# Reduction of Almost Poisson Brackets for Nonholonomic Systems

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# Outline

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- Configuration space $Q$, a smooth $n$ dimensional manifold.
- (Hyper-regular) Lagrangian $\mathcal{L}: TQ \to \mathbb{R}$.
- A non-integrable constraint distribution $\mathcal{D} \subset TQ$ defined by $k < n$ constraints that are linear and homogeneous in the velocities:

$$\sum_{s=1}^{n} \beta^i_s(q)\dot{q}^s = 0, \qquad i = 1, \dots, k.$$

$\mathcal{D}_q \subset T_qQ$ is the annhihilator of the one-forms on $Q$:

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- Conservation of Energy. Lagrange Multipliers $\lambda_i$.

# Hamiltonian Formalism

- Generalized momenta $p_s = \frac{\partial \mathcal{L}}{\partial \dot{q}^s}$. Legendre transform: $\mathrm{Leg} : TQ \to T^*Q$.
- Hamiltonian $\mathcal{H} : T^*Q \to \mathbb{R}$.
- Constraint submanifold $\mathcal{M} = \mathrm{Leg}(\mathcal{D}) \subset T^*Q$,

$$\mathcal{M} = \left\{ (p, q) \ : \ \sum_{s=1}^{n} \beta^i_s(q) \, \frac{\partial \mathcal{H}}{\partial p_s} = 0, \quad i = 1, \dots, k \right\}.$$

- Equations of motion

$$\dot{q}^s = \frac{\partial \mathcal{H}}{\partial p_s}, \qquad \dot{p}_s = -\frac{\partial \mathcal{H}}{\partial q^s} + \sum_{i=1}^{k} \lambda_i \beta^i_s(q).$$

# Intrinsic formulation

- $\Omega_Q$ the canonical symplectic form on $T^*Q$.
- Equations of motion

$$\mathbf{i}_{X_{\mathrm{nh}}}\Omega_Q = d\mathcal{H} + \sum_{i=1}^{k} \lambda_i \tau^* \beta^i .$$

  $X_{\mathrm{nh}}$ is the nonholonomic vector field, $\tau : T^*Q \to Q$ the canonical projection.
- Constraints.
  For $m \in \mathcal{M}$:

$$X_{\mathrm{nh}}(m) \in T_m\mathcal{M}, \qquad \langle \tau^*\beta^i(m), X_{\mathrm{nh}}(m) \rangle = 0, \quad i = 1, \dots, k.$$

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$$X_{\rm nh}(m) \in H_m := T_m\mathcal{M} \cap \mathrm{ann}\{\tau^*\beta^i(m), \;\; i = 1, \dots, k\} \subset T_m(T^*Q).$$

## Theorem (Weber (1986), Bates, Śniatycki (1993))

*For all $m \in \mathcal{M}$ we have the symplectic decomposition*

$$T_m(T^*Q) = H_m \oplus H_m^{\Omega_Q}.$$

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- $X_{\mathrm{nh}}(m) = \mathcal{P}_m X_{\mathcal{H}}(m)$.
- For $f_1, f_2 \in C^\infty(\mathcal{M})$ define the nonholonomic bracket

$$\{f_1, f_2\}_{\mathrm{nh}}(m) = \Omega_Q(\mathcal{P}_m X_{f_1}(m), \mathcal{P}_m X_{f_2}(m)).$$

# Properties of the Nonholonomic Bracket in $\mathcal{M}$

- Equations of motion can be written with respect to the nonholonomic bracket

$$X_{\text{nh}}(f)(m) = \{f, \mathcal{H}_{\mathcal{M}}\}_{\text{nh}}(m), \qquad \mathcal{H}_{\mathcal{M}} = \mathcal{H}\big|_{\mathcal{M}}.$$

- Cantrijn, de León, Martín De Diego (1999) show that the bracket $\{\cdot,\cdot\}_{\text{nh}}$ so defined equals that of Van der Schaft, Maschke (1992) and Marle (1998).
- Jacobi identity is satisfied if and only if the constraints are holonomic.
- The nonholonomic bracket $\{\cdot,\cdot\}_{\text{nh}}$ is a projection of the canonical Poisson tensor onto the constraint space using Lagrange D'Alembert principle.
- This formulation avoids dealing with Lagrange multipliers. The constraint forces are encoded in the bracket.

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  - Link with existence of invariant measures.
  - Chaplygin's Reducing Multiplier Theorem, Borisov & Mamaev [2002, 2005], Fedorov & Jovanović [2004], Ehlers, Koiller, Montgomery & Rios [2004], etc.

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- Restricted $G$-action, $\Phi : G \times \mathcal{M} \to \mathcal{M}$, preserves the nonholonomic bracket:

$$\{f_1 \circ \Phi_g, f_2 \circ \Phi_g\}_{\mathrm{nh}} = \{f_1, f_2\}_{\mathrm{nh}} \circ \Phi_g, \qquad \forall g \in G, f_1, f_2 \in C^\infty(\mathcal{M}).$$

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- Bracket drops to orbit space. For $F_1, F_2 \in C^\infty(\mathcal{R})$ define

$$\{F_1, F_2\}_{\mathcal{R}} := \{F_1 \circ \pi, F_2 \circ \pi\}_{\mathrm{nh}} \qquad \pi : \mathcal{M} \longrightarrow \mathcal{M}/G := \mathcal{R}.$$

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- Reduced equations on $\mathcal{R}$:

$$\dot{F} = \{F, \mathcal{H}_{\mathcal{R}}\}_{\mathcal{R}} \qquad \text{for all} \qquad F \in C^\infty(\mathcal{R}).$$

Reduced Hamiltonian $\mathcal{H}_{\mathcal{R}} : \mathcal{R} \to \mathbb{R}, \qquad \mathcal{H}_{\mathcal{M}} = \mathcal{H}_{\mathcal{R}} \circ \pi.$

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- Does the reduced space admit a foliation by even dimensional leaves? Are there Casimir functions?

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- Left invariant Hamiltonian.
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- García-Naranjo [2007], Lie algebraic expressions and study of reduced brackets.

# Reduced Brackets for LL Systems

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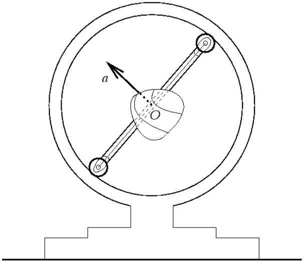
- Reduced space $\mathcal{R}$ is a linear subspace of the dual Lie algebra, $\mathcal{R} \subset \mathfrak{g}^*$.
- Reduced bracket is a projection of the Lie-Poisson bracket on $\mathfrak{g}^*$ onto $\mathcal{R}$.

# Example: The Suslov Problem.

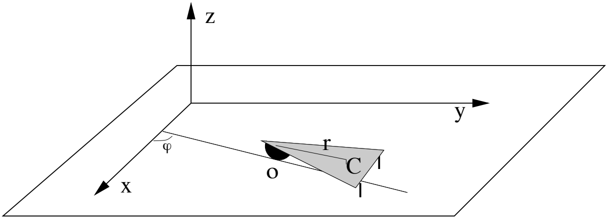
- $G = SO(3)$.




Rigid body with constraint: $a \cdot \omega = 0$.

# Example: The Chaplygin Sleigh.

- $G = SE(2)$.




Skating constraint: $\dot{y}\cos\varphi = \dot{x}\sin\varphi$.

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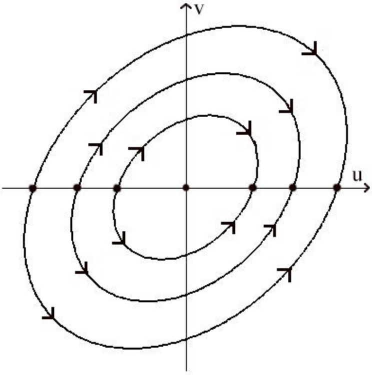
- Liouville measure, $\frac{1}{v} du \wedge dv$ is preserved on non-trivial leaves.
- Reduced Hamiltonian is:

$$\mathcal{H}_{\mathcal{R}}(u,v) = \frac{1}{2}\left(Au^2 + 2Buv + Cv^2\right), \qquad A, C, AC - B^2 > 0.$$

  ($B = 0$ for the Chaplygin sleigh).

# Qualitative Dynamics




- Hamiltonization of higher dimensional LL systems with our approach still open.
- Important work by Jovanović for multidimensional Suslov problem.

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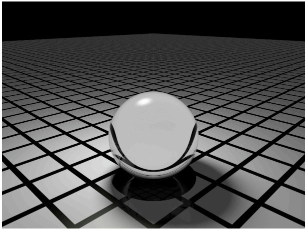
# Chaplygin Sphere



- Configuration space is $Q = SO(3) \times \mathbb{R}^2$.
- Kinetic energy Lagrangian.
- Two rolling constraints $\dot{x} = r\omega_2, \ \dot{y} = -r\omega_1$, define 8 dimensional constraint submanifold $\mathcal{M} \subset T^*Q$.

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- Goal: Understand the geometry of their bracket and tie it with the general theory of almost Poisson brackets.

# Affine Almost Poisson Brackets

- Idea: Equations of motion

$$\mathbf{i}_{X_{\mathrm{nh}}}\Omega_Q = d\mathcal{H} + \sum_{i=1}^{k} \lambda_i \tau^* \beta^i,$$

can also be written as:

$$\mathbf{i}_{X_{\mathrm{nh}}}(\Omega_Q + \Omega_0) = d\mathcal{H} + \sum_{i=1}^{k} \lambda_i \tau^* \beta^i,$$

for a two-form $\Omega_0$ satisfying $\mathbf{i}_{X_{\mathrm{nh}}}\Omega_0 = 0$.

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for a two-form $\Omega_0$ satisfying $\mathbf{i}_{X_{\text{nh}}}\Omega_0 = 0$.

- Construct bracket using the non-canonical form $\tilde{\Omega}_Q := \Omega_Q + \Omega_0$.

# Affine Almost Symplectic Structure

**Definition**

A nontrivial two-form $\Omega_0$ on $T^*Q$ defines an Affine Almost Symplectic Structure, $\tilde{\Omega}_Q := \Omega_Q + \Omega_0$, for our nonholonomic system if the following conditions hold:

- $\mathbf{i}_{X_{\mathcal{H}}} \Omega_0 = 0$.
- The form $\Omega_0$ is semi-basic. That is, if $v$ is a tangent vector to $T^*Q$ such that $\tau_* v = 0$, then $\mathbf{i}_v \Omega_0 = 0$, $(\tau : T^*Q \to Q)$.

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- *Constraints remain the same:* $X_{\mathrm{nh}}(m) \in H_m \quad \forall m \in \mathcal{M}$.
- *For all* $m \in \mathcal{M}$ *we have the symplectic decomposition*

$$T_m(T^*Q) = H_m \oplus H_m^{\tilde{\Omega}_Q} .$$

Same relevant properties as $\Omega_Q$!

# Definition of Affine Almost Poisson Brackets

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- $X_{\rm nh}(m) = \tilde{\mathcal{P}}_m X_{\mathcal{H}}(m)$.
- For $f_1, f_2 \in C^\infty(\mathcal{M})$ define the affine bracket:

$$\{f_1, f_2\tilde{\}}_{\rm nh}(m) = \tilde{\Omega}_Q(\tilde{\mathcal{P}}_m \tilde{X}_{f_1}(m), \tilde{\mathcal{P}}_m \tilde{X}_{f_2}(m)),$$

with $\tilde{X}_{f_j}$ defined by $\mathbf{i}_{\tilde{X}_{f_j}} \tilde{\Omega}_Q = df_j, \;\; j = 1, 2.$

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$$\{f_1, f_2\tilde{\}}_{\mathrm{nh}}(m) = \tilde{\Omega}_Q(\tilde{\mathcal{P}}_m \tilde{X}_{f_1}(m), \tilde{\mathcal{P}}_m \tilde{X}_{f_2}(m)),$$

  with $\tilde{X}_{f_j}$ defined by $\mathbf{i}_{\tilde{X}_{f_j}} \tilde{\Omega}_Q = df_j, \;\; j = 1, 2.$
- Equations of motion can be written with respect to the affine bracket

$$X_{\mathrm{nh}}(f)(m) = \{f, \mathcal{H}_{\mathcal{M}}\tilde{\}}_{\mathrm{nh}}(m), \qquad \mathcal{H}_{\mathcal{M}} = \mathcal{H}\big|_{\mathcal{M}}.$$

- In general $\{f_1, f_2\tilde{\}}_{\mathrm{nh}} \neq \{f_1, f_2\}_{\mathrm{nh}}$. Different way of encoding the constraint forces!
- Affine bracket is obtained by projecting a non-canonical Poisson tensor onto the constraint space using Lagrange-D'Alembert principle.

# Chaplygin Sphere revisited

- Let the two-form $\Omega_0$ on $T^*Q$ be defined by

$$\Omega_0 := mr^2 \, \mathbf{i}_{X_{\mathcal{H}}} \tau^* \nu,$$

where $\nu$ is the bi-invariant volume three-form on $SO(3)$.

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- The reduced affine bracket admits a foliation by even dimensional leaves. Conserved quantity (vertical angular momentum) is a *Casimir*.
- The reduced affine bracket is conformally Poisson. We recover Borisov and Mamaev bracket.
- After a time reparametrization, the reduced Chaplygin sphere equations can be treated as an integrable, 2 degree of freedom Hamiltonian system.

# Outline

## Conclusions.

- Reduced Nonholonomic systems can be Hamiltonian and qualitative information about the dynamics can be deduced from this description.
- Importance of Affine Brackets: Hamiltonization of Chaplygin sphere can only be achieved by reducing an affine bracket.

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Current work.

- We have shown existence of affine symplectic structures for the complete reduction of generalized Chaplygin systems with internal symmetries in higher dimensional cases.
- Hamiltonization of other higher dimensional systems with our approach is still open. Important progress done by Fedorov, Jovanovic (2004).