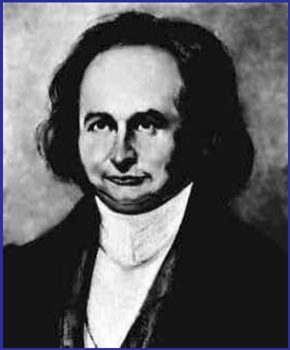


Geometric aspects of Integrability: an elementary overview

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Abstract

The talk will introduce in an elementary way both the basic ideas of Integrability and some relevant examples of Integrable Systems from a viewpoint which makes special emphasis on geometric aspects. Important particular examples from the 'basic ones' (e.g. Kepler problem, harmonic oscillator) to other more complicated ones (e.g. Calogero-Sutherland), will be presented. Then some characterizations of Integrable systems in spaces with a maximal (geometric) invariance group will be discussed, in particular those related with the Stackel separable form. A mention of some open conjectures on Integrable systems and of a novel aspect, the construction of integrable systems from coalgebras, will close the talk.

Outline

- **What is Integrability?**
  - ⋆ **Integrability and symmetry**
- **Concepts of integrability: the classical Liouville-Arnol'd theorems**
  - ⋆ **Integrability by quadratures**
  - ⋆ **Complete integrability**
  - ⋆ **Superintegrability**
- **How can you identify a system as integrable? How can you build a system which is guaranteed to be integrable?**
  - ⋆ **The Darboux 'direct approach'**
  - ⋆ **Integrability and separation of variables**
  - ⋆ **The inverse scattering method and the Lax pairs**
  - ⋆ **The Projection method**
  - ⋆ **Integrability from coalgebra structure**
- **Some conjectures and open problems**

All along the talk I will include several batches of classical examples.

What is integrability for a Hamiltonian system?

- **More general scope:** Frobenius theorem
- **Naive meaning** Possibility to explicitly integrate the equations of motion for a (hamiltonian) system, i.e., to find their solutions, as functions of the time, in a form as closed and/or explicit as possible.
  - ⋆ **Several levels of 'explicit'** Tradition was to consider a problem 'solved' as soon as it was reduced to (a sequence of) quadratures.
  - ⋆ **Pursuing the explicit integrability led to important techniques, advances and results in analysis:** e.g. elliptic functions in relation to Euler's equations for the motion of a rigid body, hyperelliptic functions in the Kowaleska top.
- **Integrability and chaotic motion are at the two ends of some spectrum** But integrability is exceptional, chaoticity is generic.
- **In all cases, integrability seems to be deeply related with some symmetry,** which might be partially hidden; the existence of constants of motion reflects this symmetry.
- **Symmetry is described by a group G, acting on M and leaving H invariant** The most important imprint of symmetry is the existence of constants of motion (Noether, Arnold): If a Hamiltonian is invariant under a Hamiltonian action of a group G, the corresponding moment is a first integral of the system

What is integrability for a Hamiltonian system?

- **Consider motion in a manifold M (dimension $2n$) with a Poisson structure, $\{\ ,\ \}$.** Hamiltonian dynamical systems, with hamiltonian $H(x)$ have the equations of motion:
  $\dot{x^j} = \{H(x), x^j\} = \mathrm{sgrad}\, H(x)$
- **Initially, Integrability meant: reducible to quadratures**
- **Integrability is linked to 'existence of constants of motion'** Constants of motion determine foliations in the phase space which is thus decomposed as a collection of lesser dimensional submanifolds and the problem is thus automatically reduced to one with less degrees of freedom.
  - ⋆ **This gives the modern way to approach this problem** Everything hinges on how many constants of motion does the given system admits and how precisely they are related, and how the phase space is foliated by the level sets of these functions.
- **Beware of the essential distinction among local 'constants of motion' and globally defined (true) constants of motion** In a small neighbourhood of any (regular) point there always exist $n-1$ local first integrals. But this propery extends to one covering the whole phase space only in exceptional situations.

Clasical examples of integrability before 1900

- **Examples for a finite number (n=2, 3) of degrees of freedom, and ODE**
- **For one degree of freedom, every hamiltonian system is integrable (H itself is a constant of motion).** Significative examples start either for a single particle in 2d or for several particles in 1d.
- **Relevant examples of integrable mechanical systems by the end of XIX century**
  - ⋆ **Motion in Euclidean space under central potentials** In particular, the Harmonic oscillator and the Kepler problem (Newton)
  - ⋆ **Motion in two Newtonian fixed centers** (Euler)
  - ⋆ **Geodesics on the triaxial ellipsoid** (Jacobi, 1838)
  - ⋆ **Motion in the sphere under a linear force** (a variant of spherical harmonic oscillator; Neumann)
  - ⋆ **Motion of three particles in 1d under a pairwise potential** $V(q) = \sum_{i<j}^{3} g_{ij}^2 (q^i - q^j)^{-2}$ (Jacobi)
  - ⋆ **Motion of a rigid body about a fixed point** in several special cases (Euler, Lagrange, Kowalewska)
  - ⋆ **Motion of a rigid body in ideal fluids in some special cases** (Clebsch, Kirchoff)

Concepts of integrability

- **Difficult to encapsulate in a single concept: several possiblities arise, each with some theoretical interest (Birkhoff 1927)**

A system of differential equations is only more or less integrable

H. Poincaré

- ⋆ **Integrability by quadratures**
- ⋆ **Partial Integrability**
- ⋆ **Complete (Liouville-Arnol'd) Integrability**
- ⋆ **Algebraic complete integrability**
- ⋆ **Superintegrability**

Integrability by quadratures (Bour 1855, Liouville 1855)

- **A system of differential equations is said to be integrable by quadratures** if its solutions can be found after a finite number of steps involving algebraic operations and integration of given functions.
- **Given a Hamiltonian system $\dot{x}^k = \omega^{kl}\partial_l H(x)$ on a symplectic space M, when is it integrable by quadratures?** A first answer is the classical:
- **Liouville theorem (Bour 1855, Liouville 1855)** If a Hamiltonian system with n degrees of freedom has n integrals of motion $F_1, \ldots, F_n$ in involution, $\{F_i, F_j\} = 0$ and functionally independent on the (intersection of) level sets of the n functions, $F_i = f_i$, then the solutions of the corresponding Hamilton's equations can be found by quadratures.
  - ⋆ **Can the involution condition be relaxed?** Yes, at the price of an additional requirement making the general result seldom applicable: if the F_i are assumed to close a **solvable** Lie algebra $\{F_i, F_j\} = c_{ij}^k F_k$, on those intersection of level sets of the n constants for which $c_{ij}^k f_k = 0$, the solutions can also be found by quadratures (Mischenko-Fomenko-Kozlov 1979).
  - ⋆ **Is n the maximum possible number of constants of motion** No, there may exist up to $2n-1$ functionally independent constants of motion (Examples: the Kepler problem or the Harmonic oscillator).
  - ⋆ **Is n the maximum number of constants of motion in involution?** Yes.

Complete Integrability (Arnol'd, 1978)

- **Liouville theorem was sharpened and generalized by Arnol'd in 1978** At present the concept on integrability behind this extension is called complete Integrability in Liouville-Arnol'd sense.
- **Arnol'd theorem** If a Hamiltonian system with n degrees of freedom has n integrals of motion $F_1, \dots, F_n$ in involution, $\{F_i, F_j\} = 0$ and functionally independent on the (intersection $M_{\mathbf{f}}$ of) level sets of the n functions, $F_i = f_i$, and the n hamiltonian fields associated to the constants F_i are complete on $M_{\mathbf{f}}$, then the flow on phase space induced by the Hamiltonians F_i leaves $M_{\mathbf{f}}$ invariant, and topologically each $M_{\mathbf{f}}$ is diffeomorphic to some product of a k-torus and, possibly a $n-k$ dimensional Euclidean space.
- **Action-angle variables follow from this construction**
  - ⋆ **In most situations, the condition that the hamiltonian fields associated to the constants F_i are complete is difficult to check** and the common practice is to omit this condition and speak (loosely) of *completely integrable systems*
- **What is the geometric meaning?** The Hamiltonian vector fields associated to the invariants of the foliation span the tangent distribution.
- **Equivalently,** there is a maximal set of Poisson commuting invariants.

Beyond Integrability: Superintegrability

- **A system with $\dim M = 2n$ may have more than n functionally independent constants of motion F_i.**
  - ⋆ **This number might range from $N = n + 1$ (minimally superintegrable systems) to $N = 2n - 1$ (maximally superintegrable systems).** Only n of them may be in involution, and as the set of constants of motion is closed under Poisson brackets, we must have

    $$\{F_i, F_j\} = \Phi_{i,j}(F_1, F_2, \dots F_N), \qquad i, j = 1, \dots N$$

- **The simplest case, but by no means the only possible case** corresponds to the $\Phi_{i,j}$ being *linear* functions: in this case constants of motion close a **Lie algebra**

  $$\{F_i, F_j\} = C^k_{i,j} F_k, \qquad i, j = 1, \dots N$$

  - ⋆ **This is however not intrinsic:** even if a particular choice of N functionally independent constants of motion F_i close a Lie algebra, a new different choice F'_i related **nonlinearly** with the F_i will *not* close a Lie algebra, but in general another type of 'higher order' algebra.

- **The 'canonical examples' of superintegrable systems**
  - ⋆ **Kepler problem** Energy, Angular momentum, Runge-Lenz vector.
  - ⋆ **Harmonic oscillator** Energy, Angular momentum, Fradkin tensor.

Integrability and separation of variables

- **The general solution of the system of Hamiltonian canonical equations** can be obtained once one has the general solution of the Hamilton-Jacobi equation
  $\frac{\partial S}{\partial t} + H(\frac{\partial S}{\partial q}, q, t) = 0$
- **One says that HJ equation admits separation of variables if the general solution of the HJ equation can be written in the form**
  $S(q; c_1, c_2, \dots, c_n) = \sum_{k=1}^{n} S_k(q^k; c_1, c_2, \dots, c_n)$
- **Liouville discussed several mechanical hamiltonian with a very specific forms of the potential.** The simplest example was a sum of 'one-dimensional potentials', each depending on a single coordinate. He found n first integrals in involution, linked by a linear relation, so $n-1$ of them functionally independent. Adding the Hamiltonian, there are n first integrals in involution. Full solution of this system requires only a sequence of one-dimensional problems which can be solved by quadratures. Similarly for other three types, to be mentioned later.
  - ⋆ **Specific form of he first integrals, quadratic in the momenta**
- **Stäckel systems (1891)** Extensions of the original Liouville system. In these cases, the n separation constants, once reinterpreted as a set of n Poisson commuting functions, display the complete integrability of the system in Liouville sense.

The Darboux direct approach to finding first integrals

- **Darboux was the first to introduce a method to find first integrals in natural Hamiltonian systems** Later used or rediscovered by many others (Eisenhart, Whittaker)
  - ⋆ **Restricted to first integrals with a dependance in the velocities (or in the momenta) prescribed a priori: linear, quadratic, . . .** . This will give **partially integrable** systems, and, eventually, to completely integrable ones.
  - ⋆ **Method: Within the prescribed form for the would-be first integral, require the vanishing of the Poisson bracket with the hamiltonian:** obtain a set of equations which determine **both the potential** for the natural hamiltonian **and the first integral** itself.
  - ⋆ **Meaningful not only for Hamiltonians in Euclidean configuration space, but also with configuration spaces of constant curvature** (say $\mathbb{S}^3$ or $\mathbb{H}^3$) In this case, the kinetic part is determined by the Riemannian metric and appears in the Hamilton-Jacobi equation through the Laplace-Beltrami operator for the space.
- **The first cases** : **linear** and **quadratic** dependence in the momenta. Fully understood.
- **The next cases** : **Higher order** dependence in the momenta.

Constants of motion linear in the velocities. Direct approach

- **Configuration space: either the Euclidean plane, or a space with constant curvature, or a Riemannian space with a general metric**
  - ⋆ **Lagrangian** $\mathcal{L} = \frac{1}{2}\, g_{\mu\nu}(q^1, q^2) v_{q^\mu} v_{q^\nu} - \mathcal{V}(q^1, q^2)$.
  - ⋆ **Hamiltonian** $\mathcal{H} = \frac{1}{2}\, g^{\mu\nu}(q^1, q^2) p_\mu p_\nu + \mathcal{V}(q^1, q^2)$.
  - ⋆ Hamilton-Jacobi equation now involves the Laplace-Beltrami operator of the metric.
- **Possible constants are of the form** $I_{\mathcal{K}} = \mathcal{K}_\mu(q^1, q^2)\, v_{q^\mu}$. This is actually a constant of motion provided $\mathcal{K}_\mu(q^1, q^2)$ satisfies some equations involving the data $g_{\mu\nu}(q^1, q^2), \mathcal{V}(q^1, q^2)$.
  - ⋆ **These equations split in two sets.** The first is independent of the potential, and determines the possible coordinate dependence of vector field $\mathcal{K}_\mu(q^1, q^2)$.
  - ⋆ **There is some geometric meaning in the equations in the first set?** $\mathcal{K}_\mu(q^1, q^2)$ is a **Killing vector field** for the metric of the space.
- **Once a Killing vector field has been chosen** the remaining equation to be satisfied by the **potential** is the usual invariance requirement. Geometrically, the potential must be invariant under the isometries generated by the Killing vector field.
- **In this case, everything works in curved spaces exactly as you may expect**

Constants of motion quadratic in the velocities. Direct approach

- **Are there constants of motion I which are quadratic in the velocities?**
  - ⋆ **Possible constants** $I_{\mathcal{K}} = \mathcal{K}_{\mu\nu}(q^1, q^2)\, v_{q^\mu} v_{q^\nu} + \mathcal{W}(q^1, q^2)$.
- **The requirement for $I_{\mathcal{K}}$ to be a constant of motion translates into some conditions on $\mathcal{K}_{\mu\nu}$ and $\mathcal{W}$** These conditions split in two subsets.
- **The first subset is independent of the potential $\mathcal{V}$: Geometric interpretation:** the tensor $\mathcal{K}_{\mu\nu}$ should be a **Killing tensor** for the metric $g_{\mu\nu}$.
  - ⋆ **After all, a rather natural extension to the first order case**
- **Once a Killing tensor $\mathcal{K}_{\mu\nu}$ has been fixed,** the second subset of conditions (which would determine $\mathcal{W}$) depend actually on the potential $\mathcal{V}$. These are a system of partial differential equations for $\mathcal{W}$; its compatibility equation is a single differential equation for the potential $\mathcal{V}$.
  - ⋆ **Essentially this equation (in the euclidean case, in general coordinates) was also given by Levi-Civita**
- **Geometric meaning** Not direct, but somehow a kind of 'second order invariance'. Here the symmetry under the integrability becomes somewhay hidden.

Killing vector / tensors and confocal coordinate systems

- **Connection between Killing vectors / tensors and 'good' coordinates**
- **Each Killing vector** determines a **coordinate web** in the configuration space associated to the one parameter subgroups of isometries; e.g. for euclidean plane these coordinates are either cartesian or polar.
  - ⋆ **Potentials allowing for a Killing vector of the metric as a constant of motion** have a special dependence on these coordinates.
- **Each Killing tensor** also determines a coordinate web in the configuration space. But coordinates are not 'group coordinates'.
  - ⋆ **Potentials allowing a $I_{\mathcal{K}}$-type quadratic constant of motion are precisely those which are separable** in the coordinate system associated to this web.
- **Coordinates are secondary, the web is the important thing**

Killing vector / tensors and confocal coordinate systems

- **Main problem: find the coordinate webs associated to the more general Killing tensor** These are precisely the coordinate systems allowing the (free) Hamilton-Jacobi equation (or equivalently, the Laplace-Beltrami operator) to be separated.
- **Which are these coordinates?** The coordinate systems associated to general Killing tensors are the **'elliptic coordinates'** and all its degenerate and/or limiting cases
- **In the euclidean plane case,** $n = 2$, this explains the four types of Liouville models: they are precisely those systems separable in **elliptic coordinates** (Euler) or in any their three degenerate and/orlimiting cases: **polar coordinates, parabolic coordinates, cartesian coordinates** .
- **In the two-dimensional sphere** $\mathbb{S}^2$ the **'elliptic' coordinates** were introduced by Jacobi, and there is a single degenerate case **spherical polar coordinates**
- **In the two-dimensional Lobachewski plane,** $\mathbb{H}^2$ there are three generic types of 'elliptic coordinates' and six possible degenerations and/or limiting cases.
- **In three dimensional Euclidean space, the coordinates separating the (flat, 3d) Laplace operator** are the **triaxial ellipsoidal coordinates** and their degenerate and/or limiting cases (prolate ellipsoidal, oblate ellipsoidal, . . .)

Constants of motion for the two 'principal cases' of superintegrability

- **The constants of motion for the two basic superintegrable systems mentioned before** are precisely either first order or second order in the velocities.
- **Harmonic Oscillator** Angular momentum (first order) and Fradkin tensor (second order).
- **Kepler-Coulomb problem** Angular momentum (first order) and Laplace-Runge-Lenz vector (second order)
  - ⋆ **Angular momentum** $J = x\dot{y} - y\dot{x}$
- **Fradkin conserved tensor and Runge-Lenz conserved vector** appear precisely as linked to irreducible sets of constants of motion coming from the separability of Hamilton-Jacobi equation for the Harmonic oscillator and the Kepler-Coulomb potentials in several coordinate systems:
  - ⋆ **Runge-Lenz vector** is related to separability of the HJ equation for the Kepler potential in a full 1-d family of parabolic coordinates, with a focus at the origin

$$A_1 = J\dot{y} - k\cos\phi, \qquad A_2 = -J\dot{x} - k\sin\phi,$$

  - ⋆ **Fradkin tensor** related to separability of Harmonic oscillator potential in a full 1-d family of cartesian coordinates with any axes orientation

$$F_{11} = (\dot{x})^2 + \omega_0^2\, x^2, \quad F_{12} = F_{21} = \dot{x}\dot{y} + \omega_0^2\, xy, \quad F_{22} = (\dot{y})^2 + \omega_0^2\, y^2.$$

More recent examples of integrability

- **Around 1900 it was being already clear that some systems were definitely not integrable:** Three body problem and even its simpler cases (Poincaré, Brunns)
  - ⋆ **This meant somehow a decline in the interest in integrable systems**
- **But several mathematicians had been studiyng some nonlinear evolution PDE equations** (not of pure mechanical type, but instead infinite dimensional), which later were shown to be also 'integrable'.
- **Korteweg-de Vries equation** ; sine-Gordon

$$u_t = u_{xxx} + u u_x; \qquad u_{tt} - u_{xx} = \sin u;$$

  - ⋆ **KdV has solitary waves as special solutions** and displayed some outstanding properties, put in a proper context by Gardner, Greene, Kruskal and Miura (1967, inverse scattering method), algebraized by Lax (1968, Lax pairs) and completed by Fadeev and Zakharov (1971, KdV as an example of a completely integrable infinite dimensional Hamiltonian system.)
- **In the 1970's, through use of these new techniques, several other systems were constructed or recognized to be integrable**

Clasical examples of integrability since 1900

- **Mostly, systems of PDE in 1+1 dimension.** Tipically, nonlinearity + dispersion conspire in a very precise way.
  - ⋆ **Bousinnesq equation** (1877)
  - ⋆ **Korteweg-De Vries equation** (1985)
  - ⋆ **Sine-Gordon equation**
  - ⋆ **Non-linear Schrödiger equation**
  - ⋆ **Classical Heisenberg ferromagnet model**
  - ⋆ **Gaudin magnet model**
- **Only a few known examples of integrable systems of PDE in 1+2 dimensions.**
  - ⋆ **Kadomtsev-Petviashvili equation** (1970, a generalization in 1+2 of the Boussinesq and KdV and equations

Still more Examples of complete integrability

- **Some new 'classical' systems of ODE**
  - ⋆ **Garnier system** (1919)
  - ⋆ **Calogero, Moser, Shuterland** (in the 1970's)
- **Tipically, systems of N particles moving in $\mathbb{R}$ and interacting via pairwise potentials**
  $H = \sum_{j=1}^{N} p_j^2 + \lambda \sum_{j<k}^{N} \Phi(q^j - q^k)$
- **Which choices for the two-particle interaction potential guarantee the complete integrability of the full system?**
- **It is possible to explicitly list those interaction potentials $\Phi(q^j - q^k)$ which will produce a completely integrable system** This is done by using the Lax pair description, and the result is (Perelomov):

$$\Phi(x) = \qquad \frac{1}{x^2}, \qquad \frac{a^2}{\sin^2(ax)}, \qquad \frac{a^2}{\sinh^2(ax)}, \qquad a^2\wp(ax)$$

- **An example known in the configuration space $\mathbb{R}^d$ with arbitrary dimension $d > 1$: any system of n interacting 'relative oscillators' in $\mathbb{R}^d$** , which is however reducible to a system of $N-1$ particles in a common fixed 'oscillator potential, (Jacobi).

The inverse scattering method and the Lax pairs

- **A set of very fruitful ideas, developed around 1970, to understand the complete integrability of Korteweg-De Vries equation** later applied with succes to understand the N-particle examples of integrable systems.
  - ⋆ **Gardner, Greene, Kruskal and Miura (1967, inverse scattering method)**
  - ⋆ **Lax (1968, Lax pairs)**
  - ⋆ **Fadeev and Zakharov (1971, KdV as an example of a completely integrable infinite dimensional Hamiltonian system.)**
- **Why the inverse scattering name?** Because the evolution for this restricted type of nonlinear PDE equations can be mapped into the evolution of scattering data for an (auxiliar) problem in quantum Mechanics, the scattering produced for an adequate potential, a problem which evolves under a *linear* Schrodinger equation.
  - ⋆ **Some traits of a 'nonlinear' version of Fourier transform** useful to deal with some restricted class of nonlinear evolution PDEs

The inverse scattering method and the Lax pairs II

- **Consider a hamiltonian system $\dot{x} = \{H(x), x\}$** and assume you could find two 'matrices' L, M, with matrix elements depending on dinamical variables, in such a way that **the evolution equation $\dot{L} = [M, L]$ is equivalent to the original equation.**
  - ⋆ **The time evolution of the matrix L is given by** $L(t) = A(t)L(0)A^{-1}(t)$, with $M = \dot{A}A^{-1}$. Hence the evolution of $L(t)$ is by a similarity, which does not change its spectrum. Hence the eigenvalues of $L(t)$ appear as constants of motion.
  - ⋆ **Then the symmetric functions of the eigenvalues, i.e., the quantities $I_k = \mathrm{tr}(L^k)$ are automatically constants of motion** and you get as many as the order of the matrices L, M.
  - ⋆ **If you can check that they are independent and in involution,** you are done.
- **Basic idea in a nutshell** To obtain first integrals (i.e, constant of motion) for the given system as eigenvalues of an operator (a matrix) which depends on the dynamical variables; when the system evolves the matrix L evolves too, but its spectrum do not change (hence the name **isospectral deformation,** coined by Moser)

The projection method (Olsianetski, Perelomov 1976)

- **Projection method is a proposal to understand a completely integrable system in terms of a 'simpler' system with more** degrees of freedom
  - ⋆ **Somehow, this method is a converse to reduction** In reduction, motion in some space is reduced to a (generally more complicated) motion in a smaller space, but at the expense of having to cope with additional forces or terms in the potential, and the original symmetry may become hidden.
- **The idea in the projection method is just the opposite:** to look for a larger space and a projection onto the original space such that the original motion appears as the projection of a (more simple) motion in the larger space.
  - ⋆ **The 'desideratum' is to obtain the given motion** as a projection of a **geodesic motion** in the larger space.

The projection method II

- **Even for an completely integrable system, to obtain explicit form of the solutions for the motion might be not easy.**
- **The projection method, when both the expressions for the 'simplest motion' and for the projection are known,** provides a complete explicit solution of the problem, which might be too dificult to obtain otherwise.
- **For instance, the problem of geodesics on the ellipsoid** can be obtained by the projection method starting from the (simplest) Clebsch system (a system with a quadratic nonlinearity) Perelomov 2000
- **Calogero system can be also obtained by the projection method**

Integrability from Coalgebra structure

- **A new way to produce integrable systems and to understand some of known ones (Ballesteros and Ragnisco, 1998)**
  - ⋆ Based on coalgebra symmetry
  - ⋆ Produce systems which are completely integrable: in particular they are completely integrable in two different ways (left and right)
  - ⋆ Direct interpretation for systems of N particles in flat or curved spaces (constant curvature and beyond)
- **A coalgebra** is an (unital, associative) Algebra A with an additional structure, the **coproduct** $\Delta : A \to A \otimes A$ which is required to be:

  coassociative $(\Delta \otimes \mathrm{id}) \circ \Delta = \Delta \circ (\mathrm{id} \otimes \Delta)$

  an algebra homomorphism for the algebra structure, $\Delta(ab) = \Delta(a)\Delta(b)$
  - ⋆ **For the Lie algebra defined by the commutator of the associative product** coproduct is a Lie algebra homomorphism, $\Delta([a,b]_A) = [\Delta(a), \Delta(b)]_{A\otimes A}$
  - ⋆ **Simplest example:** The **primitive** coproduct $\Delta(X) = 1 \otimes X + X \otimes 1$ turns any algebra into a coalgebra.
- **A Poisson coalgebra** is a Poisson algebra endowed with a coproduct Δ which is required to satisfy $\Delta(\{a,b\}_A) = \{\Delta(a), \Delta(b)\}_{A\otimes A}$.

Integrability from Coalgebra structure II

- **Coproduct is a natural structure to deal with 'composition from subsystems'** Thus it is natural to expect some coalgebra symmetry for systems of N particles interacting pairwise, where one may expect some 'composite' quantities to play a role.
- **Using coassociativity, coproduct can be iterated:**

  3-th coproduct $\Delta^{(3)} : A \rightarrow A \otimes A \otimes A$,

  4-th coproduct $\Delta^{(4)} : A \rightarrow A \otimes A \otimes A \otimes A$, ... and so on.
- **Assume the initial Poisson algebra has a Casimir $\mathcal{C}$.** Then the sucesive m-th coproducts of the Casimir determines a set of m-th Casimirs in the tensor products $A \times A$, ...

  $\mathcal{C}^{(2)} = \Delta^{(2)}(\mathcal{C}))$

  $\mathcal{C}^{(3)} = \Delta^{(3)}(\mathcal{C}))$ and so on $\mathcal{C}^{(4)}, \ldots, \mathcal{C}^{(N)}, \ldots$
- **It is clear (and can be confirmed by easy ckecking) that all these m-th Casimirs with $m = 2, 3, \ldots$ are automatically in involution**

  $\{\Delta^{(j)}(\mathcal{C})), \Delta^{(k)}(\mathcal{C}))\} = 0$
- **Furthermore, all these Casimirs automatically commute** with any element in the original algebra or with any coproduct of any element.

Integrability from Coalgebra structure: Do-it-yourself toolkit

- **Start with a Lie coalgebra** (this wil be the ‘one-particle’ basic symmetry behind the construction)
  - ⋆ **Consider say a quadratic Casimir** $\mathcal{C} \equiv \mathcal{C}^{(1)}$.
- **Take a symplectic realization of this Lie algebra,** say in terms of a pair of Darboux coordinates (q, p)
- **Now choose as ‘Hamiltonian’ the N-th coproduct of any** element in this symplectic realization. This will be a funcion of N pairs (q^i, p_i).
- **The sequence of the coproducts of the Casimir $\mathcal{C}^{(2)}, \mathcal{C}^{(3)}, \ldots, \mathcal{C}^{(N)}$provide a set of $N-1$ quantities which will be constants of motion for this hamiltonian and which will be in involution.** Together with the hamiltonian itself, these will prove the complete integrability of the system built in this way, (provide that the hamiltonian is independent of the Casimirs).
  - ⋆ **Thus the system will be completely integrable by construction.**

Integrability from Coalgebra: The Calogero-Gaudin system as an example

- **Calogero-Gaudin system.** A system of N particles moving on a 1d configuration space, with Hamiltonian

  $H(q,p) = \sum_{i<j}^{N} 2p_i p_j (1 - \cos(q^i - q^j))$

  ⋆ Related to a system first studied by Gaudin; also known as 'Gaudin magnet'.

  ⋆ Potential depends only on the relative separation among particle pairs: there is a first integral, the **total momentum** $P = \sum_i^N p_i$. But otherwise it is not clear from the outset whether or not this system has additional symmetry, and whether or not this system might be completely integrable.

- **Start with a Lie coalgebra**

  $$\{J_2, J_1\} = J_3, \quad \{J_2, J_3\} = -J_1, \quad \{J_3, J_1\} = J_2, \quad \Delta(J_i) = 1 \otimes J_i + J_i \otimes 1.$$

  ⋆ **This is isomorphic to $sl(2,R)$** and has a Casimir $\mathcal{C} \equiv \mathcal{C}^{(1)} = J_2^2 - J_1^2 - J_3^2$.

  ⋆ **Take a symplectic realization of this Lie algebra** in terms of a pair of Darboux coordinates (q,p)

  $$D(J_2) = p, \quad D(J_1) = p\cos(q), \quad D(J_3) = p\sin(q), \qquad D(\mathcal{C}) = 0.$$

Integrability from Coalgebra: The Calogero-Gaudin example II

- **Start with the image of the Casimir in the representation D:**

  $$C^{(1)}(q,p) = D(\mathcal{C}) = 0$$

- **Define a 'two-particle' first integral** as the image of the (2-nd) coproduct of the Casimir in the representation D:

  $$C^{(2)}(q^1,q^2,p_1,p_2) = (D\otimes D)\Delta^{(2)}(\mathcal{C}) = 2p_1p_2(1-\cos(q^2-q^1))$$

- **Iterate this process** and take as 'm-particle' first integral the m-th coproduct of the Casimir in the symplectic representation D. Simple computations lead to:

  $$C^{(m)}(q^1,\dots q^m;p_1,\dots p_m) = (D\otimes\cdots\otimes D)\Delta^{(m)}(\mathcal{C}) = \sum_{i<j}^{m} 2p_ip_j(1-\cos(q^i-q^j))$$

- **After N steps, one obtains a system with $N-1$ first integrals in involution: the m-th coproducts of the Casimir.** But we have not yet introduced the Hamiltonian. In this example, if we want to recover precisely the Calogero-Gaudin system, we take

  $$H = C^{(N)}(q^1,\dots q^N;p_1,\dots p_N) = \sum_{i<j}^{N} 2p_ip_j(1-\cos(q^i-q^j))$$

  ⋆ **In this case, the Hamiltonian is not independent of the 'universal' set of $N-1$ constants of motion.** But adding the total momentum $\sum_i^N p_i$ we get N **first integrals in involution.**

  ⋆ **Quite similar to the harmonic oscillator,** where 'partial' hamiltonians in 1d and 2d are the first integrals for the full oscillator in 3d.

Integrability from Coalgebra: Last comments

- **Calogero-Gaudin system is a completely integrable system: this follows quite directly from the coalgebra approach**
  - ⋆ **The symmetry underlying this complete integrability** has been identified as the **coalgebra symmetry** associated to a monoparticular $sl(2,\mathbb{R})$ in a particular symplectic realization.
- **Large room for model building; take Lie groups of higher rank (several functionally independent Casimirs), change the symplectic realization (in terms of more pairs of Darbouz coordinates), etc.**
- **Structurally robust** Allows quantum deformation for the coalgebras (basically, modifying the coproduct to something different from the primitive coproduct).
  - ⋆ **Leads to several new integrable systems in spaces with not constant curvature**

Still much room to explore

- **What means integrability in the quantum case?** The quantum analogue of classical functions in involution is a set of commuting operators, which can be seen as functions of a single operator. Hence the actual difference among the cases with one and more than one conserved quantities in involution in the quantum sense becomes blurred.
  - ⋆ **The usual way out is to understand 'quantum integrability' as the existence of n commuting operators which in the classical limit become a set of n functionally independent functions in the classical phase space** But . . . .
- **Are all completely integrable systems reducible to 'free geodesic motion' in some suitable sense?** This has been conjectured (more or less informally) by several authors (Perelomov, Marmo) and seems likely.
  - ⋆ **The projection method is a natural frame to try to establish this**
  - ⋆ **It is so actually for several cases** (i.e, the Kepler problem (linked to Moser regularization), or the Calogero system) but up to now nobody knows how to obtain in this way the N particle system with pair interaction potential the Weierstrass $\wp$ function.