Poly and Multisymplectic Geometry of Fiber Bundles

Leandro Gustavo Gomes

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de Donder-Weyl Equations and its Geometric Structure

- Main goal: Introduce a geometric structure for the de Donder-weyl equations playing the same role of symplectic forms in Hamilton equations.
- Mathematical task: define the class of differential forms that can be locally described using poly and multisymplectic canonical coordinates (Darboux-type theorem).
- de Donder-Weyl equations:

$$\frac{\partial q^i}{\partial x^\mu} = \frac{\partial \mathcal{H}}{\partial p_i^\mu} \qquad \frac{\partial p_i^\mu}{\partial x^\mu} = -\frac{\partial \mathcal{H}}{\partial q^i} \tag{1}$$

Canonical Forms

Canonical polysymplectic Form

- Defined in $T^*E \otimes \hat{T}$, π_E projection onto E:
- $\hat{\theta}_\alpha(v) \;=\; \alpha(T_\alpha \pi_E \cdot v)$
- $\hat{\omega} = -d\hat{\theta}$
- Canonical coordinates: ($\{\hat{e}_a\}$ any basis of $\hat{T}$)

$$\hat{\theta} \;=\; p_i^a \, dq^i \otimes \hat{e}_a \;, \qquad \hat{\omega} \;=\; dq^i \wedge dp_i^a \otimes \hat{e}_a \tag{2}$$

Canonical forms

Canonical Multisymplectic Form

- Defined in $\bigwedge_1^n T^*E$ with $E \xrightarrow{\pi} M$ a fiber bundle, $\dim M = n$, $\dim E = n + N$ and π_E projection onto E:
- $\theta_\alpha(v_1, \ldots, v_n) = \alpha(T_\alpha \pi_E \cdot v_1, \ldots, T_\alpha \pi_E \cdot v_n)$
- $\omega = -d\theta$
- Canonical coordinates:

$$\theta = p_i^\mu \, dq^i \wedge d^n x_\mu + p \, d^n x \,, \quad \omega = dq^i \wedge dp_i^\mu \wedge d^n x_\mu - dp \wedge d^n x$$

$\hat{\omega} \in (\bigwedge^2 V^*) \otimes \hat{T}$

- V linear space, $\hat{T}$ auxiliary linear space, $n = \dim \hat{T}$
- Projected form along $\hat{t}^* \in \hat{T}^*$

$$\omega_{\hat{t}^*} = \langle \hat{t}^*, \hat{\omega} \rangle$$

- L subspace of V, $L^\perp$ its annihilator in V^*
  - L isotropic: $\hat{\omega}^\flat(L) \subset (\bigwedge^k L^\perp) \otimes \hat{T}$
  - L maximal isotropic: $\hat{\omega}^\flat(L) = \hat{\omega}^\flat(V) \cap (\bigwedge^k L^\perp) \otimes \hat{T}$
  - L polylagrangian: $\hat{\omega}^\flat(L) = (\bigwedge^k L^\perp) \otimes \hat{T}$

Polysymplectic Forms

$\hat{\omega} \in (\bigwedge^2 V^*) \otimes \hat{T}$

- Definition: $\hat{\omega}$ is **polysymplectic** of rank N in V if there is a polylagrangian subspace L of V with codimension N.
- Theorem: $L = \sum_{\hat{t}^* \neq 0} \ker \omega_{\hat{t}^*}$ and if $n \geq 2$, then

$$L = K_1 + \ldots + K_n$$

with $K_a = \bigcap_{b \neq a} \ker \omega_{\hat{e}^b}$ and $\dim(K_a / \ker \hat{\omega}) = N$

Linear Polysymplectic Darboux Theorem

Teorema (Linear Polysymplectic Darboux Theorem)

Let $\hat{\omega} \in \left(\bigwedge^2 V^*\right) \otimes \hat{T}$ polysymplectic of rank N with polylagrangian subspace L. Fixing a basis $\{\, \hat{e}_a \,|\, 1 \leq a \leq n \,\}$ of $\hat{T}$, there is a basis

$$\{\, e_i, e_a^j \,|\, 1 \leq a \leq n \,,\, 1 \leq i, j \leq N \,\}$$

of a subspace of V complementary to $\ker \hat{\omega}$, with vectors e_a^i spanning $L/\ker \hat{\omega}$, such that with respect to the dual basis of $\mathrm{supp}\ \hat{\omega}$

$$\hat{\omega} \;=\; \left(e_i^a \wedge e^i\right) \otimes \hat{e}_a$$

Partially Horizontal Forms

$\omega \in \bigwedge_r^{k+1} W^* \quad (0 \leq r \leq k+1)$

- W linear space, V linear subspace, $T = W/V$ quotient space, $n = \dim T$, exact sequence

$$0 \longrightarrow V \longrightarrow W \stackrel{\pi}{\longrightarrow} T \longrightarrow 0$$

- $\omega \in \bigwedge_r^{k+1} W^* \subset \bigwedge^{k+1} W^*$ if ω is $(k+1-r)$-horizontal, i.e.,

$$\mathrm{i}_{v_1} \ldots \mathrm{i}_{v_{r+1}} \omega \; = \; 0 \qquad \text{for } v_1, \ldots, v_{r+1} \in V$$

The Symbol

$$\omega \in \bigwedge\nolimits_r^{k+1} W^* \quad \longrightarrow \quad \hat{\omega} \in \left(\bigwedge\nolimits^r V^*\right) \otimes \bigwedge\nolimits^{k+1-r} T^*$$

- For $\omega \in \bigwedge_r^{k+1} W^*$, define its **symbol** as the vector valued form $\hat{\omega} \in \left(\bigwedge^r V^*\right) \otimes \bigwedge^{k+1-r} T^*$ given by

$$\hat{\omega}(v_1, \ldots, v_r) \;=\; \mathrm{i}_{v_1} \ldots \mathrm{i}_{v_r} \omega \qquad \text{for } \; v_1, \ldots, v_r \in V$$

Partially Horizontal Forms

$\omega \in \bigwedge_r^{k+1} W^* \quad (0 \leq r \leq k+1)$

- L subspace of V, $L^\perp$ its anniquilator in W^*,

$$\bigwedge\nolimits_{r-1}^{k} L^\perp := \bigwedge\nolimits^{k} L^\perp \cap \bigwedge\nolimits_{r-1}^{k} W^*$$

- L isotropic: $\omega^\flat(L) \subset \bigwedge_{r-1}^{k} L^\perp$
- L maximal isotropic in V: $\omega^\flat(L) = \omega^\flat(V) \cap \bigwedge_{r-1}^{k} L^\perp$
- L multilagrangean: $\omega^\flat(L) = \bigwedge_{r-1}^{k} L^\perp$

Multisymplectic Forms

$\omega \in \bigwedge_2^{n+1} W^*$

- Definition: ω is **multisymplectic** of rank N if V has a multilagrangean subspace L with codimension N and nondegenerate.
- Theorem: If ω is multisymplectic with multilagrangean subspace L, then $\hat{\omega}$ is polysymplectic with polylagrangean subspace L and has an unidimensional kernel.

Linear Multisymplectic Darboux Theorem

Teorema (Linear Multisymplectic Darboux Theorem)

Let $\omega \in \bigwedge_2^{n+1} W^*$ *be multisymplectic of rank* N *with multilagrangean subsapce* L *of* V*. Then there is a basis* $\{\, e_i, f^\mu, e^i_\mu, e \mid 1 \leq i \leq N\, 1 \leq \mu \leq n\}$ *of a subspace of* W *complementary to* $\ker \omega$*, such that the vectors* $\{e^i_\mu, e\}$ *span* $L/\ker \omega$ *and adding the vectors* e_i*, span* V/L*, such that using the dual basis on* $\operatorname{supp} \omega$ *(*$f_\mu = f^1 \wedge \ldots \widehat{f^\mu} \ldots \wedge f^n$*)*

$$\omega = e^i \wedge e^\mu_i \wedge f_\mu \; - \; e \wedge f^1 \wedge \ldots \wedge f^n$$

$$\hat{\omega} = e^i \wedge e^\mu_i \otimes f_\mu$$

Cartan Calculus for Vertical Forms

Space of vertical forms $\Omega^r_V(P\,;\pi^*\hat{T})$

- P a fiber bundle over M and projection π
- $\hat{T}$ a vector bundle over M
- $\pi^*\hat{T}$ the pull-back bundle over P
- $\bigwedge^r VP^* \otimes \pi^*\hat{T}$ the bundle of $\pi^*\hat{T}$-valued vertical r-forms over P
- $\Omega^r_V(P\,;\pi^*\hat{T})$ the space of $\pi^*\hat{T}$-valued vertical r-forms over P

Vertical Forms

Vertical Forms

- Let $\hat{\alpha} \in \Omega_V^r(P; \pi^* \hat{T})$ be a vertical form and $\hat{t}^*$ a section of $\hat{T}^*$, then the **projection** of $\hat{\alpha}$ along $\hat{t}^*$ is the vertical form $\hat{\alpha}_{\hat{t}^*} \in \Omega_V^r(P)$ defined by:

$$\hat{\alpha}_{\hat{t}^*}(p) \;=\; \langle \hat{t}^*(\pi(p)), \hat{\alpha}(p) \rangle \qquad \text{for } \; p \in P \; .$$

Vertical Exterior Derivative

$d_V : \Omega^r_V(P\,;\pi^*\hat{T}) \to \Omega^{r+1}_V(P\,;\pi^*\hat{T})$

- for $X_0, X_1, \ldots, X_r \in \mathfrak{X}_V(P)$ e $\alpha \in \Omega^r_V(P\,;\pi^*\hat{T})$

$$d_V\alpha\,(X_0,\ldots,X_r) = \sum_{i=0}^{r}(-1)^i\,X_i\cdot\Big(\alpha(X_0,\ldots,\widehat{X}_i,\ldots,X_r)\Big)$$

$$+\sum_{0\le i<j\le r}(-1)^{i+j}\,\alpha([X_i,X_j],X_0,\ldots,\widehat{X}_i,\ldots,\widehat{X}_j,\ldots,X_r)$$

Polysymplectic fiber Bundles

$\hat{\omega} \in \Omega^2_V(P\,;\pi^*\hat{T})$

- A **Polysymplectic fiber Bundles** is a fiber bundle P over M with projection π endowed with a $\pi^*\hat{T}$-valued vertical 2-form $\hat{\omega}$ on the total space P (where $\hat{T}$ is an auxiliary vector bundle over M), called **polysymplectic form**, of rank N, if it is pointwise polysymplectic of rank N and vertically closed

$$d_V\hat{\omega} \;=\; 0$$

- **Polysymplectic manifold**: M reduces to a point.

Vertical Exterior Derivative

$d_V : \Omega_V^r(P; \pi^* \hat{T}) \to \Omega_V^{r+1}(P; \pi^* \hat{T})$

- For all Sections $\hat{t}^*$ of $\hat{T}^*$, $\quad d_V \hat{\alpha}_{\hat{t}^*} = (d_V \hat{\alpha})_{\hat{t}^*}$
- $d_V \hat{\alpha} = 0 \implies$ ker $\hat{\alpha}$ e ker $\hat{\alpha}_{\hat{t}^*}$ are involutive

Polylagrangean distribution: Integrability

Integrability of L

- Theorem: If $n \geq 3$, the polylagrangean distribution L is integrable. If $n = 2$ there is a polysymplectic form on $S0(3)$ with L non-integrable.

Polylagrangean Darboux Theorem

Teorema

Let $\hat{\omega} \in \Omega^2_V(P\,;\pi^*\hat{T})$ *be polysymplectic of rank* N *with integrable polylagrangian subbundle* L*. Fixed a base of local sections* $\{\,\hat{e}_a \,|\, 1 \leq a \leq n\,\}$ *of* $\hat{T}$*, there is a local coordinate system*

- *coordinates* r^κ *for* $\ker\hat{\omega}$
- *coordinates* p^a_i *for* $L/\ker\hat{\omega}$
- *coordinates* q^i *for* V/L

such that

$$\hat{\omega} \;=\; dp^a_i \wedge dq^i \otimes \hat{e}_a$$

Multisymplectic Fiber Bundles

$\omega \in \Omega_2^{n+1}(P)$

A **Multisymplectic Fiber Bundle** is a bundle P over M (with $\dim M = n$) endowed with a $(n-1)$-horizontal$(n+1)$-form ω on the total space P, called **multisymplectic form**, of rank N, which is pointwise multisymplectic of rank N and closed

$$d\omega \;=\; 0$$

The symbol

$\omega \in \Omega_2^{n+1}(P) \quad \longrightarrow \quad \hat{\omega} \in \Omega_V^2\left(P, \pi^*\left(\bigwedge^{n-1} T^*M\right)\right)$

- Given a $(n-1)$-horizontal$(n+1)$-form $\omega \in \Omega_2^{n+1}(P)$, define its **symbol** $\hat{\omega} \in \Omega_V^2\left(P, \pi^*\left(\bigwedge^{n-1} T^*M\right)\right)$ by

$$\hat{\omega}(X_1, X_2) = \mathrm{i}_{X_1} \mathrm{i}_{X_2} \omega \qquad \text{for } X_1, X_2 \in \mathfrak{X}_V(P)$$

The Symbol

Polysymplectic X Multisymplectic

- $d\omega \in \Omega_2^{n+2}(P) \quad \Longrightarrow \quad d_V\hat{\omega} = 0$
- Theorem: If ω is multisymplectic with multilagrangian distribution L then its symbol $\hat{\omega}$ is polysymplectic with polylagrangian distribution L.

Multisymplectic Darboux Theorem

Multisymplectic Darboux Theorem

If ω is multisymplectic then there are local **canonical coordinates** (x^μ, q^i, p_i^μ, p) such that, defining $d^n x_\mu \;:=\; \mathrm{i}_{\partial_\mu} d^n x$, we have

$$\omega \;=\; dp_i^\mu \wedge dq^i \wedge d^n x_\mu \;-\; dp \wedge d^n x$$

$$\hat{\omega} \;=\; dp_i^\mu \wedge dq^i \,\otimes\, d^n x_\mu$$

References

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