Dirac geometry, quasi-Poisson structures and moment maps

Henrique Bursztyn, IMPA

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Main goals:

Geometry underlying “exotic” moment maps

Role of Dirac/quasi-Poisson structures

Classical Scenario (Late 1960's, Kostant, Souriau, Smale...)

Momentum maps for Hamiltonian action:

$$J : M \to \mathfrak{g}^*$$

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Central role of Poisson geometry:

$\mathfrak{g}^*$ has natural Poisson structure,

$$\text{Hamiltonian } \mathfrak{g}\text{-spaces} \quad \longleftrightarrow \quad \text{Poisson maps } M \longrightarrow \mathfrak{g}^*$$

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- Symplectic groupoid actions (Mikami-Weinstein): $J : M \longrightarrow P$

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Beyond Poisson geometry: (Alekseev, Meinrenken, Kosmann-Schwarzbach...)

◇ Group-valued moment maps: $J : M \longrightarrow G$

◇ Manin pairs, quasi-Poisson group actions: $J : M \longrightarrow D/G$

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No symplectic or Poisson structures around...

The example of group-valued moment maps

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(c) $i_{u_M}\omega = \frac{1}{2} J^*(B(\cdot, u^r + u^l))$.

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Application: Moduli space of flat connections: $Hom(\pi_1(\Sigma), G)/G$.
(Atiyah-Bott, 1982)

Common features of all moment maps

1. Target of moment maps are foliated; inclusion of each leaf is a moment map (e.g. coadjoint orbits $\mathcal{O} \hookrightarrow \mathfrak{g}^*$),
2. Canonical Hamiltonian $G \times G$-space (e.g. T^*G),
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Other moment maps: *Abstract moment maps*, (Karshon, Guillemin, Ginzburg), *Optimal momentum maps* (Ortega-Ratiu), *Universal moment maps for Poisson actions* (Evens-Lu)...

Outline

1. Dirac structures (leaves, Poisson algebra, integration...)

2. Cartan-Dirac structures on Lie groups

3. Dirac morphisms and moment maps

4. Dirac structures from Manin pairs

5. Quasi-Poisson geometry

6. Final comments (spinors and volume forms, Courant morphisms...)

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Integrability: $[\![\Gamma(L),\Gamma(L)]\!]_\eta \subseteq \Gamma(L)$.

Dirac structure= almost Dirac structure + integrability.

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Integrability: $\frac{1}{2}[\pi, \pi] = \pi^\sharp(\eta)$.

Properties

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- Dirac structures integrate to **presymplectic groupoids** (B.,Crainic, Weinstein, Zhu, 2004).

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Singular foliation: Conjugacy classes

Leafwise 2-form (GHJW '97):

$$\omega(u_G, v_G)|_g := B\left(\frac{\mathrm{Ad}_g - \mathrm{Ad}_{g^{-1}}}{2} u, v\right)$$

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Momentum level set: If $J : (M, \pi) \longrightarrow \mathfrak{g}^*$ classical momentum map, then $J^{-1}(0) \hookrightarrow M$ has Dirac structure which is backward image of π, and whose forward image is π_{red} on M_{red}...

Let (P, L_P, η_P) be a Dirac manifold.

Definition: A smooth map $J : (M, L, \eta_M) \longrightarrow (P, L_P, \eta_P)$ is a (strong) Dirac morphism if

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Another example:

Theorem[B. - Crainic] There is 1-1 correspondence:

1. (M, ω, J) G-valued Hamiltonian $\mathfrak{g}$-space
2. Dirac morphisms $J : M \to G$, where G has Cartan-Dirac structure.

Moment map theory of Dirac manifold (P, L_P, η_P)

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Classical target: $\mathfrak{g}^*$ Lie-Poisson	**Dirac target:** (P, L_P, η_P)
Poisson morphisms: $J : M \to \mathfrak{g}^*$ $\mathfrak{g}$-action	**Dirac morphisms:** $J : M \to P$ L_P-action
$\iota : \mathcal{O} \hookrightarrow \mathfrak{g}^*$ Coadjoint orbit	$\iota : \mathcal{O} \hookrightarrow P$ presymplectic leaf
T^*G cotangent bundle (symplectic groupoid of $\mathfrak{g}^*$)	$(\mathcal{G}, \omega)$ presymplectic groupoid
Reduction: $J^{-1}(\mu)/G_\mu$ symplectic/Poisson reduced space	**Reduction**: $J^{-1}(x)/\mathcal{G}_x$ symplectic/Poisson reduced space

4. Dirac structures from Manin pairs

Manin pair: $(\mathfrak{d}, \mathfrak{g})$, where:

$\mathfrak{d}$ is $2n$-dim Lie algebra with $\langle \cdot, \cdot \rangle_{\mathfrak{d}}$, signature (n, n);

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Dirac structure L_S on S: $\mathfrak{g}_S \subseteq \mathfrak{d}_S \cong TS \oplus T^*S$ (w.r.t $\eta_S \in \Omega^3_{cl}(S)$).

Moment map theory: Dirac morphisms $J : M \to S = D/G$

Presymplectic groupoid: $\mathcal{G} = G \ltimes S$, $\omega \in \Omega^2(\mathcal{G})$ (explicit).

Examples

Classical momentum maps:

Manin pair: $\mathfrak{d} = \mathfrak{g} \ltimes \mathfrak{g}^*$, $[(u,\mu),(v,\nu)] = ([u,v], ad_u^*(\nu) - ad_v^*(\mu))$

$D/G = \mathfrak{g}^*$, dressing action is coadjoint action

Splitting: $T\mathfrak{g}^* \longrightarrow \mathfrak{d} \times \mathfrak{g}^*$, $\mu_x \mapsto ((0,\mu),x)$

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(similarly for G^*-valued moment maps...)

D/G-valued moment maps via Dirac geometry

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Dirac target: (P, L_P, η_P)	$(S = D/G, L_S, \eta_S)$
Dirac morphisms: $J : M \to P$ L_P-action	$J : M \to S$ $\mathfrak{g}$-action
$\iota : \mathcal{O} \hookrightarrow P$ presymplectic leaf	$\iota : \mathcal{O} \hookrightarrow S$ dressing orbit
$(\mathcal{G}, \omega)$ presymplectic groupoid	$(G \ltimes S, \omega)$ "double"
Reduction: $J^{-1}(x)/\mathcal{G}_x$ symplectic/Poisson reduced space	**Reduction**: $J^{-1}(x)/G_x$ symplectic/Poisson reduced space

5. Quasi-Poisson geometry

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Lie quasi-bialgebra: $(\mathfrak{g}, F, \chi)$, where $F : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$, $\chi \in \wedge^3 \mathfrak{g}$, so that

$$\begin{aligned} [(u,0),(v,0)]_{\mathfrak{d}} &= ([u,v],0), \\ [(v,0),(0,\mu)]_{\mathfrak{d}} &= (-\operatorname{ad}^*_\mu v, \operatorname{ad}^*_v \mu), \\ [(0,\mu)(0,\nu)]_{\mathfrak{d}} &= (\chi(\mu,\nu), F^*(\mu,\nu)), \end{aligned}$$

is a *Lie bracket* on $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$ (Drinfeld double).

5. Quasi-Poisson geometry

Consider Manin pair $(\mathfrak{d}, \mathfrak{g})$ plus lagrangian complement: $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$.

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Quasi-Poisson $\mathfrak{g}$-space: M is $\mathfrak{g}$-manifold, $\pi \in \mathfrak{X}^2(M)$ such that:

- $\frac{1}{2}[\pi,\pi] = \rho_M(\chi)$,
- $\mathcal{L}_{\rho_M(v)}\pi = -\rho_M(F(v)), \quad v \in \mathfrak{g}$.

($\rho_M : \mathfrak{g} \to \mathfrak{X}(M)$ infinitesimal action)

Moment maps for quasi-Poisson actions

Given quasi-Poisson $\mathfrak{g}$-space (M, π).

Moment map: equivariant map $J : M \longrightarrow S = D/G$ such that

$$\pi^\sharp dJ^* = \rho_M \overline{\sigma},$$

where $\overline{\sigma} = (\rho_S|_{\mathfrak{g}^*})^* : T^*S \longrightarrow \mathfrak{g}$.

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Group-valued moment maps via quasi-Poisson geometry:
(M, π), $\pi \in \mathfrak{X}^2(M)^{\mathfrak{g}}$, $J : M \longrightarrow G$

1. $\frac{1}{2}[\pi, \pi] = \rho_M(\chi_G)$, $(\chi_G \in \wedge^3 \mathfrak{g}$ Cartan trivector$)$
2. $\pi^\sharp(J^*\alpha) = \sum_i \langle \alpha, \frac{(e^i)^L + (e^i)^R}{2} \rangle \rho_M(e_i)$, $(\alpha \in \Omega^1(G), B(e_i, e^j) = \delta_i^j)$

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Equivalence of the two points of view

$(\mathfrak{d}, \mathfrak{g})$ Manin pair, $\quad S = D/G$.

Fix identifications $\mathfrak{d} \cong \mathfrak{g} \oplus \mathfrak{g}^*$ and $\mathfrak{d}_S \cong TS \oplus T^*S$.

Theorem[B., Crainic] There exists a 1-1 (functorial) correspondence between:

1. Dirac morphisms $J : M \to S$,
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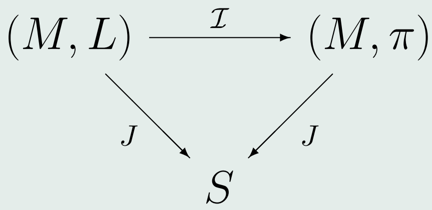

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$$\begin{array}{ccc} (M, L) & \xrightarrow{\ \mathcal{I}\ } & (M, \pi) \\ & {\scriptstyle J} \searrow \quad \swarrow {\scriptstyle J} & \\ & S & \end{array}$$

Functor preserves, e.g., reduction....

Key observation:

Dirac structure + Lagrangian complement = quasi-Poisson bivector

6. Final remarks

- Intrinsic approach to Hamiltonian spaces of Manin pairs (no choices involved) (with D. Iglesias Ponte and Pavol Severa)
- Spinorial viewpoint to Dirac geometry; construction of invariant volume forms, D.-H. theory etc... (with A. Alekseev, E. Meinrenken)

Some references...

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