Geometric nonlinear control and applications

Fátima Silva Leite
University de Coimbra - Portugal

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Outline

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- Nonlinear control systems/mechanical systems
- Controllability
- Optimal control

Geometric Nonlinear Control

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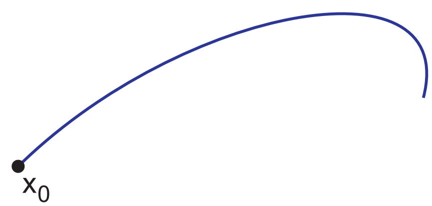
M is a smooth manifold.

A dynamical system evolving on M is a vector field $f(.)$ on M

$$\dot{x}(t) = f(x(t)), \quad x \in M.$$



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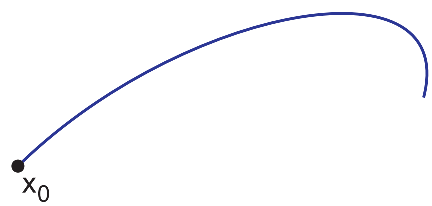
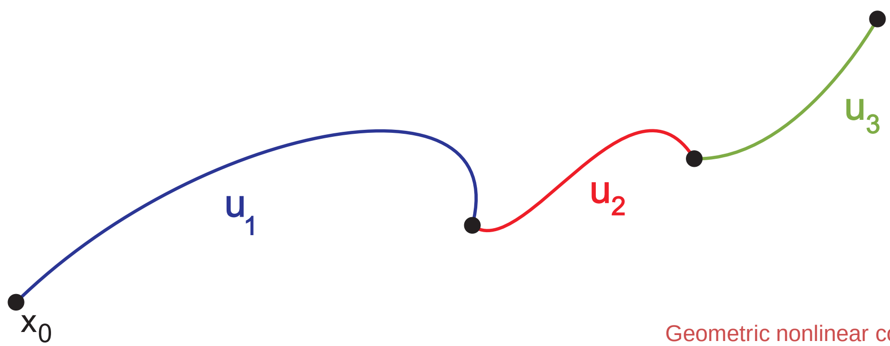


A control system evolving on M is a family of vector fields $f(., u)$ on M, parameterized by the controls u, belonging to some class $\mathcal{U}$ of admissible controls.

$$\dot{x}(t) = f(x(t), u(t)), \quad x \in M, \quad u \in \mathcal{U}$$



Matrix Lie groups

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Lie groups are smooth manifolds with the structure of a group.

(The group operations are smooth.)

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- Left-invariant vector fields

$$V(g) = g\,A, \quad g \in G, \quad A \in \mathcal{L}.$$

If $V(g) = g\,A,\; W(g) = g\,B$ are left-invariant vector fields,

$$[V, W](g) = g\,(AB - BA) \qquad \text{Lie bracket}$$

Matrix Lie groups

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Examples of classical Lie groups and Lie algebras:

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Examples of classical Lie groups and Lie algebras:

General linear group

$$GL(n,\mathbb{R}) = \{X \in gl(n,\mathbb{R}), det(X) \neq 0\}$$

$gl(n,\mathbb{R}) =$ the vector space of all real $n \times n$ matrices

Rotation group

$$SO(n,\mathbb{R}) = \{X \in GL(n,\mathbb{R}) : X^T = X^{-1}, det(X) = 1\}$$

$$so(n,\mathbb{R}) = \{A \in gl(n,\mathbb{R}) : A^T = -A\}$$

Matrix Lie groups

Examples of classical Lie groups and Lie algebras:

Euclidean group

$$SE(n,\mathbb{R}) = \left\{ \begin{bmatrix} X & v \\ 0 & 1 \end{bmatrix},\ X \in SO(n,\mathbb{R}),\ v \in \mathbb{R}^n \right\}$$

$$se(n,\mathbb{R}) = \left\{ \begin{bmatrix} A & u \\ 0 & 0 \end{bmatrix},\ A \in so(n,\mathbb{R}),\ u \in \mathbb{R}^n \right\}$$

Special Unitary group

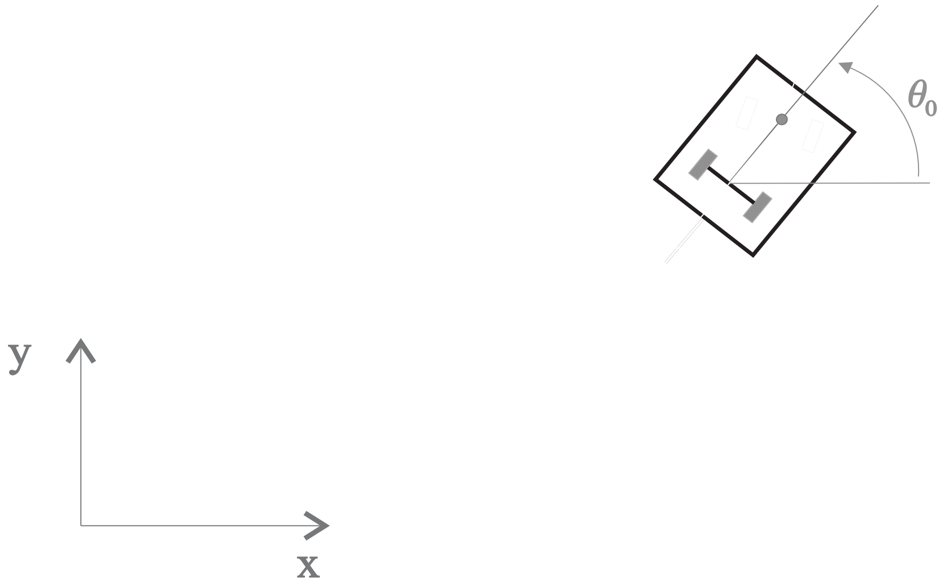
$$SU(n) = \{X \in GL(n,\mathbb{C}), X^* = X^{-1}, det(X) = 1\}$$

$$su(n) = \{A \in gl(n,\mathbb{C}), A^* = -A, \text{trace}(A) = 0\}$$

Applications

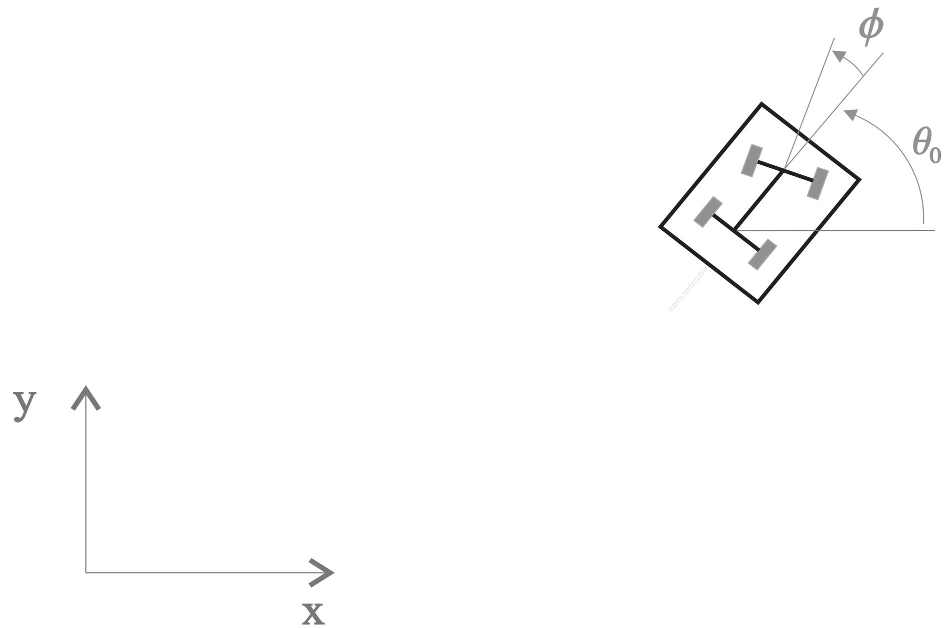
Applications

$\boldsymbol{SE(2)}$ - the configuration space of a unicycle



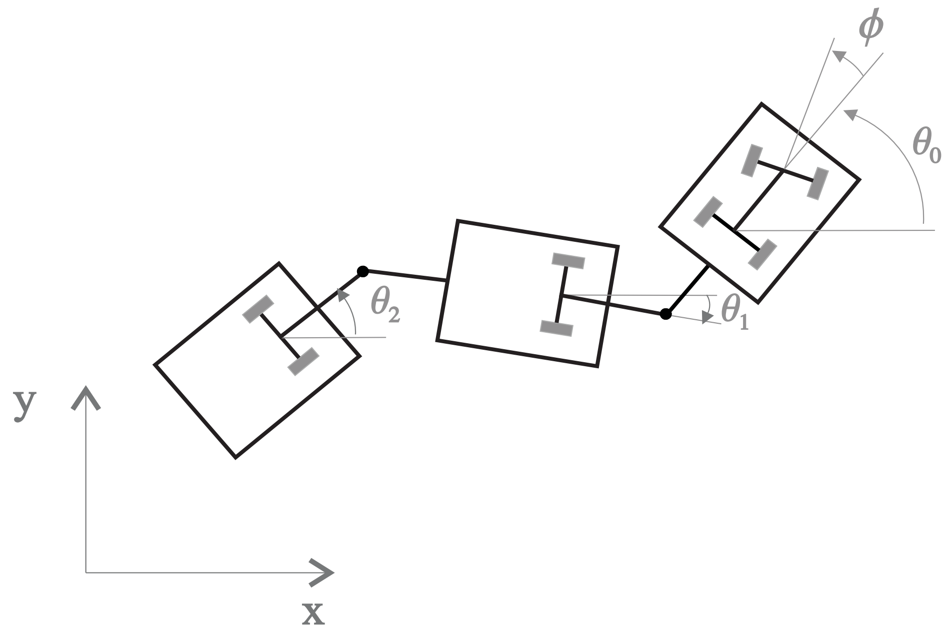
Applications

$SUP(4)$ - the configuration space of a car moving on a plane



Applications

$SUP(n+4)$ - the configuration space of a car pulling n trailers



Applications

Attitude control of satellites - $\boldsymbol{SO(3)}$

Airplane landing problem - $\boldsymbol{SE(3)}$

Quantum control of spin systems - $\boldsymbol{SU(2^n)}$

Underactuated systems on Lie groups

Underactuated systems on Lie groups

The kinematic control system is defined on a smooth n- dimensional Lie group G.

$$\Sigma : \qquad \frac{dg(t)}{dt} = g(t)\left(A_0 + \sum_{1}^{k} u_i(t) A_i\right), \qquad g(t) \in G,$$

- $u_1, \cdots, u_k$ are the control functions ($\mathcal{U}$ - piecewise constant);
- $A_0, A_1, \cdots, A_k \in \mathcal{L}$.

$$\Sigma \quad \Leftrightarrow \quad \left\{A_0 + \sum_{1}^{k} u_i(t) A_i\right\} \subset \mathcal{L}$$

▸ The system is underactuated if $k < n$.

Orbits and Attainable sets

Orbits and Attainable sets

Orbit of a control system Σ, through a point $g \in G$:

$$\mathcal{O}(g) = \{g\, exp(t_1 X_1) \cdots exp(t_N X_N) : X_i \in \Sigma, t_i \in \mathbb{R}, N \geq 0\}$$

Allowed to move forward and backward in time

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Attainable set of a control system Σ, from a point $g \in G$:

$$\mathcal{A}(g) = \{g\, exp(t_1 X_1) \cdots exp(t_N X_N) : X_i \in \Sigma, t_i \geq 0, N \geq 0\}$$

Allowed to move only forward in time!

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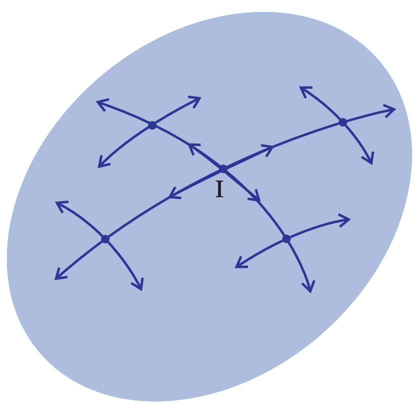
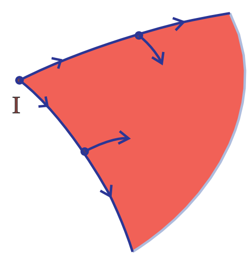


Orbit



Attainable

Orbits and Attainable sets

Orbits and Attainable sets

- $\mathcal{O}(g) = g\,\mathcal{O}(e)$
- $\mathcal{O}(e)$ is a connected Lie subgroup of G, with Lie algebra Lie(Σ)

Orbit theorem (Nagano-Sussmann)

- $\mathcal{A}(g) \subset \mathcal{O}(g)$
- $\mathcal{A}(g) = g\,\mathcal{A}(e)$
- $\mathcal{A}(e)$ is a subsemigroup of G

Motion planning

Motion planning

The objective is to drive the mechanical system from one configuration to another, without violating the constraints and using the available controls.

- Is the system controllable?
  - Doesn't care about the quality of the motion between two configurations neither the amount of control effort!

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The objective is to drive the mechanical system from one configuration to another, without violating the constraints and using the available controls.

- Is the system controllable?
  - Doesn't care about the quality of the motion between two configurations neither the amount of control effort!
- If so, what is the optimal way to control the system?
  - May require smooth motion, minimizing costs ...

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- When is a system controllable?

$$\Sigma \equiv \{A_0 + \sum_1^k u_i(t) A_i\}$$

Theorem - Σ is controllable on a Lie group G if and only if:

(1) G is connected

(2) Lie(Σ)$= \mathcal{L}$ (controllability rank condition)

(3) The attainable set $\mathcal{A}(e)$ is a subgroup of G.

(For systems without drift, (3) follows from (1)+(2))

Rolling motions

Rolling motions

M and N are smooth manifolds, with the same dimension, embedded in the Euclidean space $\mathbb{R}^n$.

M rolls upon N, without slip or twist, along a curve $\alpha \in M$.

Rolling is a rigid motion in the embedding space, subject to holonomic and nonholonomic constraints.

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The rolling motion is then described by the action of the Euclidean group $\mathbf{SE}_n = \mathbf{SO}_n \ltimes \mathbb{R}^n$ on $\mathbb{R}^n$:

$$\mathbf{SE}_n := \{X = (R, s) : \; R \in \mathbf{SO}_n, s \in \mathbb{R}^n\}$$

$$\begin{array}{ccc} \mathbf{SE}_n \times \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ (X, p) & \longmapsto & X \circ p = Rp + s \end{array}$$

Rolling maps

Rolling maps

(Sharpe, Differential geometry, Springer 1997)

A rolling map of a manifold M upon N, without twist or slip, along a piecewise smooth curve $\alpha : [0,\tau] \longrightarrow M$ is a mapping

$$\begin{array}{rcccl} X : & [0,\tau] & \longrightarrow & \mathbf{SE}_n = \mathbf{SO}_n \ltimes \mathbb{R}^n \\ & t & \longmapsto & X(t) = (R(t), s(t)) \end{array}$$

satisfying the following conditions:

- rolling conditions (holonomic constraints)
- no-slip and no-twist conditions (nonholonomic constraints)

Rolling conditions

Rolling conditions

- $X(t) \circ \alpha(t) =: \alpha_{\mathrm{dev}}(t) \in N$
  - α is the rolling curve on M
  - α_{dev} is the development of α on N

- $T_{X(t)\circ\alpha(t)}(X(t) \circ M) = T_{\alpha_{\mathrm{dev}}} N$
  - The tangent spaces coincide at each contact point

No-slip condition

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The manifolds N and $X(t) \circ M$ have the same velocity at the point of contact:

No-slip condition

The manifolds $\boldsymbol{N}$ and $\boldsymbol{X(t)} \circ \boldsymbol{M}$ have the same velocity at the point of contact:

$$\dot{\boldsymbol{\alpha}}_{\mathbf{dev}}(\boldsymbol{t}) = \boldsymbol{X}(\boldsymbol{t}) \circ \dot{\boldsymbol{\alpha}}(\boldsymbol{t})$$

No-slip condition

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$$\Updownarrow$$

No-slip condition

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$$\dot{\alpha}_{\mathbf{dev}}(t) = X(t) \circ \dot{\alpha}(t)$$

$$\Updownarrow$$

$$\boxed{\dot{R}(t)R^{\top}(t)(\alpha_{\mathbf{dev}}(t) - s(t)) + \dot{s}(t) = \mathbf{0}}$$

No-twist conditions

No-twist conditions

- Tangential part

$$\boxed{\dot{R}(t)R^{\top}(t)\, T_{\alpha_{\text{dev}}(t)}N \subset (T_{\alpha_{\text{dev}}(t)}N)^{\perp}}$$

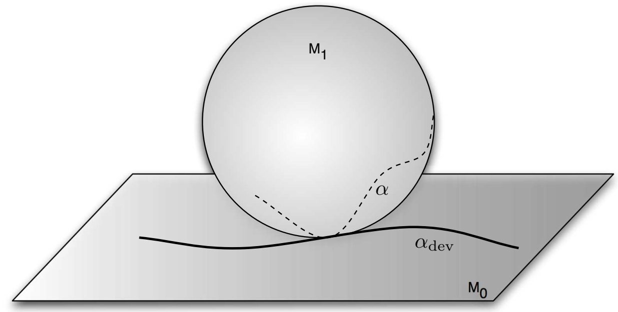
- Normal part

$$\boxed{\dot{R}(t)R^{\top}(t)\, (T_{\alpha_{\text{dev}}(t)}N)^{\perp} \subset T_{\alpha_{\text{dev}}(t)}N}$$

The rolling sphere

The rolling sphere

S^n rolls upon $T^{\text{aff}}_{p_0}S^n$, without slip or twist, along a curve α.
(p_0 is the south pole).



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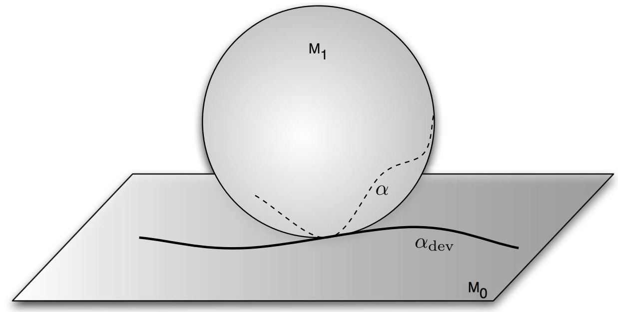

Kinematic equations:

$$\begin{cases} \dot{s}(t) = u(t) \\ \dot{R}(t) = R(t)\underbrace{(\sum_{i=1}^{n} u_i(t) A_{n+1,i})}_{A(t)} \end{cases}$$

$$u = \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix}^\top, \qquad A = \left[\begin{array}{ccc|c} & & & -u_1 \\ & & & \vdots \\ & & & -u_n \\ \hline u_1 & \cdots & u_n & 0 \end{array}\right]$$

Controllability of the kinematics

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Rewrite the equations as a left-invariant control system on $G = \mathbb{R}^n \times \mathbf{SO}_{n+1}$, with Lie algebra $\mathcal{L}(G) = \mathbb{R}^n \oplus \mathfrak{so}_{n+1}$:

$$\dot{g} = \sum_{i=1}^{n} u_i(t)(F_i(g) + H_i(g)),$$

$$g = (s, R), \qquad F_i(g) = \frac{\partial}{\partial x_i}(s) \oplus 0, \qquad H_i(g) = 0 \oplus RA_{n+1,i}.$$

Now prove the controllability rank condition:

$$\text{Lie}\{F_1(e) + H_1(e), \cdots, F_n(e) + H_n(e)\} = \mathcal{L}.$$

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Theorem:
The kinematic equations for rolling S^n upon its affine tangent space at a point are controllable on $\mathbb{R}^n \times \mathbf{SO}_{n+1}$.

(Jurdjevic, Zimmerman (2004))

Constructive proof of controllability

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The rolling sphere S^n is a complete nonholonomic system.

Constructive proof of controllability

Question: How to steer the rolling sphere from one initial configuration to any other configuration, without violating the nonholonomic constraints?

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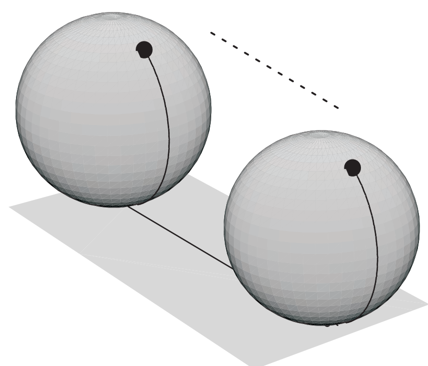
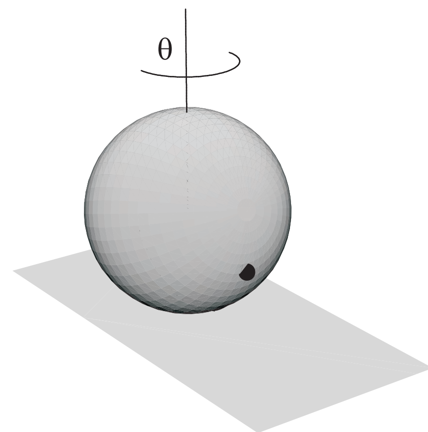
Forbidden motions

Slips



Slips are pure translations

Twists



Twists are pure rotations that stabilize $\boldsymbol{p_0}$

Constructive proof of controllability

Question: How to steer the rolling sphere from one initial configuration to any other configuration, without violating the nonholonomic constraints?

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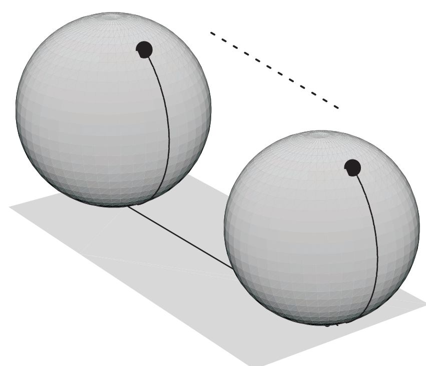

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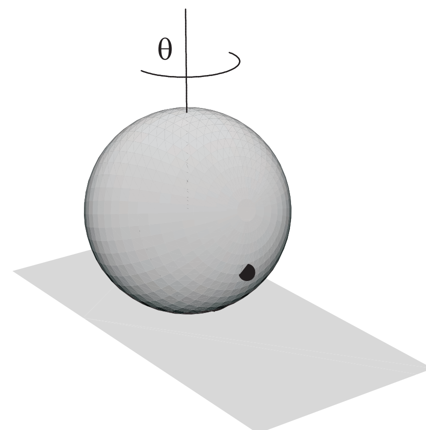

Twists are pure rotations that stabilize p_0

Answer: Miming slips and twists by rolling without slip or twist!

How to mime a twist?

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For S^2, a twist is a rotation around the z-axis

$$z(\varphi) = \mathrm{e}^{-\varphi A_{12}} = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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$$z(\varphi) = e^{-\varphi A_{12}} = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Any rotation around the z-axis can be obtained by rotating around the x-axis and the y-axis. (Euler's theorem)

$$z(\varphi) = x(\frac{\pi}{2})\, y(\frac{\varphi}{2})\, x(-\pi)\, y(-\frac{\varphi}{2})\, x(\frac{\pi}{2})$$

Miming a twist

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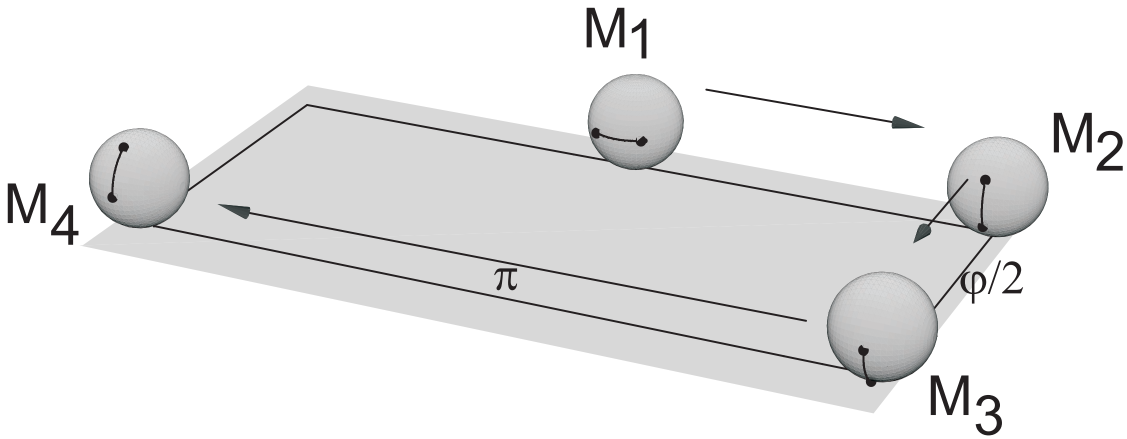
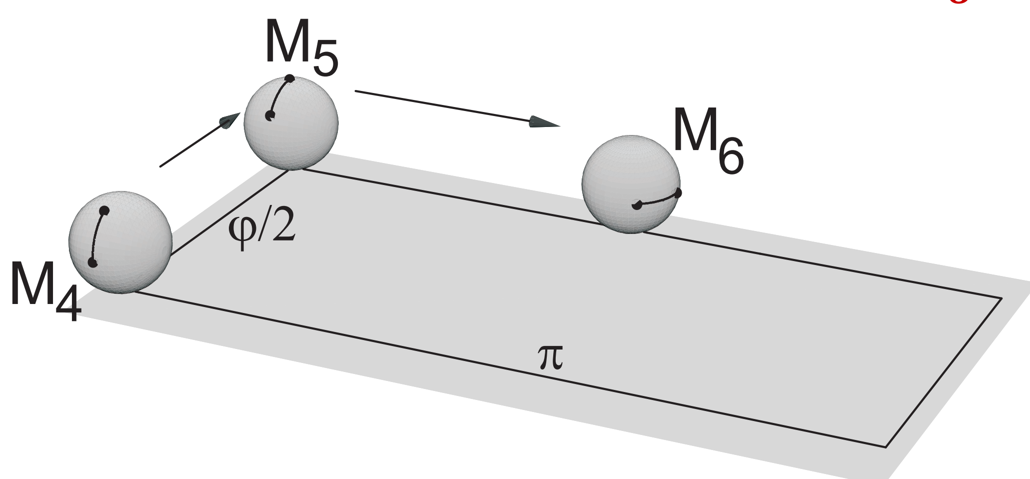
$$M_1 = S^2$$



$$M_6 = z(\varphi)S^2$$



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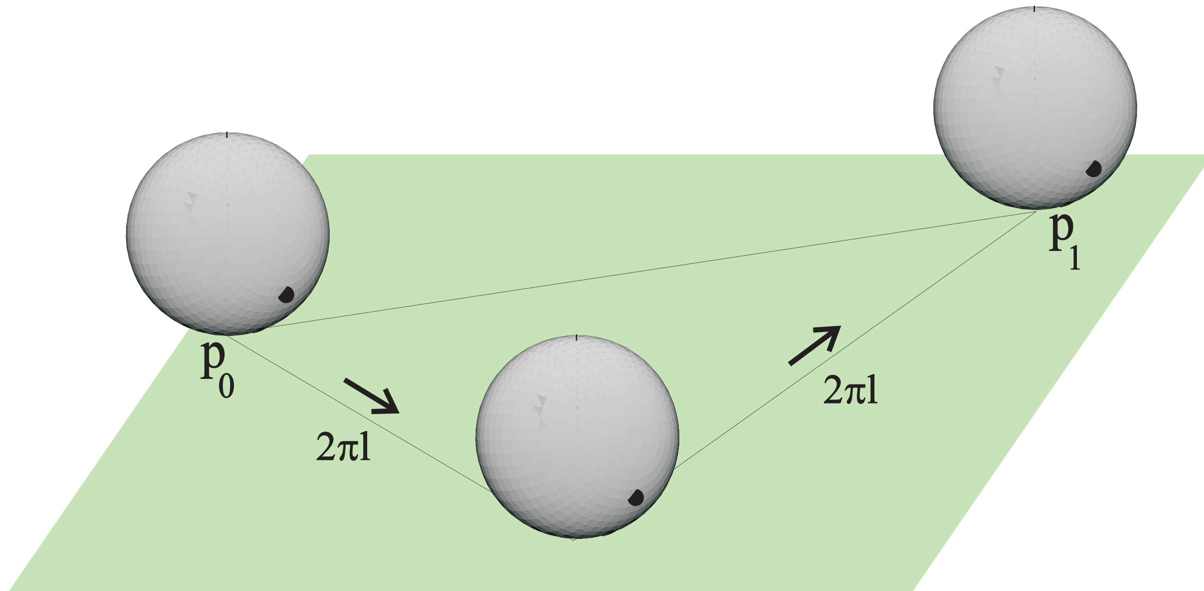

Controllability of the kinematic equations

Controllability of the kinematic equations

For the n-dimensional case, twists belong to the stabilizer subgroup of p_0.

A twist can be accomplished by rolling without slip or twist due to convenient Cartan decompositions of $\mathfrak{so}_{n+1}$ and corresponding $\mathbf{SO}_{n+1}$.

Controllability of the kinematic equations

The structure of the skewsymmetric matrix $A(t)$ plays an important role:

$$A = \left[\begin{array}{ccc|c} & & & -u_1 \\ & & & \vdots \\ & & & -u_n \\ \hline u_1 & \cdots & u_n & 0 \end{array}\right] = \sum_{i=1}^{n} u_i\, A_{n+1,i}$$

$$\mathfrak{so}_{n+1} = \mathcal{T} \oplus \mathcal{P}, \qquad \text{Cartan decomposition}$$

where

$$\mathcal{T} \simeq \mathfrak{so}_n, \quad \mathcal{P} = span\{A_{n+1,i}\}.$$

Generalized Euler's theorem:

$R \in \exp\left(\mathfrak{so}_n\right)$, can be written as

$$R = \prod \exp\left(t_j B_j\right),\ B_j \in \{A_{n+1,i}\},\ 1 \le i \le n.$$

Optimal Control Problems

Optimal Control Problems

$J(u) = \int_0^{t_1} \varphi(q(t), u(t))\, dt \longrightarrow \min$ (cost functional)

$\dot{q} = f(q, u), \quad q \in M, \quad u \in \mathcal{U}$ (a control system)

$q(0) = q_0, \quad q(t_1) = q_1$ (boundary conditions)

Agrachev, Jurdjevic, Sachkov, Sussmann, ...

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Hamiltonian systems play a very important role in optimal control theory.

Hamiltonian equations

Hamiltonian equations

Solutions of any optimal control problem are described by trajectories of a Hamiltonian system, intrinsically associated to the problem by a procedure that is a geometric elaboration of the Lagrange multipliers rule.

$$H(p,q,u) = p\,f(q,u) - \varphi(q,u) \qquad \text{(Hamiltonian)}$$

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$$H(p,q,u) = p\, f(q,u) - \varphi(q,u) \qquad \text{(Hamiltonian)}$$

$$\begin{cases} \dot{p} = -\frac{\partial H}{\partial q}(p,q,u) \\ \\ \dot{q} = +\frac{\partial H}{\partial p}(p,q,u) \end{cases} \qquad \text{(Hamiltonian equations on } T^*M\text{)}$$

$+$ one extra equation

$$\frac{\partial H}{\partial u}(p,q,u) = 0$$

Optimal control on Lie groups

Optimal control on Lie groups

The cotangent bundle T^*G of a Lie group G has a natural trivialization : $T^*G \cong \mathcal{L}^* \times G$.

For a left-invariant optimal control problem on a Lie group G, the Hamiltonian doesn't depend on $g \in G$

$$\begin{cases} \dot{a} = \left(\text{ad}\,\dfrac{\partial H}{\partial a}\right)^* a, & a \in \mathcal{L}^* \\ \dot{g} = g\,\dfrac{\partial H}{\partial a}, & g \in G. \end{cases}$$

(Hamiltonian equations on $\mathcal{L}^* \times G$)

Optimal control on Lie groups

If $G \subset GL(n)$ is compact, its Lie algebra has an invariant inner product $< .,. >$:

$$< A, B >= -\text{tr}(AB)$$

and this allow the identification $\mathcal{L} \equiv \mathcal{L}^*$:

$$\begin{array}{ccc} \mathcal{L} & \leftrightarrow & \mathcal{L}^* \\ A & \leftrightarrow & a =< A, . > \end{array}$$

in particular,

$$\left(\text{ad}\,\frac{\partial H}{\partial a}\right)^* : \mathcal{L}^* \to \mathcal{L}^* \quad \text{is identified with} \quad \text{ad}\,\frac{\partial H}{\partial A} : \mathcal{L} \to \mathcal{L}.$$

$$\begin{cases} \dot{A} = \left[A, \dfrac{\partial H}{\partial A}\right], & A \in \mathcal{L} \\ \dot{g} = g\,\dfrac{\partial H}{\partial A}, & g \in G. \end{cases}$$

(Hamiltonian equations on $\mathcal{L} \times G$)

Euler’s elastic problem

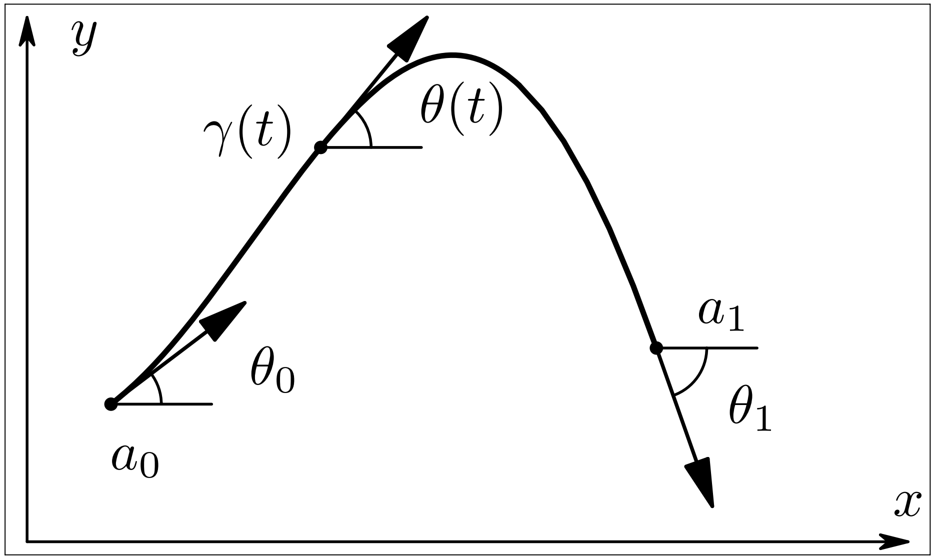
Euler's elastic problem

This is a problem studied by Euler in 1744:

Given
- $\mathbf{2}$ points $\boldsymbol{a_0 = (x_0, y_0)}$, $\boldsymbol{a_1 = (x_1, y_1)}$ in the plane
- $\mathbf{2}$ unit vectors $\boldsymbol{v_0}$, $\boldsymbol{v_1}$, attached to $\boldsymbol{a_0}$ and $\boldsymbol{a_1}$ respect.

Find the profile of the elastic rod with fixed endpoints $\boldsymbol{a_0}$ and $\boldsymbol{a_1}$ and fixed tangents $\boldsymbol{v_0}$ and $\boldsymbol{v_1}$ at these endpoints.

Euler's elastic problem



- $\gamma(t) = (x(t), y(t))$, $t \in [0, t_1]$ - the arc-length parametrization of the elastic rod, with fixed length t_1
- $\theta(t)$ - the angle between the velocity vector and the positive direction of the x-axis

Euler's elastic problem

Then the elastic problem can be stated as follows:

$$\dot{x} = \cos\theta,$$

$$\dot{y} = \sin\theta,$$

$$\dot{\theta} = u,$$

$$(x, y, \theta)(0) = (x_0, y_0, \theta_0), \qquad (x, y, \theta)(t_1) = (x_1, y_1, \theta_1),$$

where $v_0 = (\cos\theta_0, \sin\theta_0)$, $v_1 = (\cos\theta_1, \sin\theta_1)$. The elastic energy of the rod is measured by the integral

$$J = \frac{1}{2}\int_0^{t_1} k^2\,dt \to \min,$$

where k is the curvature of the rod. For an arc-length parametrized curve, the curvature is, up to sign, equal to the angular velocity, thus $k^2 = \dot{\theta}^2 = u^2$, and we obtain the cost functional for the optimal control problem:

$$J = \frac{1}{2}\int_0^{t_1} u^2\,dt \to \min.$$

The rolling sphere (again)

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Problem - Given 2 admissible configurations, roll the sphere upon the tangent plane from the first configuration to the second, so that the curve traced in the plane by the contact point be the shortest possible.

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$$J(u) = \frac{1}{2}\int_0^{t_1} (u_1^2 + u_2^2)\, dt \longrightarrow \min$$ (cost functional)

subject to:

$$\dot{s}(t) = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

(Kinematic equations)

$$\dot{R}(t) = R(t) \begin{bmatrix} 0 & 0 & u_1 \\ 0 & 0 & u_2 \\ -u_1 & -u_2 & 0 \end{bmatrix}$$

$$X(0) = X_0 = (s_0, R_0)$$
$$X(t_1) = X_1 = (s_1, R_1)$$

(boundary conditions)

The rolling sphere / Euler's elastica

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A remarkable result

The point of contact of the sphere rolling optimally traces Euler elastica on the plane!

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