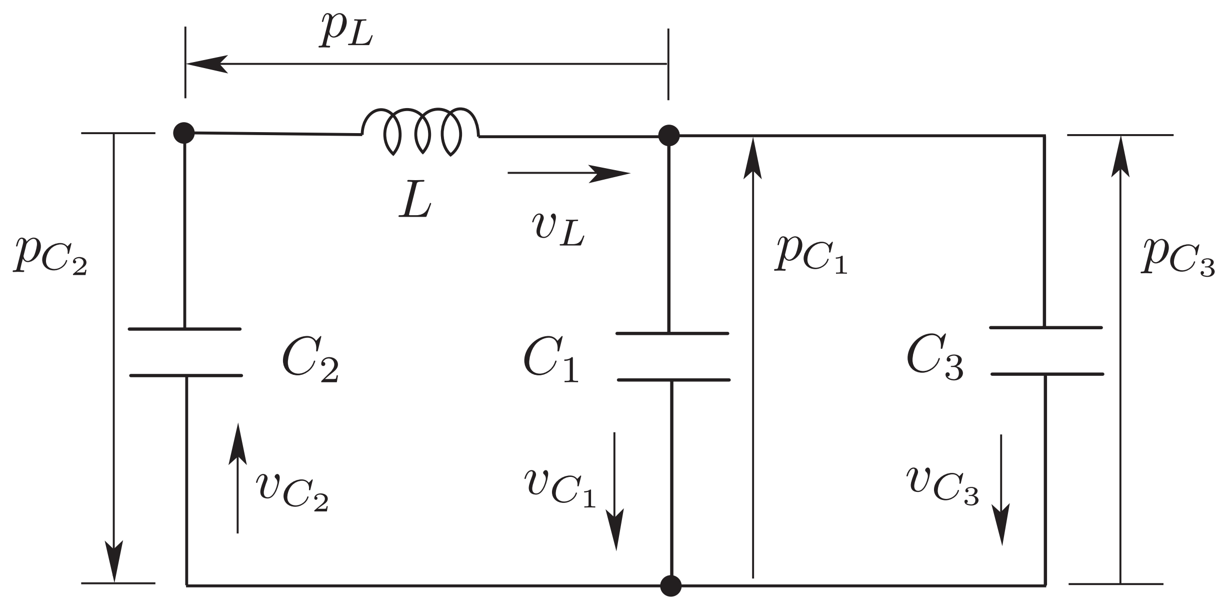
where $\Delta_0 = \Delta$ by definition.
We have the following expressions for the constraint submanifolds $M_k$.

$$M_1 = \{(q, v, p) \mid p = \varphi(v), v \in \Delta\}$$

$$\begin{aligned} M_{2k-1} = \{(q, v, p) \mid \ & q \in \psi^{-1}(\varphi(\Delta_{k-2}) + \Delta^\circ), \\ & p = \varphi(v),\ v \in \Delta_{k-1}\},\ \ k \geq 2 \end{aligned}$$

$$\begin{aligned} M_{2k} = \{(q, v, p) \mid \ & q \in \psi^{-1}(\varphi(\Delta_{k-1}) + \Delta^\circ), \\ & p = \varphi(v),\ v \in \Delta_{k-1}\},\ \ k \geq 1. \end{aligned}$$

**A Concrete Example.** We shall illustrate our method with a 4 -port L-C circuit studied by Yoshimura and Marsden, with configuration space $E = \mathbb{R}^4$.




Notation:

- $q = (q_L, q_{C_1}, q_{C_2}, q_{C_3}) \in E$ : charge space,
- $v = (v_L, v_{C_1}, v_{C_2}, v_{C_3}) \in T_qE$ : current space,
- $p = (p_L, p_{C_1}, p_{C_2}, p_{C_3}) \in T_q^*E$ : flux linkage space.

The Lagrangian of the L-C circuit is $L : TE \to \mathbb{R}$,

$$L(q, v) = \frac{1}{2}L(v_L)^2 - \frac{1}{2}\frac{(q_{C_1})^2}{C_1} - \frac{1}{2}\frac{(q_{C_2})^2}{C_2} - \frac{1}{2}\frac{(q_{C_3})^2}{C_3}.$$

The KCL constraints $\Delta \subseteq TE$ for the current $v$ are,

$$\begin{aligned} -v_L + v_{C_2} &= 0 \\ -v_{C_1} + v_{C_2} - v_{C_3} &= 0 \end{aligned}$$

The constraint KCL space is, for each $q \in E$,

$$\Delta(q) = \{v \in T_qE \mid \langle \omega^a, v \rangle = 0,\ a = 1,\, 2\},$$

where

$$\omega^a_k = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 \end{pmatrix}$$

The constraint KVL space is the annihilator $\Delta^\circ$ of $\Delta$, defined, for each $q \in E$,

$$\Delta^\circ(q) = \{p \in T^*_qE \mid \langle p, v \rangle = 0, \text{ for all } v \in \Delta(q)\}.$$

We remember that the Dirac structure $\bar{D}_\Delta \subseteq TM \oplus T^*M$ on $M = TE \oplus T^*E$ associated to the space $\Delta(q)$ is, for each $(q, v, p) \in M$, given by

$$\bar{D}_\Delta(q, v, p) = \{(q, v, p, \dot{q}, \dot{v}, \dot{p}, \alpha, \gamma, \beta) \in TM \oplus T^*M \mid \dot{q} \in \Delta(q),\ \alpha + \dot{p} \in \Delta^\circ(q),\ \beta = \dot{q},\ \gamma = 0\},$$

the corresponding energy function $\mathcal{E} : M \to \mathbb{R},$ is given by $\mathcal{E}(q, v, p) = pv - L(q, v),$ we can easily verify that the Dirac equations

$$(q, v, p, \dot{q}, \dot{v}, \dot{p}) \oplus d\mathcal{E} \in \bar{D}_\Delta(q, v, p),$$

are equivalent to the Implicit Differential Equations, given in coordinates by

$$\begin{aligned}
\dot{q}_L = v_L,\ \dot{q}_{C_1} = v_{C_1},\ \dot{q}_{C_2} = v_{C_2},\ \dot{q}_{C_3} = v_{C_3} \\
\dot{p}_L + \frac{q_{C_1}}{C_1} + \dot{p}_{C_1} + \frac{q_{C_2}}{C_2} + \dot{p}_{C_2} = 0 \\
\dot{p}_L + \frac{q_{C_2}}{C_2} + \dot{p}_{C_2} + \frac{q_{C_3}}{C_3} + \dot{p}_{C_3} = 0 \\
p_L = Lv_L,\ p_{C_1} = p_{C_2} = p_{C_3} = 0 \\
v_L = v_{C_2},\ v_{C_1} = v_{C_2} - v_{C_3}
\end{aligned}$$

The CA algorithm for this example:

The expressions for $\varphi(v) = \frac{\partial L}{\partial v}$ and $\psi(q) = \frac{\partial L}{\partial q}$ are in this case,

$$\varphi(v_L, v_{C_1}, v_{C_2}, v_{C_3}) = L v_L \underline{e}^0,$$
$$\psi(q_L, q_{C_1}, q_{C_2}, q_{C_3}) = -\frac{q_{C_1}}{C_1}\underline{e}^1 - \frac{q_{C_2}}{C_2}\underline{e}^2 - \frac{q_{C_3}}{C_3}\underline{e}^3,$$

with $\{\bar{e}_0, \bar{e}_1, \bar{e}_2, \bar{e}_3\}$ and $(\underline{e}^0, \underline{e}^1, \underline{e}^2, \underline{e}^3)$ bases of $E$ and $E^*$, respectively.
We get,

$$M_1 = \{(q_L, q_{C_1}, q_{C_2}, q_{C_3}, v_L, v_{C_1}, v_{C_2}, v_{C_3}, p_L, p_{C_1}, p_{C_2}, p_{C_3}) \mid p_L = L_{v_L},\ p_{C_1} = p_{C_2} = p_{C_3} = 0, v_L = v_{C_2},\ v_{C_1} = v_{C_2} - v_{C_3}\}.$$

$$M_2 = \{(q_L, q_{C_1}, q_{C_2}, q_{C_3}, v_L, v_{C_1}, v_{C_2}, v_{C_3}, p_L, p_{C_1}, p_{C_2}, p_{C_3}) \mid p_L = L_{v_L},\ p_{C_1} = p_{C_2} = p_{C_3} = 0, v_L = v_{C_2},\ v_{C_1} = v_{C_2} - v_{C_3},\ \frac{q_{C_1}}{C_1} = \frac{q_{C_3}}{C_3}\};$$

and

$$M_3 = \{(q_L, q_{C_1}, q_{C_2}, q_{C_3}, v_L, v_{C_1}, v_{C_2}, v_{C_3}, p_L, p_{C_1}, p_{C_2}, p_{C_3}) \mid$$
$$p_L = L_{v_L},\ p_{C_1} = p_{C_2} = p_{C_3} = 0,$$
$$v_L = v_{C_2},\ v_{C_1} = v_{C_2} - v_{C_3},\ \frac{q_{C_1}}{C_1} = \frac{q_{C_3}}{C_3},\ \frac{v_{C_1}}{C_1} = \frac{v_{C_3}}{C_3}\}.$$

Finally, we conclude that $M_3 = M_4$ and the algorithm stops. We can parametrize $M_3$, which has dimension 4, and we obtain an ODE equivalent to equations of motion:

$$\begin{aligned}
\dot{q}_L &= \frac{p_L}{L}\\
\dot{q}_{C_1} &= \frac{C_1}{L(C_1 + C_3)} p_L\\
\dot{q}_{C_2} &= \frac{p_L}{L}\\
\dot{p}_L &= -\frac{q_{C_1}}{C_1} - \frac{q_{C_2}}{C_2}
\end{aligned}$$

The solutions should be tangent to $W_3 = TM_3 \cap E_{\bar{D}_\Delta}$, that is

$$\dot{p}_L = L\dot{v}_L,\ \dot{p}_{C_1} = \dot{p}_{C_2} = \dot{p}_{C_3} = 0,\ \dot{v}_L = \dot{v}_{C_2},$$
$$\dot{v}_{C_1} = \dot{v}_{C_2} - \dot{v}_{C_3},\ \frac{\dot{q}_{C_1}}{C_1} = \frac{\dot{q}_{C_3}}{C_3},\ \frac{\dot{v}_{C_1}}{C_1} = \frac{\dot{v}_{C_3}}{C_3}.$$

The CAD algorithm for this example:

Applying the CA algorithm the final constraint submanifold is $M_3 = M_4$. We have proven that $W_3 = TM_3 \cap E_{\bar{D}_\Delta}$ is given by

$$\dot{p}_L = L\dot{v}_L,\ \dot{p}_{C_1} = \dot{p}_{C_2} = \dot{p}_{C_3} = 0,\ \dot{v}_L = \dot{v}_{C_2},$$
$$\dot{v}_{C_1} = \dot{v}_{C_2} - \dot{v}_{C_3},\ \frac{\dot{q}_{C_1}}{C_1} = \frac{\dot{q}_{C_3}}{C_3},\ \frac{\dot{v}_{C_1}}{C_1} = \frac{\dot{v}_{C_3}}{C_3}.$$

For a given $x_0 \in M_3$,

$$\begin{aligned} x_0 =& (q_{L0}, q_{C_10}, q_{C_20}, q_{C_30}, v_{L0}, \\ & v_{C_10}, v_{C_20}, v_{C_30}, p_{L0}, p_{C_10}, p_{C_20}, p_{C_30}), \end{aligned}$$

the elements of the symplectic leaf $x_0 + \tilde{W}_3$ are characterized by the conditions defining $M_3$ plus the condition obtained by integrating with respect to time the conditions

$$\begin{aligned} -v_L + v_{C_2} =& 0 \\ -v_{C_1} + v_{C_2} - v_{C_3} =& 0, \end{aligned}$$

applied to $(q_L - q_{L0}, q_{C_1} - q_{C_10}, q_{C_2} - q_{C_20}, q_{C_3} - q_{C_30})$, that is

$$\begin{aligned} -(q_L - q_{L0}) + (q_{C_2} - q_{C_20}) &= 0 \\ -(q_{C_1} - q_{C_10}) + (q_{C_2} - q_{C_20}) - (q_{C_3} - q_{C_30}) &= 0. \end{aligned}$$

For simplicity, we will assume on that the conditions

$$\begin{aligned} q_{L0} - q_{C_2 0} &= 0 \\ q_{C_1 0} - q_{C_2 0} + q_{C_3 0} &= 0, \end{aligned}$$

are satisfied.

We can conclude that $x_0 + \tilde{W}_3$ has dimension 2 and the projection $\bar{\pi}(x_0 + \tilde{W}_3) \subseteq T^*E$ is the subspace of $T^*E$ defined by:

$$\begin{aligned} q_{C_1} = q_{C_2} = q_{C_3} &= 0 \\ p_{C_1} = p_{C_2} = p_{C_3} &= 0. \end{aligned}$$

We can use the variables $(q_L, p_L)$ to parametrize $\bar{\pi}(x_0 + \tilde{W}_3)$, namely,

$$
\begin{aligned}
v_L &= \frac{p_L}{L},\, p_{c_1} = 0,\, p_{c_2} = 0,\, p_{c_3} = 0,\\
v_{C_2} &= \frac{p_L}{L},\\
v_{C_1} &= \frac{C_1}{C_1 + C_3}\frac{1}{L}p_L,\\
v_{C_3} &= \frac{C_3}{C_1 + C_3}\frac{1}{L}p_L,\\
v_{C_3} &= \frac{C_3}{C_1 + C_3}\frac{1}{L}p_L,\\
q_{c_3} &= \frac{C_3}{C_1 + C_3}q_L\\
q_{c_1} &= \frac{C_1}{C_1 + C_3}q_L\\
q_{C_2} &= q_L\\
\nu_L &= 0,\, \nu_{C_1} = 0,\, \nu_{C_2} = 0,\, \nu_{C_3} = 0.
\end{aligned}
$$

We can calculate the matrix

$\Sigma_{ij} = \{d\epsilon_i, d\epsilon_j\} = \{\epsilon_i, \epsilon_j\}$

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -L & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{C_1} & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{C_3} & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \\
0 & \frac{1}{C_1} & 0 & -\frac{1}{C_3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{C_1} & 0 & -\frac{1}{C_3} & \\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
L & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -\frac{1}{C_1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & \frac{1}{C_3} & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},$$

with inverse $\Sigma^{ij}$,. Since $c = 3$ we have $\lambda^i_{(3)} = \Sigma^{ij}\{\mathcal{E}, \epsilon_j\}$, $j = 1, \ldots, 14$, where the column vector $[\{\mathcal{E}, \epsilon_j\}]$ is

$$[\{\mathcal{E}, \epsilon_j\}] = \left[0, -\frac{q_{C_1}}{C_1}, -\frac{q_{C_2}}{C_2}, -\frac{q_{C_3}}{C_3}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right]^T,$$

$\epsilon_j$ are chosen such that the equations $\epsilon_j = 0$, $j = 1, \ldots, 14$, define $x_0 + \tilde{W}_3$. In fact, we have chosen

$$\begin{aligned}
\epsilon_1 &= p_L - Lv_L \\
\epsilon_2 &= p_{C_1} \\
\epsilon_3 &= p_{C_2} \\
\epsilon_4 &= p_{C_3} \\
\epsilon_5 &= v_L - v_{C_2} \\
\epsilon_6 &= v_{C_1} - v_{C_2} + v_{C_3} \\
\epsilon_7 &= \frac{q_{C_1}}{C_1} - \frac{q_{C_3}}{C_3} \\
\epsilon_8 &= \frac{v_{C_1}}{C_1} - \frac{v_{C_3}}{C_3} \\
\epsilon_9 &= q_L - q_{C_2} \\
\epsilon_{10} &= q_{C_1} - q_{C_2} + q_{C_3} \\
\epsilon_{11} &= \nu_L \\
\epsilon_{12} &= \nu_{C_1} \\
\epsilon_{13} &= \nu_{C_2} \\
\epsilon_{14} &= \nu_{C_3}.
\end{aligned}$$

The Hamiltonian vector field representing the evolution in $x_0 + \tilde{W}_3$:

$$X_{\tilde{\mathcal{E}}} = d\mathcal{E}^\sharp + \lambda^j_{(3)} X_{\epsilon_j}.$$

For instance, we will calculate $\dot{p}_L$ and $\dot{v}_L$:

$$\begin{aligned}
\dot{p}_L &= X_{\tilde{\mathcal{E}}}(p_L) \\
&= (d\mathcal{E})^\sharp(p_L) + \lambda^j_{(3)} X_{\epsilon_j}(p_L) \\
&= \{p_L, \mathcal{E}\} + \lambda^j_{(3)} \{p_L, \epsilon_j\} \\
&= 0 + \lambda^9_{(3)}(-1) \\
&= -\frac{q_{C_1}}{C_1} - \frac{q_{C_2}}{C_2}
\end{aligned}$$

$$\begin{aligned}
\dot{v}_L &= X_{\tilde{\mathcal{E}}}(v_L) \\
&= (d\mathcal{E})^\sharp(v_L) + \lambda^j_{(3)} X_{\epsilon_j}(v_L) \\
&= \{v_L, \mathcal{E}\} + \lambda^j_{(3)} \{v_L, \epsilon_j\} \\
&= 0 + \lambda^{11}_{(3)} \\
&= \frac{1}{L}(-\frac{q_{C_1}}{C_1} - \frac{q_{C_2}}{C_2})
\end{aligned}$$

The complete coordinate expression of $X_{\tilde{\mathcal{E}}}$:

$$\begin{bmatrix} \dot{q}_L \\ \dot{q}_{C_1} \\ \dot{q}_{C_2} \\ \dot{q}_{C_3} \\ \dot{v}_L \\ \dot{v}_{C_1} \\ \dot{v}_{C_2} \\ \dot{v}_{C_3} \\ \dot{p}_L \\ \dot{p}_{C_1} \\ \dot{p}_{C_2} \\ \dot{p}_{C_3} \\ \dot{\nu}_L \\ \dot{\nu}_{C_1} \\ \dot{\nu}_{C_2} \\ \dot{\nu}_{C_3} \end{bmatrix} = \begin{bmatrix} v_L \\ v_{c_1} \\ v_{c_2} \\ v_{c_3} \\ \frac{1}{L} \\ \frac{C_1}{C_1+C_3}\frac{1}{L}\left(-\frac{q_{C_1}}{C_1} - \frac{q_{C_2}}{C_2}\right) \\ \frac{1}{L}\left(-\frac{q_{C_1}}{C_1} - \frac{q_{C_2}}{C_2}\right) \\ \frac{C_3}{C_1+C_3}\frac{1}{L}\left(-\frac{q_{C_1}}{C_1} - \frac{q_{C_2}}{C_2}\right) \\ -\frac{q_{C_1}}{C_1} - \frac{q_{C_2}}{C_2}, \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Equations of motion in the symplectic manifold $x_0 + \tilde{W}_3$.

$$X_{d\mathcal{E}|W_{3x}} = (\omega_D|W_{3x})^\sharp\,(d\mathcal{E}|W_{3x})\,.$$

We are going to use the previous parametrization of $x_0 + \tilde{W}_3$ with coordinates $q_L$, $p_L$, where the symplectic form is given by $dq_L \wedge dp_L$.
The Lagrangian is given

$$L(q,v) = \frac{1}{2}L(v_L)^2 - \frac{1}{2}\frac{(q_{C_1})^2}{C_1} - \frac{1}{2}\frac{(q_{C_2})^2}{C_2} - \frac{1}{2}\frac{(q_{C_3})^2}{C_3}.$$

In terms of the parametrization, the energy function $\mathcal{E} = p_L v_L - L(q,v)$ restricted to $x_0 + \tilde{W}_3$ is

$$\left(\mathcal{E}|x_0 + \tilde{W}_c\right)(q_L, p_L) = \frac{1}{2}\frac{p_L^2}{L} + \frac{1}{2}q_L^2\left(\frac{1}{C_1 + C_3} + \frac{1}{C_2}\right).$$

Then the vector field $X_{d\mathcal{E}|x_0+\tilde{W}_c}$ is given in coordinates $q_L$, $p_L$ by

$$\begin{aligned}
\dot{q}_L &= \frac{\partial H}{\partial p_L} = \frac{p_L}{L} \\
\dot{p}_L &= -\frac{\partial H}{\partial q_L} = -q_L\left(\frac{1}{C_1 + C_3} + \frac{1}{C_2}\right)
\end{aligned}$$

**Final Remark:** the previous equations can easily be deduced by elementary rules of circuit theory. For instance, one can first replace the capacitors $C_1$ and $C_3$ by a single capacitor with capacity $C_1 + C_3$. Then one obtains the simplest L-C circuit with inductance $L$ and a capacitor which is the series $C$ of $C_2$ and $C_1 + C_3$,

$$C = \frac{C_2(C_1 + C_3)}{C_2 + C_1 + C_3};$$

which is very easy to solve, and we obtain a solution which is equivalent to

$$\dot{q}_L = \frac{\partial H}{\partial p_L} = \frac{p_L}{L}$$

$$\dot{p}_L = -\frac{\partial H}{\partial q_L} = -q_L \left( \frac{1}{C_1 + C_3} + \frac{1}{C_2} \right)$$

*But, the theory developed in this work reveals a certain Hamiltonian structure behind a given L-C circuit.*

## 3. Further Generalizations

Some basic facts concerning general Implicit Differential Equations (IDE) and Constraint Algorithms.

$$\varphi(x, \dot{x}) = 0$$

Basic questions not completely answered yet: existence, uniqueness or extension of solutions.

Partial results in this direction have been established: CeEt2006 and references therein.

One of the common features of those results: how to transform certain types of IDE into an equivalent ODE depending on parameters on a certain *final constraint manifold*?

This is achieved by the reduction or constraint or Gotay-Nester algorithm, depending on the context and author, but there is a natural common idea to all of them. For instance, we can see the following references to see how the algorithm works in different contexts:

- CeEt, Desingularization of implicit analytic differential equations, *J. Phys. A* **39**, 10975–11001, 2006.
- Gotay, Hinds, and Nester, Presymplectic manifolds and the Dirac-Bergman theory of constraints, *J. Math. Phys.* **19**, 2388–2399, 1978.
- Pritchard, F., On implicit systems of differential equations, *J. Diff. Equations* **194**, 328–363, 2003.
- Rabier, P. and Rheinboldt, W. , A geometric treatment of implicit differential-algebraic equations, *J. Differential Equations* **109**, 110–146, 1994.


*From the geometric point of view the basic idea of this algorithm is very simple and natural and is already contained in the Dirac theory of constraints.*