Vakonomic Mechanics on Lie affgebroids

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Outline

1. Lie affgebroids
2. Lagrangian formalism on Lie affgebroids
3. Vakonomic mechanics on Lie affgebroids

Lie affgebroids

- Notation

$\tau_{\mathcal{A}} : \mathcal{A} \to Q$ affine bundle with associated vector bundle
$\tau_V : V \to Q$

$\tau_{\mathcal{A}^+} : \mathcal{A}^+ = Aff(\mathcal{A}, \mathbb{R}) \to Q$ the affine dual bundle

$1_{\mathcal{A}} \in \Gamma(\mathcal{A}^+)$, $1_{\mathcal{A}}(x)(a_x) = 1$, for $a_x \in \mathcal{A}_x$, with $x \in Q$

$\tau_{\widetilde{\mathcal{A}}} : \widetilde{\mathcal{A}} = (\mathcal{A}^+)^* \to Q$ the bidual bundle

$i_{\mathcal{A}} : \mathcal{A} \to \widetilde{\mathcal{A}}$, $i_{\mathcal{A}}(a)(\varphi) = \varphi(a)$, the inclusion with associated linear map $i_V : V \to \widetilde{\mathcal{A}}$

Definition

Lie affgebroid structure on $\mathcal{A}$:

$[\![\cdot,\cdot]\!]_V : \Gamma(V) \times \Gamma(V) \to \Gamma(V)$ Lie bracket

$D : \Gamma(\mathcal{A}) \times \Gamma(V) \to \Gamma(V)$ $\mathbb{R}$-linear action

$\rho_{\mathcal{A}} : \mathcal{A} \to TQ$ affine map, the *anchor map*

such that

$$D_X[\![\bar{Y},\bar{Z}]\!]_V = [\![D_X\bar{Y},\bar{Z}]\!]_V + [\![\bar{Y},D_X\bar{Z}]\!]_V$$

$$D_{X+\bar{Y}}\bar{Z} = D_X\bar{Z} + [\![\bar{Y},\bar{Z}]\!]_V$$

$$D_X(f\bar{Y}) = fD_X\bar{Y} + \rho_{\mathcal{A}}(X)(f)\bar{Y}$$

for $X \in \Gamma(\mathcal{A})$, $\bar{Y},\bar{Z} \in \Gamma(V)$, $f \in C^\infty(Q)$

Definition

Lie algebroid structure on a vector bundle $\tau_E : E \to Q$:

$[\![\cdot,\cdot]\!]_E : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ Lie bracket

$\rho_E : E \to TQ$ bundle map, the *anchor map*

such that

$$[\![X, fY]\!]_E = f[\![X, Y]\!]_E + \rho_E(X)(f)Y$$

for $X, Y \in \Gamma(E)$, $f \in C^\infty(Q)$

Example (The tangent bundle of a manifold)

$\pi_Q : E = TQ \to Q$

$[\![\cdot,\cdot]\!]_E = [\cdot,\cdot]$

$\rho_E = Id$

- Lie algebroid $(E, [\![\cdot,\cdot]\!]_E, \rho_E)$ is Lie affgebroid with $D = [\![\cdot,\cdot]\!]_E$

$$D_X[\![\bar{Y},\bar{Z}]\!]_V = [\![D_X\bar{Y},\bar{Z}]\!]_V + [\![\bar{Y},D_X\bar{Z}]\!]_V$$
$$D_{X+\bar{Y}}\bar{Z} = D_X\bar{Z} + [\![\bar{Y},\bar{Z}]\!]_V$$
$$D_X(f\bar{Y}) = fD_X\bar{Y} + \rho_{\mathcal{A}}(X)(f)\bar{Y}$$

- $(E, [\![\cdot,\cdot]\!]_E, \rho_E)$ Lie algebroid

the differential of E $\quad d^E : \Gamma(\wedge^k E^*) \longrightarrow \Gamma(\wedge^{k+1} E^*)$

$$
\begin{aligned}
(d^E\mu)(X_0,\ldots,X_k) &= \sum_{i=0}^{k}(-1)^i \rho_E(X_i)(\mu(X_0,\ldots,\widehat{X}_i,\ldots,X_k)) \\
&+ \sum_{i<j}(-1)^{i+j}\mu([\![X_i,X_j]\!]_E, X_0,\ldots,\widehat{X}_i,\ldots,\widehat{X}_j,\ldots,X_k)
\end{aligned}
$$

$\mu \in \Gamma(\wedge^k E^*)$, $X_0,\ldots,X_k \in \Gamma(E)$

- $\xi \in \Gamma(E^*)$ is a *1-cocyle* iff $d^E\xi = 0$

- $(\mathcal{A}, [\![\cdot,\cdot]\!]_V, D, \rho_{\mathcal{A}})$ Lie affgebroid $\Rightarrow$ $(V, [\![\cdot,\cdot]\!]_V, \rho_V)$ Lie algebroid
- $(\mathcal{A}, [\![\cdot,\cdot]\!]_V, D, \rho_{\mathcal{A}})$ Lie affgebroid

  $$\Downarrow$$

  $(\widetilde{\mathcal{A}}, [\![\cdot,\cdot]\!]_{\widetilde{\mathcal{A}}}, \rho_{\widetilde{\mathcal{A}}})$ Lie algebroid $+\ 1_{\mathcal{A}} \in \Gamma(\mathcal{A}^+)$ 1-cocycle

  Conversely, $(U, [\![\cdot,\cdot]\!]_U, \rho_U)$ Lie algebroid and $\phi : U \to \mathbb{R}$ 1-cocycle, $\phi_{|U_x} \neq 0$

  $$\Downarrow$$

  $\mathcal{A} = \phi^{-1}\{1\}$ Lie affgebroid with $(\widetilde{\mathcal{A}}, [\![\cdot,\cdot]\!]_{\widetilde{\mathcal{A}}}, \rho_{\widetilde{\mathcal{A}}}) \approx (U, [\![\cdot,\cdot]\!]_U, \rho_U)$, $1_{\mathcal{A}} \approx \phi$ and $V = \phi^{-1}\{0\}$

Example (The 1-jet bundle of a fibration)

$\tau : Q \to \mathbb{R}$ fibration

$\tau_{1,0} : J^1\tau \to Q$ affine bundle modelled on $\pi = (\pi_Q)_{|V\tau} : V\tau \to Q$

$$\eta = \tau^*(dt),\ t \text{ usual coordinate on } \mathbb{R}$$

$$\Downarrow$$

$$J^1\tau \cong \{v \in TQ \,|\, \eta(v) = 1\} \quad V\tau = \{v \in TQ \,|\, \eta(v) = 0\}$$

$$\pi_Q^* : (J^1\tau)^+ = T^*Q \to Q$$

$$\Downarrow$$

$\pi_Q : \widetilde{J^1\tau} \cong TQ \to Q$ with the standard Lie algebroid structure

$1_{J^1\tau} = \eta$

The Lagrangian formalism on Lie affgebroids

$(\tau_{\mathcal{A}} : \mathcal{A} \to Q, \tau_V : V \to Q, ([\![\cdot,\cdot]\!]_V, D, \rho_{\mathcal{A}}))$ Lie affgebroid

$(\widetilde{\mathcal{A}}, [\![\cdot,\cdot]\!]_{\widetilde{\mathcal{A}}}, \rho_{\widetilde{\mathcal{A}}})$ Lie algebroid over Q

- (x^i) local coordinates on Q

  $\{e_0, e_\alpha\}$ local basis of $\Gamma(\widetilde{\mathcal{A}})$ adapted to $1_{\mathcal{A}}$ $(1_{\mathcal{A}}(e_0)=1,\ 1_{\mathcal{A}}(e_\alpha)=0)$

$$[\![e_0, e_\alpha]\!]_{\widetilde{\mathcal{A}}} = C^\gamma_{0\alpha} e_\gamma \quad [\![e_\alpha, e_\beta]\!]_{\widetilde{\mathcal{A}}} = C^\gamma_{\alpha\beta} e_\gamma$$

$$\rho_{\widetilde{\mathcal{A}}}(e_0) = \rho^i_0 \frac{\partial}{\partial x^i} \qquad \rho_{\widetilde{\mathcal{A}}}(e_\alpha) = \rho^i_\alpha \frac{\partial}{\partial x^i}$$

$$\Downarrow$$

(x^i, y^0, y^α) local coordinates on $\widetilde{\mathcal{A}}$

(x^i, y^α) local coordinates on $\mathcal{A}$ $(y^0 = 1)$

- $\tau_{\mathcal{A}} : \mathcal{A} \to Q$ fibration

$$\Downarrow$$

$$\mathcal{T}_a^{\widetilde{A}}\mathcal{A} = \{(\tilde{a}, X_a) \in \widetilde{A} \times T_a\mathcal{A} \,|\, \rho_{\widetilde{A}}(\tilde{a}) = (T_a\tau_{\mathcal{A}})(X_a)\}$$

$$\tau_{\widetilde{A}}^{\tau_{\mathcal{A}}} : \mathcal{T}^{\widetilde{A}}\mathcal{A} \to \mathcal{A}, \quad \tau_{\widetilde{A}}^{\tau_{\mathcal{A}}}(\tilde{a}, X_a) = \pi_{\mathcal{A}}(X_a) = a$$

$(\mathcal{T}^{\widetilde{A}}\mathcal{A}, [\![\cdot,\cdot]\!]_{\widetilde{A}}^{\tau_{\mathcal{A}}}, \rho_{\widetilde{A}}^{\tau_{\mathcal{A}}})$ Lie algebroid over $\mathcal{A}$ called *the prolongation of $\widetilde{A}$ over $\tau_{\mathcal{A}}$* or *the $\widetilde{A}$-tangent bundle to $\mathcal{A}$*

$\{\mathcal{X}_0, \mathcal{X}_\alpha, \mathcal{V}_\alpha\}$ local basis of sections of $\mathcal{T}^{\widetilde{A}}\mathcal{A}$:

$$\mathcal{X}_0(a) = \left(e_0(\tau_{\mathcal{A}}(a)), \rho_0^i \frac{\partial}{\partial x^i}_{|a}\right)$$

$$\mathcal{X}_\alpha(a) = \left(e_\alpha(\tau_{\mathcal{A}}(a)), \rho_\alpha^i \frac{\partial}{\partial x^i}_{|a}\right) \qquad \mathcal{V}_\alpha(a) = \left(0, \frac{\partial}{\partial y^\alpha}_{|a}\right)$$

$$[\![\mathcal{X}_0, \mathcal{X}_\alpha]\!]^{\tau_{\tilde{\mathcal{A}}}^{\mathcal{A}}} = C_{0\alpha}^{\gamma}\mathcal{X}_\gamma \qquad [\![\mathcal{X}_\alpha, \mathcal{X}_\beta]\!]^{\tau_{\tilde{\mathcal{A}}}^{\mathcal{A}}} = C_{\alpha\beta}^{\gamma}\mathcal{X}_\gamma$$

$$[\![\mathcal{X}_0, \mathcal{V}_\alpha]\!]^{\tau_{\tilde{\mathcal{A}}}^{\mathcal{A}}} = [\![\mathcal{X}_\alpha, \mathcal{V}_\beta]\!]^{\tau_{\tilde{\mathcal{A}}}^{\mathcal{A}}} = [\![\mathcal{V}_\alpha, \mathcal{V}_\beta]\!]^{\tau_{\tilde{\mathcal{A}}}^{\mathcal{A}}} = 0$$

$$\rho^{\tau_{\tilde{\mathcal{A}}}^{\mathcal{A}}}(\mathcal{X}_0) = \rho_0^i \frac{\partial}{\partial x^i} \qquad \rho^{\tau_{\tilde{\mathcal{A}}}^{\mathcal{A}}}(\mathcal{X}_\alpha) = \rho_\alpha^i \frac{\partial}{\partial x^i}$$

$$\rho^{\tau_{\tilde{\mathcal{A}}}^{\mathcal{A}}}(\mathcal{V}_\alpha) = \frac{\partial}{\partial y^\alpha}$$

- $\phi_0 : \mathcal{T}^{\widetilde{\mathcal{A}}}\mathcal{A} \to \mathbb{R} \qquad \Rightarrow \phi_0 \in \Gamma((\mathcal{T}^{\widetilde{\mathcal{A}}}\mathcal{A})^*)$
  $\phi_0(\tilde{a}, X_a) = 1_{\mathcal{A}}(\tilde{a}) \qquad \phi_0 = \mathcal{X}^0$
  $1_{\mathcal{A}}$ is a 1-cocycle $\Rightarrow \phi_0$ is a 1-cocycle
- *the vertical endomorphism* $S : \mathcal{A} \to \mathcal{T}^{\widetilde{\mathcal{A}}}\mathcal{A} \otimes (\mathcal{T}^{\widetilde{\mathcal{A}}}\mathcal{A})^*$
  $$S = (\mathcal{X}^\alpha - y^\alpha \mathcal{X}^0) \otimes \mathcal{V}_\alpha$$
- $\xi \in \Gamma(\mathcal{T}^{\widetilde{\mathcal{A}}}\mathcal{A})$ is a *second order differential equation* (SODE) on $\mathcal{A}$ if
  $$\phi_0(\xi) = 1 \text{ and } S\xi = 0$$

- $L : \mathcal{A} \to \mathbb{R}$ Lagrangian function

- *the Poincaré-Cartan 1-section and 2-section*

$$\Theta_L = L\phi_0 + (d^{\mathcal{T}^{\tilde{A}}\mathcal{A}} L) \circ S \in \Gamma((\mathcal{T}^{\tilde{A}}\mathcal{A})^*)$$
$$\Omega_L = -d^{\mathcal{T}^{\tilde{A}}\mathcal{A}} \Theta_L \in \Gamma(\wedge^2 (\mathcal{T}^{\tilde{A}}\mathcal{A})^*)$$

- $\gamma : I \subseteq \mathbb{R} \to \mathcal{A}$ is *a solution of the Euler-Lagrange equations* iff

  i) γ is admissible (i.e. $(i_{\mathcal{A}}(\gamma(t)), \dot{\gamma}(t)) \in \mathcal{T}^{\tilde{A}}_{\gamma(t)}\mathcal{A}, \forall t$)

  ii) $i_{(i_{\mathcal{A}}(\gamma(t)), \dot{\gamma}(t))} \Omega_L(\gamma(t)) = 0$

or locally iff $\gamma(t) = (x^i(t), y^\alpha(t))$ and

$$\frac{dx^i}{dt} = \rho_0^i + \rho_\alpha^i y^\alpha \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial y^\alpha}\right) = \rho_\alpha^i \frac{\partial L}{\partial x^i} + (C_{0\alpha}^\gamma + C_{\beta\alpha}^\gamma y^\beta) \frac{\partial L}{\partial y^\gamma}$$

Example (The 1-jet bundle of a fibration)

Particular case: $\mathcal{A} = J^1\tau \quad \tau : Q \to \mathbb{R}$

$$\Downarrow$$

$$\alpha = i \qquad y^\alpha = \dot{x}^i$$

$$\rho^i_0 = 0 \quad \rho^i_j = \delta^i_j \quad C^\gamma_{0\alpha} = C^\gamma_{\alpha\beta} = 0$$

Classical non-autonomous Euler-Lagrange equations

$$\frac{dx^i}{dt} = \dot{x}^i \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = \frac{\partial L}{\partial x^i}$$

- L is *regular* iff the matrix $(W_{\alpha\beta}) = \left(\frac{\partial^2 L}{\partial y^\alpha \partial y^\beta}\right)$ is regular or,

equivalently, (Ω_L, ϕ_0) is a cosymplectic structure on $\mathcal{T}^{\tilde{A}}\mathcal{A}$, i.e.,

$$\{\phi_0 \wedge \Omega_L \wedge \ldots^{(n} \cdots \wedge \Omega_L\}(a) \neq 0, \quad \text{for all} \quad a \in \mathcal{A}$$
$$d^{\mathcal{T}^{\tilde{A}}\mathcal{A}}\phi_0 = 0 \qquad d^{\mathcal{T}^{\tilde{A}}\mathcal{A}}\Omega_L = 0$$

$\triangleright$ If L is regular

$\Downarrow$

the Reeb section of (Ω_L, ϕ_0), R_L, is the unique Lagrangian SODE associated with L:

$$i_{R_L}\Omega_L = 0 \text{ and } i_{R_L}\phi_0 = 1$$

$\Downarrow$

the integral curves of the vector field $\rho^{\mathcal{T}\mathcal{A}}_{\tilde{\mathcal{A}}}(R_L)$ are the solutions of the Euler-Lagrange equations associated with L

Vakonomic Mechanics on Lie affgebroids

$\tau_{\mathcal{A}} : \mathcal{A} \to Q$ Lie affgebroid of rank n, dim $Q = m$

$L : \mathcal{A} \to \mathbb{R}$ Lagrangian function on $\mathcal{A}$

$\mathcal{M} \subset \mathcal{A}$ embedded submanifold, *the constraint submanifold*

$\tau_{\mathcal{M}} = \tau_{\mathcal{A}|\mathcal{M}} : \mathcal{M} \to Q$ surjective submersion

dim $\mathcal{M} = n + m - \bar{m}$

$$\mathcal{A}^+ \xleftarrow{\ \mathrm{pr}_1\ } \mathcal{A}^+ \oplus \mathcal{A} \xrightarrow{\ \mathrm{pr}_2\ } \mathcal{A}$$

$$\mathcal{A}^+ \xleftarrow{\ \pi_1\ } W_0 = pr_2^{-1}(\mathcal{M}) = \mathcal{A}^+ \times_Q \mathcal{M} \xrightarrow{\ \pi_2\ } \mathcal{M}$$

$\pi_1 = (\mathrm{pr}_1)_{|W_0}$ $\pi_2 = (\mathrm{pr}_2)_{|W_0}$

$\nu : W_0 = \mathcal{A}^+ \times_Q \mathcal{M} \to Q$ the canonical projection

$$\begin{array}{ccc} \mathcal{T}^{\tilde{A}}W_0 & \xrightarrow{\mathcal{T}\pi_1} & \mathcal{T}^{\tilde{A}}\mathcal{A}^+ \\ \downarrow & & \downarrow \\ W_0 & \xrightarrow{\pi_1} & \mathcal{A}^+ \end{array}$$

$(\mathcal{T}\pi_1, \pi_1)$ Lie algebroid morphism

$\mathcal{T}\pi_1 = (Id, T\pi_1)$

$\Omega = (\mathcal{T}\pi_1, \pi_1)^*\Omega_{\tilde{\mathcal{A}}}$ is a presymplectic section on $\mathcal{T}^{\tilde{A}}W_0$

$\Omega_{\tilde{\mathcal{A}}}$ being the canonical symplectic section on $\mathcal{T}^{\tilde{A}}\mathcal{A}^+$

$\eta : \mathcal{T}^{\tilde{A}}W_0 \to \mathbb{R} \qquad \Rightarrow \eta \in \Gamma((\mathcal{T}^{\tilde{A}}W_0)^*)$

$\eta(\tilde{a}, X) = 1_{\mathcal{A}}(\tilde{a})$

$1_{\mathcal{A}}$ is a 1-cocycle $\Rightarrow \eta$ is a 1-cocycle

- The Pontryagin Hamiltonian $\quad H_0 : W_0 = \mathcal{A}^+ \times_Q \mathcal{M} \to \mathbb{R}$

$$H_0(\varphi, a) = \varphi(a) - \tilde{L}(a), \quad \tilde{L} = L_{|\mathcal{M}}$$

$$\Downarrow$$

$$\Downarrow$$

$$\Omega_0 = \Omega + d^{\mathcal{T}^{\widetilde{A}}W_0} H_0 \wedge \eta \text{ is a presymplectic section on } \mathcal{T}^{\widetilde{A}}W_0$$

$$\Downarrow$$

$(\mathcal{T}^{\widetilde{A}}W_0, \Omega_0, \eta)$ is a precosymplectic system

(i.e., $d^{\mathcal{T}^{\widetilde{A}}W_0}\Omega_0 = 0$ and $d^{\mathcal{T}^{\widetilde{A}}W_0}\eta = 0$)

Definition

The *vakonomic problem on Lie affgebroids* is find the solutions for the equations

$$i_X\Omega_0 = 0 \text{ and } i_X\eta = 1, \; X \in \Gamma(\mathcal{T}^{\widetilde{A}}W_0)$$

- (x^i) local coordinates on Q
  $\{e_0, e_\alpha\}$ local basis of sections of $\widetilde{\mathcal{A}}$ adapted to $1_{\mathcal{A}}$
  (x^i, y^0, y^α) local coordinates on $\widetilde{\mathcal{A}}$

$$\mathcal{M} \equiv \{(x^i, y^\alpha) \mid y^B = \Psi^B(x^i, y^b),\ B = 1, \ldots, \bar{m}\}$$

$y^\alpha = (y^B, y^b)$ $1 \leq \alpha \leq n$, $1 \leq B \leq \bar{m}$, $\bar{m} + 1 \leq b \leq n$

$$\Downarrow$$

(x^i, y^b) are local coordinates on $\mathcal{M}$

$(x^i, y_0, y_\alpha, y^b)$ local coordinates for $W_0 = \mathcal{A}^+ \times_Q \mathcal{M}$

$\{\mathcal{Y}_0, \mathcal{Y}_\alpha, \mathcal{P}^0, \mathcal{P}^\alpha, \mathcal{V}_b\}$ a local basis of sections of $\mathcal{T}^{\widetilde{\mathcal{A}}} W_0$:

$$\mathcal{Y}_0(\varphi, \mathrm{a}) = \left(e_0(x), \rho_0^i \frac{\partial}{\partial x^i}_{|\varphi}, 0\right), \quad \mathcal{Y}_\alpha(\varphi, \mathrm{a}) = \left(e_\alpha(x), \rho_\alpha^i \frac{\partial}{\partial x^i}_{|\varphi}, 0\right)$$

$$\mathcal{P}^0(\varphi, \mathrm{a}) = \left(0, \frac{\partial}{\partial y_0}_{|\varphi}, 0\right), \qquad \mathcal{P}^\alpha(\varphi, \mathrm{a}) = \left(0, \frac{\partial}{\partial y_\alpha}_{|\varphi}, 0\right)$$

$$\mathcal{V}_b(\varphi, \mathrm{a}) = \left(0, 0, \frac{\partial}{\partial y^b}_{|\mathrm{a}}\right), \quad \text{where } (\varphi, \mathrm{a}) \in W_0 \text{ and } \nu(\varphi, \mathrm{a}) = x$$

▷ The above equations only have sense in the points of the submanifold W_1 of W_0 satisfying the equations

$$y_b = \frac{\partial \tilde{L}}{\partial y^b} - y_B \frac{\partial \Psi^B}{\partial y^b}, \quad \bar{m}+1 \leq b \leq n$$

A solution of the vakonomic problem is of the form

$$X_{(\Upsilon_0,\Upsilon^b)} = \mathcal{Y}_0 + \Psi^B \mathcal{Y}_B + y^b \mathcal{Y}_b + \Upsilon_0 \mathcal{P}^0 + \Big[\rho^i_\alpha \Big(\frac{\partial \tilde{L}}{\partial x^i} - y_D \frac{\partial \Psi^D}{\partial x^i}\Big) + y_\gamma (C^\gamma_{0\alpha} + \psi^D C^\gamma_{D\alpha} + y^d C^\gamma_{d\alpha})\Big] \mathcal{P}^\alpha + \Upsilon^b \mathcal{V}_b$$

Therefore, the vakonomic equations are

$$\left\{ \begin{array}{l} \dot{x}^i = \rho^i_0 + \Psi^B \rho^i_B + y^b \rho^i_b \\ \dot{y}_B = \Big(\frac{\partial \tilde{L}}{\partial x^i} - y_D \frac{\partial \Psi^D}{\partial x^i}\Big)\rho^i_B - y_\gamma (C^\gamma_{B0} + \Psi^D C^\gamma_{BD} + y^b C^\gamma_{Bb}) \\ \frac{d}{dt}\Big(\frac{\partial \tilde{L}}{\partial y^b} - y_B \frac{\partial \Psi^B}{\partial y^b}\Big) = \Big(\frac{\partial \tilde{L}}{\partial x^i} - y_B \frac{\partial \Psi^B}{\partial x^i}\Big)\rho^i_b - y_\gamma (C^\gamma_{b0} + \Psi^D C^\gamma_{bD} + y^d C^\gamma_{bd}) \end{array} \right.$$

Example (The 1-jet bundle of a fibration)

Particular case: $\mathcal{A} = J^1\tau \quad \tau : Q \to \mathbb{R}$

$$\Downarrow$$

The vakonomic equations associated with a constrained system $(L, \mathcal{M})$ on $J^1\tau$ are

$$\left\{ \begin{array}{l} \dot{p}_B = \dfrac{\partial \tilde{L}}{\partial q^B} - p_D \dfrac{\partial \Psi^D}{\partial q^B} \\ \dfrac{d}{dt}\Big(\dfrac{\partial \tilde{L}}{\partial \dot{q}^b} - p_B \dfrac{\partial \Psi^B}{\partial \dot{q}^b}\Big) = \dfrac{\partial \tilde{L}}{\partial q^b} - p_B \dfrac{\partial \Psi^B}{\partial q^b} \\ \dot{q}^B = \Psi^B(t, q^i, \dot{q}^b) \end{array} \right.$$

▷ We discuss the regularity of the vakonomic system $(L, \mathcal{M})$
Denote $H_1 = (H_0)_{|W_1} : W_1 \to \mathbb{R}$

$$W_1' = \{w \in W_1 \mid H_1(w) = 0\}$$

$\nu_1' : W_1' \to Q$ the restriction of $\nu : W_0 \to Q$ to W_1'
$\tau_{\widetilde{\mathcal{A}}}^{\nu_1'} : \mathcal{T}^{\widetilde{\mathcal{A}}} W_1' \to W_1'$ the prolongation of $\widetilde{\mathcal{A}}$ over ν_1'
Ω_1' (resp., η_1') the restriction of Ω_0 (resp., η) to $\mathcal{T}^{\widetilde{\mathcal{A}}} W_1'$

Theorem

If (Ω_1', η_1') is a cosymplectic structure on $\mathcal{T}^{\widetilde{\mathcal{A}}} W_1'$, then there exists a unique section $\zeta_1 \in \Gamma(\mathcal{T}^{\widetilde{\mathcal{A}}} W_1')$ whose integral curves are solutions of the vakonomic equations. In fact, ζ_1 is the Reeb section of the cosymplectic structure (Ω_1', η_1'), that is,

$$i_{\zeta_1} \Omega_1' = 0 \quad \text{and} \quad i_{\zeta_1} \eta_1' = 1$$

Definition

The vakonomic system $(L, \mathcal{M})$ on the Lie affgebroid $\mathcal{A}$ is said to be *regular* if the pair (Ω_1', η_1') is a cosymplectic structure on the Lie algebroid $\mathcal{T}^{\widetilde{A}} W_1' \to W_1'$.

Proposition

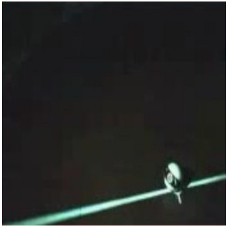
(Ω_1', η_1') is a cosymplectic structure on $\mathcal{T}^{\widetilde{A}} W_1'$ iff for all system of coordinates $(x^i, y_0, y_\alpha, y^b)$ on W_0 we have that

$$\det \left(\frac{\partial^2 \tilde{L}}{\partial y^b \partial y^c} - y_B \frac{\partial^2 \Psi^B}{\partial y^b \partial y^c} \right) \neq 0, \quad \text{for all point in } W_1'$$

- The vakonomic bracket associated with a regular vakonomic system on a Lie affgebroid

- Optimal control systems on a Lie affgebroid as a vakonomic systems on a Lie affgebroid

- Example: An homogeneous rolling ball without sliding on a rotating table with time-dependent angular velocity



$$\tau_{\widetilde{\mathcal{A}}} = \pi_{\mathbb{R}^3} \circ \mathrm{pr}_1 : \widetilde{\mathcal{A}} = T\mathbb{R}^3 \times \mathbb{R}^3 \to Q = \mathbb{R}^3$$

$$\tau_{\widetilde{\mathcal{A}}}(t, x, y; \dot{t}, \dot{x}, \dot{y}, \omega_x, \omega_y, \omega_z) = (t, x, y)$$

Lie algebroid

$$\{e_0 = \frac{\partial}{\partial t}, e_1 = \frac{\partial}{\partial x}, e_2 = \frac{\partial}{\partial y},$$

$$e_3 = (1, 0, 0), e_4 = (0, 1, 0), e_5 = (0, 0, 1)\}$$

local basis of sections of $\widetilde{\mathcal{A}} = T\mathbb{R}^3 \times \mathbb{R}^3$

(t, x, y) standard coordinates on Q

$(t, x, y; \dot{t}, \dot{x}, \dot{y}, \omega_x, \omega_y, \omega_z)$ induced coordinates on $\widetilde{\mathcal{A}}$

$\phi : T\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ 1-cocycle, $\phi(t, x, y; \dot{t}, \dot{x}, \dot{y}, \omega_x, \omega_y, \omega_z) = \dot{t}$

$$\Downarrow$$

$\tau_{\mathcal{A}} : \mathcal{A} = \phi^{-1}\{1\} \equiv \mathbb{R} \times T\mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R}^3$ Atiyah-Lie affgebroid modelled over $\tau_V : V = \phi^{-1}\{0\} \equiv \mathbb{R} \times T\mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R}^3$

The kinetic energy is $K : \mathcal{A} \to \mathbb{R}$

$K(t, x, y; \dot{x}, \dot{y}, \omega_x, \omega_y, \omega_z) = \frac{1}{2}(m\dot{x}^2 + m\dot{y}^2 + mk^2(\omega_x^2 + \omega_y^2 + \omega_z^2))$

$(\omega_x, \omega_y, \omega_z) \equiv$ components of the angular velocity of the ball

Constraints: $\dot{x} - r\omega_y = -\Omega(t)y \quad \dot{y} + r\omega_x = \Omega(t)x$

which define an affine subbundle $\mathcal{B}$ of $\mathcal{A}$

Assume full control over the motion of the center of the ball and consider the cost function

$$L(t, x, y; \dot{x}, \dot{y}, \omega_x, \omega_y, \omega_z) = \frac{1}{2}\left((\dot{x})^2 + (\dot{y})^2\right)$$

and the following optimal control problem: *Given points $q_0, q_1 \in Q$, find an optimal control curve $(t, x(t), y(t))$ on the reduced space that steer the system from q_0 to q_1, minimizes $\int_0^1 \frac{1}{2}\left((\dot{x})^2 + (\dot{y})^2\right)\, dt$, subject to the constraints defined above joint with $\omega_z = cte$*

A necessary condition for optimality of this problem is given for the corresponding vakonomic equations.

Denote $y^1 = \dot{x},\ y^2 = \dot{y},\ y^3 = \omega_x,\ y^4 = \omega_y,\ y^5 = \omega_z$

The vakonomic problem is given by the Lagrangian

$$L(t, x, y; y^1, y^2, y^3, y^4, y^5) = \frac{1}{2}((y^1)^2 + (y^2)^2)$$

and the constraint submanifold $\mathcal{M}$ is defined by

$$y^3 = \Psi^3(t, x, y, y^1, y^2) = \frac{1}{r}(-y^2 + \Omega(t)x)$$
$$y^4 = \Psi^4(t, x, y, y^1, y^2) = \frac{1}{r}(y^1 + \Omega(t)y)$$
$$y^5 = \Psi^5(t, x, y, y^1, y^2) = c$$

Then, the vakonomic equations are

$$\left\{\begin{array}{rcl} \dot{y}_3 &=& -\frac{1}{r}(y^1 + \Omega(t)y)y_5 + cy_4, \\ \dot{y}_4 &=& -\frac{1}{r}(y^2 - \Omega(t)x)y_5 - cy_3, \\ \dot{y}_5 &=& \frac{1}{r}(y^1 + \Omega(t)y)y_3 - \frac{1}{r}(-y^2 + \Omega(t)x)y_4, \\ \dfrac{d}{dt}(ry^1 - y_4) &=& -\Omega(t)y_3, \\ \dfrac{d}{dt}(ry^2 + y_3) &=& -\Omega(t)y_4, \\ y^1 = \dot{x}\,, & & y^2 = \dot{y} \end{array}\right.$$

THANKS FOR YOUR ATTENTION!!!!