# Nonholonomic Mechanics: A Lie algebroid perspective

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## A. Weinstein (96)

Generalized theory of Lagrangian Mechanics on Lie algebroids

A program to develop formalisms of the dynamical behavior of Lagrangian and Hamiltonian systems on Lie algebroids (in particular, nonholonomic Mechanics) and discrete Mechanics on Lie groupoids

Belgium (F. Cantrijn, B. Langerock, T. Mestdag and W. Sarlet)

Poland (K. Grabowska, J. Grabowski, P. Urbanski)

Portugal (J.M. Nunes da Costa, P. Santos)

Romania (M. Popescu, P. Popescu)

Spain (P. Balseiro, J.F. Cariñena, D. Iglesias, M. de León, JCM, D. Martín de Diego, E. Martínez, E. Padrón and D. Sosa)

UK (M. Crampin, D. Saunders)

USA (J. Cortés)

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Several reasons for discussing Nonholonomic Mechanics on Lie algebroids

1. The inclusive nature of the Lie algebroid framework (under the same umbrella, one can consider standard constrained mechanical systems, nonholonomic systems on Lie algebras, constrained systems evolving on semidirect products or nonholonomic systems with symmetries)
2. The reduction of a nonholonomic mechanical system on a Lie algebroid is a nonholonomic mechanical system on a Lie algebroid. However, the reduction of an standard nonholonomic system on the tangent (cotangent) bundle of the configuration manifold is not, in general, an standard nonholonomic system
3. The theory of Lie algebroids gives a natural interpretation for the use of quasi-coordinates (velocities) in nonholonomic Mechanics

# The plain of the talk

1. Almost Lie algebroids
   - 1.1 Definition, examples and almost differential
   - 1.2 Linear almost Poisson structure on the dual bundle to an almost Lie algebroid
   - 1.3 The hamiltonian dynamics on an almost Lie algebroid
2. Nonholonomic Mechanical systems subjected to linear constraints on a Lie algebroid
   - 2.1 Definition and some examples
   - 2.2 Nonholonomic mechanical systems and almost Lie algebroids. The nonholonomic bracket and the constrained Hamiltonian dynamics
   - 2.3 The Lie algebroid nonholonomic momentum map
   - 2.4 Nonholonomic Mechanics, almost Lie algebroids and Hamilton-Jacobi Theory
   - 2.5 Invariant volumes and nonholonomic mechanical systems on Lie algebroids

- Present and future work

## 1.1 Definition, examples and almost differential

$\tau_E : E \longrightarrow Q$ a real vector bundle over $Q$

$$\Gamma(E) \equiv C^\infty(Q) - \text{modulo of sections of } \tau_E : E \longrightarrow Q$$

$$(\llbracket\cdot,\cdot\rrbracket, \rho) \text{ an almost Lie algebroid structure on } E$$

$$\Updownarrow$$

- $\llbracket\cdot,\cdot\rrbracket$ a skew-symmetric $\mathbb{R}$-linear bracket on $\Gamma(E)$
- $\rho : E \longrightarrow TQ$ a vector bundle morphism (the anchor map)
- X,Y$\in \Gamma(E)$, $f \in C^\infty(Q) \Rightarrow \llbracket X, fY\rrbracket = f\llbracket X, Y\rrbracket + \rho(X)(f)Y$

$\llbracket\cdot,\cdot\rrbracket$ satisfies the Jacobi identity $\Longrightarrow (\llbracket\cdot,\cdot\rrbracket, \rho)$ a Lie algebroid structure on $E$

$$(\llbracket\cdot,\cdot\rrbracket, \rho) \text{ a Lie algebroid structure}$$

$$\Downarrow$$

$$\rho : \Gamma(E) \rightarrow \mathfrak{X}(Q) \text{ is a Lie algebra morphism}$$

K. Mackenzie, London Math. Soc. Lecture Note Series **213** Cambridge University Press, 2005

local expressions

$(q^i) \equiv$ local coordinates on $Q$

$\{e_\alpha\}$ a local basis of $\Gamma(E)$

$\rho(e_\alpha) = \rho^i_\alpha \frac{\partial}{\partial q^i}, \quad [\![ e_\alpha, e_\beta ]\!] = C^\gamma_{\alpha\beta} e_\gamma$

$C^\gamma_{\alpha\beta}, \rho^i_\alpha \equiv$ local structure functions of $\tau_E : E \to Q$

$([\![\cdot,\cdot]\!], \rho)$ a Lie algebroid structure

$$\Downarrow$$

$$\rho^j_\alpha \frac{\partial \rho^i_\beta}{\partial q^j} - \rho^j_\beta \frac{\partial \rho^i_\alpha}{\partial q^j} = \rho^i_\gamma C^\gamma_{\alpha\beta}, \quad \forall \alpha, \beta$$

$$\sum_{\text{cyclic}(\alpha,\beta,\gamma)} [\rho^i_\alpha \frac{\partial C^\nu_{\beta\gamma}}{\partial q^i} + C^\nu_{\alpha\mu} C^\mu_{\beta\gamma}] = 0, \quad \forall v$$

Local structure equations

EXAMPLES:

The Atiyah algebroid associated with a principal $G$-bundle

$p : Q \to Q/G$ a principal $G$-bundle

$\tau : TQ/G \to Q/G, \quad [v_q] \to p(q)$ is a real vector bundle over $Q/G$

$\Gamma(TQ/G) = \{X \in \mathfrak{X}(Q) / X \text{ is } G\text{-invariant}\}$

- $X, Y \in \mathfrak{X}(Q)$ are $G$-invariant $\Rightarrow [X, Y]$ also is $G$-invariant

$$\Downarrow$$

$[\![\cdot,\cdot]\!] : \Gamma(TQ/G) \times \Gamma(TQ/G) \to \Gamma(TQ/G)$ a Lie bracket on $\Gamma(TQ/G)$

- $X \in \mathfrak{X}(Q)$ is $G$-invariant $\Rightarrow X$ is $p$-projectable

$$\Downarrow$$

$\rho : TQ/G \to T(Q/G), \quad [v_q] \to (T_q p)(v_q)$ the anchor map

$\tau : TQ/G \to Q/G$ the Atiyah algebroid

associated with the principal $G$-bundle $p : Q \to Q/G$

PARTICULAR CASES:

- $G = \{e\} \Longrightarrow TQ/G = TQ$ (tangent bundles)

  $\Gamma(TQ) = \mathfrak{X}(Q) \quad [\![\cdot,\cdot]\!] = [\cdot,\cdot]$ the standard Lie bracket of vector fields on $Q$

  $\rho : TQ \to TQ$ is the identity map

- $G = Q \Longrightarrow TG/G \cong (G \times \mathfrak{g})/G \cong \mathfrak{g}$ (Lie algebras)

  $[\![\cdot,\cdot]\!]$ is the Lie bracket $[\cdot,\cdot]_{\mathfrak{g}}$, $\quad \rho_{\mathfrak{g}} : \mathfrak{g} \to \{\text{a point}\}$ is the zero map

- $p : Q = G \times N \to N$ a trivial principal $G$-bundle

$$\Downarrow$$

$$TQ/G \cong \frac{TG \times TN}{G} \cong \frac{G \times \mathfrak{g} \times TN}{G} \cong \mathfrak{g} \times TN$$

$$\tau : TQ/G \cong \mathfrak{g} \times TN \to Q/G = N$$

(product of a tangent bundle and a Lie algebra)

$$\Gamma(\mathfrak{g} \times TN) \equiv C^\infty(N, \mathfrak{g}) \oplus \mathfrak{X}(N)$$
$$\xi, \xi' \in \mathfrak{g}, \quad X, X' \in \mathfrak{X}(N) \Rightarrow [\![(\xi, X), (\xi', X')]\!] = ([\xi, \xi'], [X, X'])$$
$$\rho(\xi, X) = X$$

The Lie algebroid associated with an right (left) infinitesimal action:

$\mathfrak{g}$ a real Lie algebra of finite dimension

$\Phi : \mathfrak{g} \to \mathfrak{X}(Q)$ an infinitesimal right (left) action of $\mathfrak{g}$ on $Q$

$$\Phi([\xi,\eta]_{\mathfrak{g}}) = (-)[\Phi(\xi),\Phi(\eta)], \quad \forall \xi,\eta \in \mathfrak{g}$$

$\tau : E = Q \times \mathfrak{g} \to Q$ is a Lie algebroid

$\rho : Q \times \mathfrak{g} \to TQ; \quad (q,\xi) \in Q \times \mathfrak{g} \to (-)\Phi(\xi)(q) \in T_q Q$

$\xi,\eta \in \mathfrak{g} \Longrightarrow [\![\xi,\eta]\!] = [\xi,\eta]$

$(E, [\![\cdot,\cdot]\!], \rho)$ the Lie algebroid associated with $\Phi$

## The typical example of an almost Lie algebroid

- $(E, [\![\cdot,\cdot]\!], \rho)$ a Lie algebroid over a manifold $Q$
- $\tau_D : D \to Q$ a vector subbundle of $E$ over $Q$
- $\mathcal{P} : E \to D$ a vector bundle morphism over the identity of $Q$

  $\mathcal{P}_{|D} = Id$

$$\Downarrow$$

$([\![\cdot,\cdot]\!]_{\mathcal{P}}, \rho_P)$ is an almost Lie algebroid structure on $\tau_D : D \to Q$

$$[\![X, Y]\!]_{\mathcal{P}} = \mathcal{P}([\![X, Y]\!]), \quad \rho_{\mathcal{P}}(X) = \rho(X) \quad X, Y \in \Gamma(D)$$

$([\![\cdot,\cdot]\!]_{\mathcal{P}}, \rho_{\mathcal{P}})$ the almost Lie algebroid structure on $D$ associated with the morphism $\mathcal{P}$

## The almost differential of an almost Lie algebroid

$(E, [\![\cdot,\cdot]\!], \rho)$ an almost Lie algebroid over $Q$

$d : \Gamma(\wedge^k E^*) \longrightarrow \Gamma(\wedge^{k+1} E^*)$

$$
\begin{aligned}
(d\alpha)(X_0, \ldots, X_k) &= \sum_{i=0}^{k} (-1)^i \rho(X_i)(\alpha(X_0, \ldots, \widehat{X}_i, \ldots, X_k)) \\
&\quad + \sum_{i<j} (-1)^{i+j} \alpha([\![X_i, X_j]\!], X_0, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_k)
\end{aligned}
$$

$X_0, \ldots, \ldots X_k \in \Gamma(E)$

$([\![\cdot,\cdot]\!], \rho)$ a Lie algebroid structure on $E$

$$\Downarrow$$

$$d^2 = 0$$

$(E, [\![\cdot,\cdot]\!], \rho)$ an almost Lie algebroid over $Q$

$\tau^* : E^* \to Q$ the dual bundle to $E$

$$\Downarrow$$

$\{\cdot,\cdot\} : C^\infty(E^*) \times C^\infty(E^*) \to C^\infty(E^*)$ a linear almost Poisson bracket on E*

- $\{\cdot,\cdot\}$ is $\mathbb{R}$-bilinear and skew-symmetric
- $\{\cdot,\cdot\}$ satisfies the Leibniz rule

$$\{F.F', G\} = F\{F', G\} + F'\{F, G\}$$

- $\{\cdot,\cdot\}$ is linear: $X, Y \in \Gamma(E) \Longrightarrow \{\hat{X}, \hat{Y}\}$ is a linear function on $E^*$

$$\hat{X} : E^* \to \mathbb{R} \text{ is a linear function}$$

$$\hat{X}(\alpha) = \alpha(X(\tau^*(\alpha))), \quad \forall \alpha \in E^*$$

$$\{\hat{X}, \hat{Y}\} = -\widehat{[\![X, Y]\!]}, \quad \{\hat{X}, g \circ \tau^*\} = -\rho(X)(g) \circ \tau^*$$

$$\{f \circ \tau^*, g \circ \tau^*\} = 0$$

$$X, Y \in \Gamma(E), \quad f, g \in C^\infty(Q)$$

$([\![\cdot, \cdot]\!], \rho)$ is a Lie algebroid structure

$\Downarrow$

$\{\cdot, \cdot\}$ satisfies the Jacobi identity

$\{\cdot, \cdot\}$ is a linear Poisson bracket on $E^*$

Coste, Dazord, Weinstein (1987), Courant (1990)

$\Pi$ the almost Poisson 2-vector associated with $\{\cdot,\cdot\}$

Local expression:

$(q^i) \equiv$ local coordinates on $Q$

$\{e_\alpha\}$ a local basis of $\Gamma(E)$

$(q^i, p_\alpha) \equiv$ local coordinates on $E^*$

$(\rho^i_\alpha, C^\gamma_{\alpha\beta})$ local structure functions of $E$

$$\Pi = \rho^i_\alpha \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_\alpha} - \frac{1}{2} C^\gamma_{\alpha\beta} p_\gamma \frac{\partial}{\partial p_\alpha} \wedge \frac{\partial}{\partial p_\beta}$$

$(E, [\![\cdot,\cdot]\!], \rho)$ an almost Lie algebroid over $Q$

$H : E^* \longrightarrow R$ a Hamiltonian function

$\mathcal{H}_H^\Pi = -i(dH)\Pi \in \mathfrak{X}(E^*) \equiv$ the Hamiltonian vector field of $H$

Solutions of the Hamilton equations for H $\equiv$ Integral curves of $\mathcal{H}_H^\Pi$

Local expressions:

$$\mathcal{H}_H^\Pi = \rho_\alpha^i \frac{\partial H}{\partial p_\alpha} \frac{\partial}{\partial q^i} - (\rho_\alpha^i \frac{\partial H}{\partial q^i} + C_{\alpha\beta}^\gamma p_\gamma \frac{\partial H}{\partial p_\beta}) \frac{\partial}{\partial p_\alpha}$$

**Hamilton equations:**

$$\frac{dq^i}{dt} = \rho_\alpha^i \frac{\partial H}{\partial p_\alpha}, \quad \frac{dp_\alpha}{dt} = -(\rho_\alpha^i \frac{\partial H}{\partial q^i} + C_{\alpha\beta}^\gamma p_\gamma \frac{\partial H}{\partial p_\beta})$$

## UNCONSTRAINED (HOLONOMIC) MECHANICS

$\Downarrow$

## LIE ALGEBROIDS

A Weinstein (1996)
E Martínez (2001), M Popescu, P Popescu (2001)
M de León, JCM, E Martínez (2005)
K Grabowska, P Urbanski, J Grabowski (2006)

Klein's formalism for Lagrangian Mechanics on Lie algebroids
Symplectic formalism for Hamiltonian Mechanics on Lie algebroids

## CONSTRAINED (NONHOLONOMIC) MECHANICS

$\Downarrow$

ALMOST LIE ALGEBROIDS

J Cortés, E Martínez (2004)
T Mestdag (2005), T Mestdag, B Langerock (2005)
J Cortés, M de León, JCM, E Martínez (2005)
J Cortés, M de León, JCM, D Martín de Diego, E Martínez (2006)
M de León, JCM, D Martín de Diego (2007)
K Grabowska, J Grabowski (2008)

## 2.1 Definition and some examples

$\mathcal{G} : E \times_Q E \to \mathbb{R}$ a bundle metric on a Lie algebroid $(E, [\![\cdot\,,\cdot]\!], \rho)$ of rank $n$

$V : Q \to \mathbb{R}$ a real function on $Q$

- The Lagrangian function $L : E \to \mathbb{R}$:

$$L(a) = \frac{1}{2}\mathcal{G}(a,a) - V(\tau(a)), \quad a \in E,$$

- The nonholonomic constraints are determined by a subbundle $D$ of $E$ with rank $r$

Some examples:

- Nonholonomic mechanical systems on Lie algebras

  $E = \mathfrak{g}$ a real Lie algebra of finite dimension
  $D = \mathfrak{h}$ a vector subspace of $\mathfrak{g}$

  From reduction of nonholonomic LL mechanical systems
  Fedorov, Zenkov (2005)

  Explicit examples: Suslov system, Chaplygin sleigh or multidimensional generalizations
  Bloch, Fedorov, Jovanovic, Kozlov, Zenkov, ...

- Nonholonomic mechanical systems on right action Lie algebroids

  $E = Q \times \mathfrak{g}$, with $Q$ a manifold and $\mathfrak{g}$ a real Lie algebra of finite dimension
  $D$ is a vector subbundle of $E$

  From reduction of nonholonomic LR mechanical systems

  Explicit examples: Veselova system or multidimensional generalizations
  Veselov, Veselova (1989), Jovanovic (1999), Fedorov, Jovanovic (2004)

- Nonholonomic mechanical systems on left action Lie algebroids

  $E = \mathfrak{g} \times Q$, with $\mathfrak{g}$ a real Lie algebra of finite dimension and $Q$ a manifold
  $D$ is a vector subbundle of $E$

  From reduction of nonholonomic mechanical systems with semi-direct product symmetry
  Tai (2004)

  Explicit examples: Chaplygin giro
  Markeev (2002), Tai (2004)

- Nonholonomic mechanical systems on Atiyah algebroids

  $E = TQ/G$, with $p : Q \to Q/G$ a principal $G$-bundle
  $D = \tilde{D}/G$, with $\tilde{D}$ a $G$-invariant distribution on $Q$

  From reduction of nonholonomic mechanical systems which are invariant under the action of a symmetry Lie group $G$

  Explicit examples: the snakeboard and the two-wheeled planar mobile robot

## First conclusions:

✓ • The inclusive nature of the Lie algebroid framework for nonholonomic Mechanics

✓ • The reduction of a nonholonomic mechanical system on Lie algebroid is a nonholonomic mechanical system on a Lie algebroid

$\mathcal{G} : E \times_Q E \to \mathbb{R}$ a bundle metric on a Lie algebroid $(E, [\![\cdot\,,\cdot]\!], \rho)$ of rank $n$

$V : Q \to \mathbb{R}$ a real function on $Q$

$$\Downarrow$$

$L : E \to \mathbb{R}$ the Lagrangian function of mechanical type on $E$
$(L, D)$ a nonholonomic mechanical system on $E$

$$\Downarrow$$

$$E = D \oplus D^{\perp}$$

The orthogonal projector

$$\mathcal{P} : E \to D$$

$$\Downarrow$$

The almost Lie algebroid structure $([\![\cdot, \cdot]\!]_{\mathcal{P}}, \rho_{\mathcal{P}})$ on $D$ associated with the projector $\mathcal{P}$

$$[\![X,Y]\!]_{\mathcal{P}} = \mathcal{P}[\![X,Y]\!], \quad \rho_P(X) = \rho(X)$$
$$X, Y \in \Gamma(D)$$

$$\Downarrow$$

The nonholonomic bracket $\{\cdot,\cdot\}_{D^*} \equiv$( the linear almost Poisson bracket on $D^*$)

$$\{F_{D^*}, G_{D^*}\}_{D^*} = \{F_{D^*} \circ i_D^*, G_{D^*} \circ i_D^*\} \circ \mathcal{P}^*$$

$\{\cdot,\cdot\} \equiv$ the linear Poisson bracket on $E^*$

$i_D : D \longrightarrow E$ the canonical inclusion

$i_D^* : E^* \longrightarrow D^*$ the dual projection

$\mathcal{P}^* : D^* \longrightarrow E^*$ the dual morphism to $\mathcal{P}$

$\Pi \equiv$ the linear Poisson structure on $E^*$

$\Pi_{D^*} \equiv$ the linear almost Poisson structure on $D^*$

Particular case:

$E = TQ$. Then the linear Poisson structure on $E^* = T^*Q$ is the canonical symplectic structure. Thus, $D$ is a distribution on $Q$ and $\{\ ,\ \}_{D^*}$ is (or isomorphic to) the nonholonomic bracket studied by several authors: A.J. Van der Schaft, B.M. Maschke 1994; Ch.M. Marle 1998; W.S. Koon, J.E. Marsden 1998; L. Bates 1998; A. Ibort, M. de León, JCM, D. Martín de Diego 1999; F. Cantrijn, M. de León, D. Martín de Diego 1999; F. Cantrijn, M. de León, JCM, D. Martín de Diego 2000; Juan-Pablo Ortega, V. Planas-Bielsa 2004; H. Cendra, S. Grillo 2006; and others

## Unconstrained level

- The Hamiltonian function

$$H : E^* \to \mathbb{R}$$
$$\alpha \to \frac{1}{2}\mathcal{G}(\alpha,\alpha) + V \circ \tau_E^*$$

- Hamiltonian dynamics

$$\mathcal{H}_H^\Pi = -i(dH)\Pi$$

- Legendre transformation

$$Leg_L = \flat_{\mathcal{G}} : E \to E^*$$

$$\flat_{\mathcal{G}}(a)(b) = \mathcal{G}(a,b), \ \ a, b \in E_x$$

- Lagrangian dynamics

The Euler-Lagrange vector field $\xi_L \in \mathfrak{X}(E)$

$$(T\flat_{\mathcal{G}})(\xi_L) = \mathcal{H}_H^\Pi$$

## Constrained level

- The Hamiltonian function

$h : D^* \to \mathbb{R}$

$h = H \circ \mathcal{P}^*$

- Hamiltonian dynamics

$$\mathcal{H}_h^{\Pi_{D^*}} = -i(dh)\Pi_{D^*}$$

- Legendre transformation

$$Leg_{(L,D)} : D \to D^*$$

$$Leg_{(L,D)} = i_D^* \circ Leg_L \circ i_D$$

- Lagrangian dynamics

$$\xi_{(L,D)} = T\mathcal{P} \circ \xi_L \circ i_D \in \mathfrak{X}(D)$$

the nonholonomic

Euler-Lagrange vector field

## The local expressions

An important fact: to choose a local basis of $\Gamma(E)$ which is adapted to the nonholonomic problem:

$$
\begin{array}{rcl}
(q^i) & \equiv & \text{local coordinates on } Q \\
\{e_a\} & \equiv & \text{an orthonormal local basis of } \Gamma(D) \\
\{e_A\} & \equiv & \text{an orthonormal local basis of } \Gamma(D^\perp) \\
\{e_\alpha\} = \{e_a, e_A\} & \equiv & \text{an orthonormal local basis of } \Gamma(E) \\
(q^i, v^a, v^A) & \equiv & \text{local coordinates on } E \\
(q^i, p_a, p_A) & \equiv & \text{local coordiantes on } E^*
\end{array}
$$

## Unconstrained level

- The Hamiltonian function
  $H(q^i, p_\alpha) = \frac{1}{2}(p_\alpha^2) + V(q)$
- The Hamilton equations
  $$\frac{dq^i}{dt} = \rho^i_\alpha p_\alpha$$
  $$\frac{dp_\alpha}{dt} = -C^\gamma_{\alpha\beta} p_\gamma p_\beta - \rho^i_\alpha \frac{\partial V}{\partial q^i}$$
- The Legendre transformation
  $Leg_L(q^i, v^\alpha) = (q^i, v^\alpha)$
- The Lagrangian function
  $L(q^i, v^\alpha) = \frac{1}{2}(v^\alpha)^2 - V(q)$
- Lagrangian dynamics
  $$\frac{dq^i}{dt} = \rho^i_\alpha v^\alpha$$
  $$\frac{dv^\alpha}{dt} = -C^\gamma_{\alpha\beta} v^\gamma v^\beta - \rho^i_\alpha \frac{\partial V}{\partial q^i}$$

## Constrained level

- The Hamiltonian function
  $h(q^i, p_a) = \frac{1}{2}(p_a)^2 + V(q)$
- The Hamilton equations
  $$\frac{dq^i}{dt} = \rho^i_a p_a, \quad p_A = 0$$
  $$\frac{dp_a}{dt} = -C^c_{ab} p_c p_b - \rho^i_a \frac{\partial V}{\partial q^i}$$
- The Legendre transformation
  $Leg_{(L,D)}(q^i, v^a) = (q^i, v^a)$
- The Lagrangian function
  $L(q^i, v^a) = \frac{1}{2}(v^a)^2 - V(q)$
- Lagrangian dynamics
  $$\frac{dq^i}{dt} = \rho^i_a v^a, \quad v^A = 0$$
  $$\frac{dv^a}{dt} = -C^c_{ab} v^c v^b - \rho^i_a \frac{\partial V}{\partial q^i}$$

## The expression of the dynamical equations may be very simple !

In fact, this expressions depends (essentially) on the nature of the local structure functions of $E$ with respect to the local basis of sections $\{e_\alpha\}$

# A new conclusion

In order to simplify the resolution of the dynamical equations one may take local coordinates (quasi-coordinates or quasi-velocities) which are adapted to the nonholonomic problem

$$\Updownarrow$$

To choose an appropriated local basis of sections of the Lie algebroid

## Complete Lifts of sections in a Lie algebroid

J. Grabowski, P. Urbanski (1997)

$(E, [\![\cdot,\cdot]\!]_E, \rho_E)$ a Lie algebroid over $Q$

$X : Q \to E \in \Gamma(E)$

$$\Downarrow$$

$X^c \in \mathfrak{X}(E)$ the complete lift to $E$ of $X$

$(q^A, v^i) \equiv$ local fibred coordinates on $E$

$X = X^i e_i$

$$X^c = X^i(\rho_E)_i^A \frac{\partial}{\partial q^A} + ((\rho_E)_j^A \frac{\partial X^k}{\partial q^A} - X^i C_{ij}^k) v^j \frac{\partial}{\partial v^k}$$

A particular case:

$E = TQ (\Longrightarrow (\rho_E)_\beta^A = \delta_\beta^A, \quad C_{\alpha\beta}^\gamma = 0)$

$X = X^B \frac{\partial}{\partial q^B} \in \mathfrak{X}(Q) = \Gamma(E), \quad X^c = X^B \frac{\partial}{\partial q^B} + \frac{\partial X^B}{\partial q^A} v^A \frac{\partial}{\partial v^B}$

The standard complete lift of $X \in \mathfrak{X}(Q)$

$(L, D)$ a nonholonomic mechanical system on a Lie algebroid $\tau : E \to Q$

$\mathfrak{g}$ a real Lie algebra of finite dimension

$\psi : Q \times \mathfrak{g} \to E$ a vector bundle morphism over the identity of $Q$

$$\xi \in \mathfrak{g} \Rightarrow \psi_\xi : Q \to E \in \Gamma(E)$$

$$\mathfrak{g}^D = \cup_{x \in Q} \mathfrak{g}_x^D$$

$$\mathfrak{g}_x^D = \{\xi \in \mathfrak{g} / \psi_\xi(x) \in D_x\}$$

$\sigma : Q \to Q \times \mathfrak{g}$ a section of the trivial vector bundle $Q \times \mathfrak{g} \to Q$

$$\sigma(x) = (x, \tilde{\sigma}(x)), \text{ for all } x \in Q$$

$$\tilde{\sigma}(x) \in \mathfrak{g}_x^D, \text{ for all } x \in Q$$

**The nonholonomic momentum equation**

$$\mathcal{H}_H^{\Pi_{D^*}}(\widehat{\psi \circ \sigma}) \circ \flat_{\mathcal{G}} = (\psi \circ \sigma)^c(L)$$

A consequence

$\xi \in \mathfrak{g}$ a horizontal symmetry ($\psi_\xi(x) \in D_x, \forall x \in Q$)
$L$ is invariant under the action $\psi$ ($\psi_\xi^c(L) = 0, \forall \xi \in \mathfrak{g}$)

$$\Downarrow$$

$\widehat{\psi \circ \sigma}$ is a constant of the motion for the constrained Hamiltonian dynamics

$(L, D)$ a nonholonomic mechanical system on a Lie algebroid $\tau : E \to Q$

$\Pi_{D^*}$ the almost Poisson 2-vector associated with the nonholonomic bracket on $D^*$

$d^D$ the almost differential on the almost Lie algebroid $\tau_D : D \to Q$

$h$ the constrained Hamiltonian, $\mathcal{H}_h^{\Pi_{D^*}}$ the Hamiltonian vector field of $h$

$$\alpha \in \Gamma(D^*)$$

The vector field $\mathcal{H}_{h,\alpha}^{\Pi_{D^*}}$ on $Q$

$$\mathcal{H}_{h,\alpha}^{\Pi_{D^*}}(q) = (T_{\alpha(q)}\tau_{D^*})(\mathcal{H}_h^{\Pi_{D^*}}(\alpha(q))) \quad \text{for all } q \in Q$$

### Theorem

Let $\alpha : Q \to D^*$ be a section of the vector bundle $\tau^* : E^* \to Q$ such that $d^D\alpha = 0$. Under these hypotheses, the following are equivalent:

(i) If $c : I \to Q$ is an integral curve of the vector field $\mathcal{H}_{h,\alpha}^{\Pi_{D^*}}$ then $\alpha \circ c : I \to D^*$ is a solution of the constrained Hamilton equations for $h$.

(ii) $\alpha$ satisfies the Hamilton-Jacobi equation

$$d^D(h \circ \alpha) = 0.$$

The local expressions:

$$\alpha(q^i) = (q^i, \alpha_\gamma(q^i))$$

$$d^D\alpha = 0 \Leftrightarrow C^\delta_{\gamma\nu}\alpha_\delta = \rho^i_\gamma \frac{\partial \alpha_\nu}{\partial q^i} - \rho^i_\nu \frac{\partial \alpha_\gamma}{\partial q^i}$$

$$d^D(h \circ \alpha) = 0 \Leftrightarrow \rho^i_\gamma(\frac{\partial h}{\partial q^i} \circ \alpha + \frac{\partial \alpha_\nu}{\partial q^i}(\frac{\partial h}{\partial p_\nu} \circ \alpha)) = 0$$

Applications: The snakeboard

### The Poisson character of the Hamiltonian dynamics

The unconstrained (constrained) Hamiltonian dynamics of an unconstrained (constrained) mechanical system on a (an almost) Lie algebroid $E$ is (almost) Poisson (the Poisson structure is, in general, degenerate)

$$\Downarrow$$

This dynamics does not preserve, in general, a volume form on the phase space $E^*$

### Problem

To discuss the existence of an invariant volume for a (nonholonomic) mechanical system on a Lie algebroid

work in preparation (with Y Fedorov)

$(D, [\![\cdot,\cdot]\!]_D, \rho_D)$ an almost Lie algebroid over $Q$, $\quad rank D = n$

$\nu$ a volume form on $Q$

$\Omega \in \Gamma(\Lambda^n D^*)$ a volume form on the vector bundle $\tau : D \to Q$

**Definition**

The modular section $\mathcal{M}^{(\nu,\Omega)} \in \Gamma(D^*)$ of $D$ with respect to $\nu$ and $\Omega$ is given by

$$\mathcal{M}^{(\nu,\Omega)}(X) = div_\nu(\rho_D(X)) - div_\Omega X$$

for all $X \in \Gamma(D)$

$$d(i(\rho_D(X))\nu) = div_\nu(\rho(X))\nu, \qquad d^D(i(X)\Omega) = (div_\Omega X)\Omega$$

**Definition**

$D$ is unimodular if $\mathcal{M}^{(\nu,\Omega)} = 0$, for some $\nu$ and $\Omega$

- $\mathcal{M}^{(e^{\sigma}\nu, e^{\mu}\Omega)} = \mathcal{M}^{(\nu,\Omega)} + d^D(\sigma - \mu), \quad \sigma, \mu \in C^{\infty}(Q)$
- $D$ is a Lie algebroid over $Q \Rightarrow \mathcal{M}^{(\nu,\Omega)} \in \Gamma(E^*)$ is the modular section of $D$ introduced by Evens, Lu and Weinstein (1999)

$$d^D(\mathcal{M}^{(\nu,\Omega)}) = 0$$

$$\Downarrow$$

The cohomology class of $\mathcal{M}^{(\nu,\Omega)}$ does not depend on the chosen volumes $\nu$ and $\Omega$

The modular class of the Lie algebroid $D$

The local expression:

$(q^i)$ local coordinates on $Q$
$\{e_\alpha\}$ a local basis of $\Gamma(D)$

$$\nu = dq^1 \wedge \cdots \wedge dq^m, \quad \Omega = e^1 \wedge \cdots \wedge e^n$$

$$\mathcal{M}^{(\nu,\Omega)} = (C^\beta_{\alpha\beta} + \frac{\partial \rho^i_\alpha}{\partial q^i})e^\alpha$$

$(L, D)$ a nonholonomic mechanical system on the Lie algebroid $\tau : E \to Q$, with $V = 0$

$$h : D^* \to \mathbb{R} \text{ the constrained Hamiltonian}$$

**Theorem**

The almost Lie algebroid is $\tau_D : D \to Q$ is unimodular if and only if the constrained Hamiltonian dynamics preserves a volume form on $D^*$, that is, there exists a volume form $\Psi$ on $D^*$ such that

$$div_\Psi \mathcal{H}_h^{\Pi_{D^*}} = 0$$

**Important remark**

The fact that the constrained Hamiltonian dynamics preserves a volume form does not depend on the bundle metric!

Some applications: the unconstrained case ($E = D$)

**$E = TP/G \simeq TQ \times \mathfrak{g}$ the Atiyah algebroid associated with a trivial principal $G$-bundle $P = Q \times G \to Q$**

The Hamiltonian dynamics preserves a volume form on $E^* = T^*Q/G$ if and only if $\mathfrak{g}$ is unimodular

An Extension of the Liouville Theorem ($E = TQ$) and of a result by Kozlov (1988) for Hamiltonian systems on the dual space of a Lie algebra

**$E = Q \times \mathfrak{g}$ the Lie algebroid associated with a right (left) infinitesimal action $\Phi : \mathfrak{g} \to \mathfrak{X}(Q)$**

If $\mathfrak{g}$ is unimodular and $\Phi$ preserves a volume form on $Q$ then $E = Q \times \mathfrak{g}$ is unimodular and the Hamiltonian dynamics preserves a volume form on $E^* = Q \times \mathfrak{g}^*$

More work must be done

- What happens if $V \neq 0$?
- Applications to constrained dynamics ($D$ is an almost Lie algebroid). Compare with previous results of Bloch, Fedorov, Jovanovic, Zenkov for the standard and Lie algebra case

# More things on our present and future work

Collaboration with some people

(P. Balseiro, D. Chinea, D. Iglesias, M. de León, D. Martín de Diego, E. Padrón, D. Sosa,....)

- To discuss in more detail the reduction procedure for nonholonomic Mechanics on Lie algebroids as it has been done in

  J. Koiller (1992)

  AM Bloch, PS Krishnaprasad, J. Marsden and R.M. Murray (1996)

  F. Cantrijn, M. de León, JCM and D. Martín de Diego (1998,99)

  for the standard case
- Generalized nonholonomic mechanics and Leibniz algebroids

  (we lose the skew-symmetric character of the geometric objects of the theory; talk by P. Balseiro)
- Constrained mechanics and (almost) Lie affgebroids

  (extension to the affine setting of the previous results on nonholonomic mechanical systems on Lie algebroids)

# Lie groupoids and Discrete Mechanics

Lie algebroids are the infinitesimal invariants of Lie groupoids

- Unconstrained case: First steps in this direction by
  JCM, D. Martín de Diego and E. Martínez (2006)
  following previous contributions by
  J.E. Marsden, M. West (2001)
  for the standard case and by
  A. I. Bobenko and Y. B. Suris (1999), J.E. Marsden, S. Pekarsky and S. Shkoller (1999)
  for the Lie group case
- Constrained case: First steps in this direction by
  D. Iglesias, JCM, D. Martín de Diego and E. Martínez (2008)
  following previous contributions by
  J. Cortés, S. Martínez (2001)
  for the standard case and by
  Y. Fedorov, D. Zenkov (2005), R. McLachlan, M. Perlmutter (2006) and Y. Fedorov (2007)
  for the Lie group case

THANKS !