

The geometry of nonholonomic mechanical systems

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The birth of the theory of dynamics of nonholonomic systems occurred at the time when the universal and brilliant analytical formalism created by Euler and Lagrange was found, to general amazement, to be inapplicable to the very simple mechanical problems of rigid bodies rolling without slipping on a plane. Lindelöf's error, detected by Chaplygin, became famous and rolling systems attracted the attention of many eminent scientist of the time...

Neimark and Fufaev, 1972

- What is nonholonomic mechanics?
- The geometry of nonholonomic systems
- Some future work

What is nonholonomic mechanics?

Nonholonomic constraints

$(q^A), 1 \leq A \leq n$ coordinates on a configuration space Q

Constraints $\phi_i(q^A, \dot{q}^A, t) = 0, 1 \leq i \leq m$

{
time-dependent (rheonomic)
time-independent (scleronomic)

What is nonholonomic mechanics?

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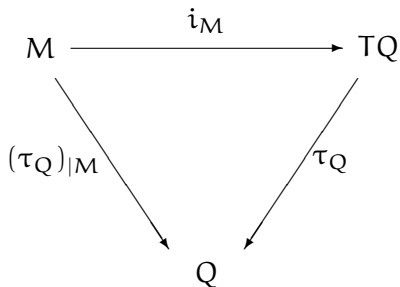
Holonomic or Nonholonomic

- 1 **Holonomic.** A holonomic constraint is derived from a constraint in configuration space.
 - **Example:** particle constrained to move on a sphere has the constraint
$$\sum_{A=1}^n (q^A)^2 = r^2, \quad \mathbf{q} \cdot \dot{\mathbf{q}} = 0.$$
- 2 **Semi-Holonomic.** The constraint is integrable. These constraints typically imply conservation laws given by a foliation of Q by integral manifolds.
 - **Example:** vertical disk rolling on a straight line without slipping. Constraint $\dot{\phi} = \dot{x}$, implies $\phi = x + \text{constant}$.
- 3 **Nonholonomic.** The constraint is not integrable. Cannot be reduced to semi-holonomic constraints and does not impose restrictions on the configuration space.
 - **Example:** Nonholonomic particle $\dot{\phi} = \dot{z} - y\dot{x} = 0$.



Geometrizing...

The constraints are globally described by a submanifold M of the velocity phase space TQ .



- Linear constraints** : M is a vector subbundle of TQ D
- Affine constraints** : M is an affine subbundle of TQ

Lagrange-d'Alembert principle



J. L. Lagrange (1736–1813)



Jean Le Rond d'Alembert (1717–1783)

$$\phi^i = \mu_{iA}^i(q) \dot{q}^A, 1 \leq i \leq m$$

Admissible infinitesimal virtual variation $\delta q^A \longrightarrow \mu_{iA} \delta q^A = 0$.

Definition

A nonholonomic constraint is said to be ideal if the infinitesimal work of the constraint force vanishes for any admissible infinitesimal virtual displacement.

Lagrange-D'Alembert's equations

$$L : TQ \rightarrow \mathbb{R}$$

(Q, L, D) a nonholonomic mechanical system

Lagrange-D'Alembert's equations

$$\left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^A} - \frac{\partial L}{\partial q^A} \right) \delta q^A = 0$$

with $\delta q \in D$.

$$\begin{aligned} \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^A} - \frac{\partial L}{\partial q^A} &= \lambda_i \mu_A^i \\ \mu_{iA}(q) \dot{q}^A &= 0 \end{aligned}$$

Nonlinear constraints. Chetaev's principle



P. Appell (1855–1930)



N.G. Chetaev (1902–1959)

P. Appell: Sur les liaisons exprimées par des relations non linéaires entre les vitesses, *C. R. Acad. Sci. Paris* 152 (1911), 1197–1200.

P. Appell: Exemple de mouvement d'un point assujéti à une liaison exprimée par une relation non linéaire entre les composantes de la vitesse, *Rend. Circ. mat. Palermo* 32 (1911), 48–50.

$$\phi^i(t, q, \dot{q}) = 0, \quad 1 \leq i \leq m$$

Some physical properties of the constraints should impose restrictions to the set of possible values of the constraint forces.

$$\begin{aligned} \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^A} - \frac{\partial L}{\partial q^A} &= \lambda_i \frac{\partial \phi^i}{\partial \dot{q}^A} \\ \phi^i(t, q, \dot{q}) &= 0 \end{aligned}$$

Simple examples show that Chetaev's rule cannot be used in general.

C.-M. Marle: Various approaches to nonholonomic systems. *Rep. Math. Phys.* 42 No 1/2 (1998), 211–229.

Cendra, Hernán; Ibort, Alberto; de León, Manuel; Martín de Diego, David A generalization of Chetaev's principle for a class of higher order non-holonomic constraints. *J. Math. Phys.* 45 (2004), no. 7, 2785–2801.

Cendra, H.; Grillo, S.: Generalized nonholonomic mechanics, servomechanisms and related brackets. *J. Math. Phys.* 47 (2006), no. 2, 022902, 29 pp.

and next talks....

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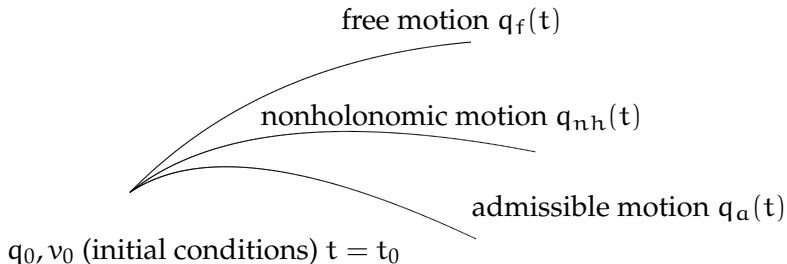
The principle of least constraint

“The motion of a system of particles connected together in any way, and whose motions are subject to arbitrary external restrictions , always takes place in the most complete agreement possible with free motion or under the weakest possible constraint. The measure of the constraint applied to the system at each elementary interval of time is the sum of products of the mass of each particle with the square of its departure from the free motion”



K. F. Gauss (1777–1855)

K. F. Gauss
Über ein neues Grundgesetz der Mechanik,
Journal de Crelle, Vol. IV (1829)



$$Z(\ddot{q}_a(t_0)) = \frac{1}{2} \frac{\partial^2 L}{\partial \dot{q}^A \partial \dot{q}^B} (\ddot{q}_a^A - \ddot{q}_f^A)(\ddot{q}_a^B - \ddot{q}_f^B)|_{t_0}$$

measure of deviations of motions

Gauss's principle of least constraint

Among admissible motions the one that deviates least from the free motion is the nonholonomic motion



O. Hölder (1859–1937)

$$\mathcal{C}^2(x, y) = \{c : [0, 1] \longrightarrow \mathbb{Q} \mid c \text{ is } C^2, c(0) = x, \text{ and } c(1) = y\}.$$

$$\mathcal{V}_c = \{X \in T_c \mathcal{C}^2(x, y) / X(t) \in D_{c(t)}, \forall t \in [0, 1]\}$$

Action functional

$$\begin{aligned} \mathcal{J} &: \mathcal{C}^2(x, y) \longrightarrow \mathbb{R} \\ c &\mapsto \int_0^1 L(\dot{c}(t)) dt \end{aligned}$$

Hölder's principle

An admissible path c ($\dot{c}(t) \in D_{c(t)}$), is a solution of the nonholonomic problem if

$$d\mathcal{J}(c)(X) = 0, \text{ for all } X \in \mathcal{V}_c$$

Geodesics in non-holonomic geometry.

Von

J. L. Synge in Dublin (Ireland).

Synopsis:

1. Introduction.
2. The constraint.
3. Equations of constrained geodesics derived from a variational principle.
4. Second form of the equations of constrained geodesics.
5. Parallel (Γ) propagation.
6. Geodesic stability for a general correspondence.
7. Isometric and normal correspondences.
8. Dynamical significance of the paper.

1. Introduction.

The stability of geodesics in Riemannian space was first discussed simultaneously and independently by Levi-Civita¹⁾ and myself²⁾. The tensorial equation obtained may be written

$$(1.1) \quad \bar{\eta}^r + G_{msn}^r x^s \eta^s x^n = 0,$$

where η^r is the infinitesimal vector joining a point on the fundamental geodesic to the corresponding point of a neighbouring geodesic, $x^r = \frac{dx^r}{ds}$, where x^r is the coordinate system, G_{msn}^r is the mixed curvature tensor

¹⁾ „Sur l'écart géodésique," *Math. Annalen* 97 (1928), p. 291—320. Cf. also Levi-Civita, „The Absolute Differential Calculus," English Translation (1927).

²⁾ „On the Geometry of Dynamics," *Phil. Trans. Roy. Soc., A*, 236 (1926), p. 81—106. That paper will be referred to as GD. The dynamical problem of „Stability in the action sense," discussed in GD., Chap. IX, is precisely the geometrical problem of the stability of geodesics in Riemannian space.



J. L. Synge: Geodesics in nonholonomic geometry. *Math. Annalen* 99 (1928), 738–751.

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F. Cantrijn, J. Cortés M. de León, D. Martín de Diego: On the geometry of generalized Chaplygin systems, *Mathematical Proceedings of the Cambridge Philosophical Society.* 132(2):323–351, 2002.

The nonholonomic connection

(Q, g) Riemannian manifold. ∇ Levi-Civita connection

$$\left\{ \begin{array}{l} L(v) = \frac{1}{2}g(v, v), v \in T_x Q \\ \mathcal{D} \end{array} \right. \Leftrightarrow \nabla_{\dot{c}(t)} \dot{c}(t) \in \mathcal{D}_{\dot{c}(t)}^\perp$$

$\dot{c}(t) \in \mathcal{D}_{c(t)}$, where \mathcal{D}^\perp is the g -orthogonal complement of \mathcal{D}

$$P: TQ \rightarrow \mathcal{D}$$

$$Q: TQ \rightarrow \mathcal{D}^\perp$$

Nonholonomic equations $\bar{\nabla}_X Y = \nabla_X Y + (\nabla_X Q)(Y)$

$$\bar{\nabla}_{\dot{c}(t)} \dot{c}(t) = 0$$

Proposition:

For all $Z \in X(Q)$ and $X, Y \in \mathcal{D}$ we have that

$$Z(g(X, Y)) = g(\bar{\nabla}_Z X, Y) + g(X, \bar{\nabla}_Z Y)$$

Reduction of Nonholonomic systems with Symmetries

Arch. Rational Mech. Anal. 118 (1992) 113–148. © Springer-Verlag 1992

Reduction of Some Classical Non-Holonomic Systems with Symmetry

JAIR KOELLER

Communicated by R. McGEHEE

Abstract

Two types of nonholonomic systems with symmetry are treated: (i) the configuration space is a total space of a G -principal bundle and the constraints are given by a connection; (ii) the configuration space is G itself and the constraints are given by left-invariant forms. The proofs are based on the method of quasicordinates. In passing, a derivation of the Maurer-Cartan equations for Lie groups is obtained. Simple examples are given to illustrate the algorithmical character of the main results.

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One of the interesting occurrences of symmetry in mechanics is the rolling of a solid body without slipping along a two dimensional surface (possibly of a complex profile). The results of this process are studied by the mechanics of nonholonomic systems Recently, deep and interesting connections of this subject with Lie groups were discovered

A. T. FOMENKO, Visual and hidden Symmetry in geometry, in *Comp. Math. Appl.*, 17, 1989.



Define the connection on Q/G :

$$\tilde{\nabla}_X Y = Z \quad \text{with} \quad Z = \rho_*(\bar{\nabla}_{X^h} Y^h)$$

$$\tilde{\nabla} \tilde{g} = 0$$

$\tilde{\nabla}_{\dot{c}(t)} \dot{c}(t) = 0 \longrightarrow c^h(t)$ solution of the nonholonomic problem.

An example: the two wheeled robot



$$\begin{aligned}\dot{x} \sin \theta - \dot{y} \cos \theta &= 0 \\ \dot{x} \cos \theta + \dot{y} \sin \theta + \omega \dot{\theta} + \rho \dot{\phi}_1 &= 0 \\ \dot{x} \cos \theta + \dot{y} \sin \theta - \omega \dot{\theta} + \rho \dot{\phi}_2 &= 0\end{aligned}$$

$$\begin{aligned}T &= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + m_0 l \dot{\theta} (\dot{y} \cos \theta - \dot{x} \sin \theta) \\ &\quad + \frac{1}{2} J \dot{\theta}^2 + \frac{1}{2} J_2 (\dot{\phi}_1^2 + \dot{\phi}_2^2)\end{aligned}$$

$$\begin{aligned}
g &= m dx \otimes dx + m dy \otimes dy + m_0 l \cos \theta dy \otimes d\theta + m_0 l \cos \theta d\theta \otimes dy \\
&\quad - m_0 l \sin \theta dx \otimes d\theta - m_0 l \sin \theta d\theta \otimes dx + J d\theta \otimes d\theta \\
&\quad + J_2 d\phi_1 \otimes d\phi_1 + J_2 d\phi_2 \otimes d\phi_2
\end{aligned}$$

Configuration space $Q = SE(2) \times T^2: (x, y, \theta, \phi_1, \phi_2)$

$$\begin{aligned}
\Upsilon_g(q) &= (a + x \cos \alpha - y \sin \alpha, b + x \sin \alpha + y \cos \alpha, \alpha + \theta, \phi_1, \phi_2) \\
\rho: Q &\rightarrow T^2
\end{aligned}$$

$$\left\{ \begin{aligned}
 \ddot{x} &= -\frac{m\rho^2 \sin(2\theta)}{2J_2 + m\rho^2} \dot{x}\dot{\theta} + \frac{m\rho^2 \cos(2\theta)}{2J_2 + m\rho^2} \dot{y}\dot{\theta} + \frac{l\rho^2 m_0 \cos \theta}{2J_2 + m\rho^2} \dot{\theta}^2 \\
 &\quad + \frac{J_2 \rho \sin \theta}{2J_2 + m\rho^2} \dot{\phi}_1 \dot{\theta} + \frac{J_2 \rho \sin \theta}{2J_2 + m\rho^2} \dot{\phi}_2 \dot{\theta} \\
 \ddot{y} &= \frac{m\rho^2 \cos(2\theta)}{2J_2 + m\rho^2} \dot{x}\dot{\theta} + \frac{m\rho^2 \sin(2\theta)}{2J_2 + m\rho^2} \dot{y}\dot{\theta} \\
 &\quad + \frac{l\rho^2 m_0 \sin \theta}{2J_2 + m\rho^2} \dot{\theta}^2 - \frac{J_2 \rho \cos \theta}{2J_2 + m\rho^2} \dot{\phi}_1 \dot{\theta} - \frac{J_2 \rho \cos \theta}{2J_2 + m\rho^2} \dot{\phi}_2 \dot{\theta} \\
 \ddot{\theta} &= -\frac{l\rho^2 m_0 \cos \theta}{2J_2 \omega^2 + j\rho^2} \dot{x}\dot{\theta} - \frac{l\rho^2 m_0 \sin \theta}{2J_2 \omega^2 + j\rho^2} \dot{y}\dot{\theta} \\
 \ddot{\phi}_1 &= \frac{\rho(l\omega(2J_2 + m\rho^2)m_0 \cos \theta + m(2J_2 \omega^2 + J\rho^2) \sin \theta)}{(2J_2 \omega^2 + j\rho^2)(2J_2 + m\rho^2)} \dot{x}\dot{\theta} \\
 &\quad + \frac{\rho(l\omega(2J_2 + m\rho^2)m_0 \sin \theta - m(2J_2 \omega^2 + J\rho^2) \cos \theta)}{(2J_2 \omega^2 + j\rho^2)(2J_2 + m\rho^2)} \dot{y}\dot{\theta} - \frac{l\rho m_0}{2J_2 + m\rho^2} \dot{\theta}^2 \\
 \ddot{\phi}_2 &= -\frac{\rho(l\omega(2J_2 + m\rho^2)m_0 \cos \theta - m(2J_2 \omega^2 + J\rho^2) \sin \theta)}{(2J_2 \omega^2 + j\rho^2)(2J_2 + m\rho^2)} \dot{x}\dot{\theta} \\
 &\quad - \frac{\rho(l\omega(2J_2 + m\rho^2)m_0 \sin \theta + m(2J_2 \omega^2 + J\rho^2) \cos \theta)}{(2J_2 \omega^2 + j\rho^2)(2J_2 + m\rho^2)} \dot{y}\dot{\theta} - \frac{l\rho m_0}{2J_2 + m\rho^2} \dot{\theta}^2
 \end{aligned} \right.$$

Geodesic equations on T^2 are:

$$\begin{cases} \ddot{\phi}_1 + \mathcal{K}_1(\dot{\phi}_1)^2 - (\mathcal{K}_1 + \mathcal{K}_2)\dot{\phi}_1\dot{\phi}_2 + \mathcal{K}_2(\dot{\phi}_2)^2 = 0 \\ \ddot{\phi}_2 + \mathcal{K}_2(\dot{\phi}_1)^2 - (\mathcal{K}_1 + \mathcal{K}_2)\dot{\phi}_1\dot{\phi}_2 + \mathcal{K}_1(\dot{\phi}_2)^2 = 0 \end{cases}$$

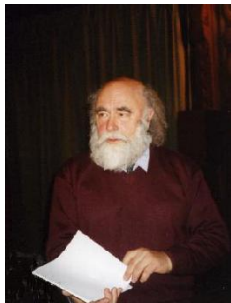
$$\dot{x} = -\frac{\rho \cos \theta}{2}(\dot{\phi}_1 + \dot{\phi}_2)$$

$$\dot{y} = -\frac{\rho \sin \theta}{2}(\dot{\phi}_1 + \dot{\phi}_2)$$

$$\dot{\theta} = -\frac{\rho}{2\omega}(\dot{\phi}_1 - \dot{\phi}_2)$$

Using the Differential Geometry on the tangent bundle

A. M. Vershik and L.D. Faddeev: Differential Geometry and Lagrangian mechanics with constraints. *Soviet Physics - Doklady* 17 (1) (1972), 34-36.



On the geometry of non-holonomic Lagrangian systems

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We present a geometric framework for non-holonomic Lagrangian systems in terms of distributions on the configuration manifold. If the constrained system is regular, an almost product structure on the phase space of velocities is constructed such that the constrained dynamics is obtained by projecting the free dynamics. If the constrained system is singular, we develop a constraint algorithm which is very similar to that developed by Dirac and Bergmann, and later globalized by Gotay and Nester. Special attention to the case of constrained systems given by connections is paid. In particular, we extend the results of Koiller for Chaplygin systems. An application to the so-called non-holonomic geometry is given. © 1996 American Institute of Physics. [S0022-2488/96/02407-3]

1. INTRODUCTION

A non-holonomic Lagrangian system consists of a regular Lagrangian $L(q^A, \dot{q}^A)$ defined on the phase space of velocities TQ of a configuration manifold Q with local coordinates $(q^A), 1 \leq A \leq n = \dim Q$, subjected to constraints defined by m local functions $\phi_i(q^A, \dot{q}^A)$. That means that the only allowable velocities are those verifying that $\phi_i = 0$. We only consider the case of linear constraints, say those of the form $\phi_i(q^A, \dot{q}^A) = (\mu_i)_A(q) \dot{q}^A$. By applying a suitable Hamilton's principle, we arrive to the constrained Euler-Lagrange equations,

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^A} \right) - \frac{\partial L}{\partial q^A} = -\lambda^i (\mu_i)_A,$$

where λ^i , $1 \leq i \leq m$, are some Lagrange multipliers to be determined (see, for instance, Valcovici,¹ Pars,² Neimark and Fufaev,³ Vershik and Faddeev,⁴ Saletan and Cromer,⁵ Rumiantsev,⁶ Pirronneau,⁷ Vershik and Gershkovich,⁸ Massa and Pagan^{23a)}). In some of them, a more general type of constraints was discussed. We notice that Hamilton's principle in the non-holonomic framework is not a variational principle. We remit to the excellent book by Rosenberg¹⁸ for a detailed discussion on that subject.

In the last years, there is an increasing interest in non-holonomic mechanics, and other approaches from a geometrical point of view have appeared: Weber,¹² Pitsaas,^{13,14} Marle,¹⁵ Massa and Pagan,¹⁶ Bates and Snitsycki,¹⁹ Giachetta,¹⁷ Koiller,¹⁸ Cariñena and Rañada,¹⁹ Rañada,²⁰ Dizord,²¹ Cariñena and Rañada,²² Sarlet, Cantrijn and Saunders,^{22,24} Sarlet,^{22,26} de León and M. de Diego.²⁷⁻³¹

Our approach is a globalization of the one by Cariñena and Rañada.¹⁹ In order to globalize their picture, we will consider a distribution D of codimension m defined on Q . The constraints

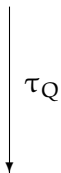
^{a)}Electronic mail: mdeleon@pirm1.csic.es

^{b)}Electronic mail: ccdm05@cc.csic.es

Unconstrained Lagrangian systems

TQ

TQ



Q

$z \in T_x Q$



x

(q^A, \dot{q}^A)



(q^A)

Vertical lift $z \in T_x Q$

$$\begin{aligned} T_x Q &\longrightarrow T_z TQ \\ X &\longmapsto (X^v)_z \quad (t \rightarrow z + tX) \\ \text{(in coordinates)} \quad X = X^A \frac{\partial}{\partial q^A} &\longmapsto X = X^A \frac{\partial}{\partial \dot{q}^A} \end{aligned}$$

Liouville vector field Δ

$$\Delta(z) = (z^v)_z \quad \Delta = \dot{q}^A \frac{\partial}{\partial \dot{q}^A}$$

Vertical endomorphism S

$$\begin{aligned} T_z TQ &\longrightarrow T_z TQ \\ Y &\longmapsto (T\tau_Q(z)(Y))^v \\ S(X^A \frac{\partial}{\partial q^A} + \tilde{X}^A \frac{\partial}{\partial \dot{q}^A}) &= X^A \frac{\partial}{\partial \dot{q}^A} \quad S = \frac{\partial}{\partial \dot{q}^A} \otimes dq^A \end{aligned}$$

$$L : TQ \longrightarrow \mathbb{R}$$

Poincaré-Cartan 1-form $\alpha_L = S^*(dL)$

Poincaré-Cartan 2-form $\omega_L = -d\alpha_L$

Energy function $E_L = \Delta L - L$

L is regular $\iff \left(\frac{\partial^2 L}{\partial \dot{q}^A \partial \dot{q}^B} \right)$ is regular

$i_{\Gamma_L} \omega_L = dE_L$ Γ_L Euler-Lagrange vector field

- 1 $S\Gamma_L = \Delta$ (Γ_L is a SODE)

$$\Gamma_L = \dot{q}^A \frac{\partial}{\partial q^A} + \Gamma^A \frac{\partial}{\partial \dot{q}^A}$$

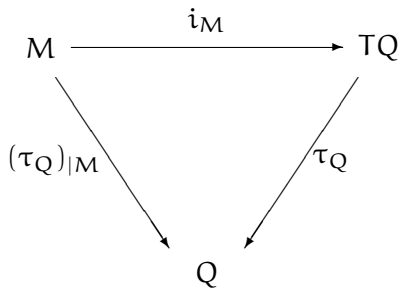
$$\left\{ \begin{array}{l} \frac{dq^A}{dt} = \dot{q}^A \\ \frac{d\dot{q}^A}{dt} = \Gamma^A(q^A, \dot{q}^A) \end{array} \right. \leftrightarrow \frac{d^2 q^A}{dt^2} = \Gamma^A(q^A, \frac{dq^A}{dt})$$

- 2 The solutions of Γ_L are the solutions of Euler-lagrange equations

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^A} \right) - \frac{\partial L}{\partial q^A} = 0$$

Euler-Lagrange equations

Nonholonomic constraints



M submanifold of TQ .

$$\phi^1 = 0, \dots, \phi^m = 0$$

Admissibility condition

$$\det \frac{\partial(\phi^1, \dots, \phi^m)}{\partial(\dot{q}_1, \dots, \dot{q}_m)} \neq 0 \iff \forall x \in M, \quad \dim T_x M^o = \dim S^*(T_x M^o)$$

$$\dot{q}^i = \Psi^i(q^A, \dot{q}^a), \quad 1 \leq i \leq m, m+1 \leq a \leq n \text{ and } 1 \leq A \leq n$$

Linear constraints on the velocities

$$\phi^i(q^A, \dot{q}^A) = \mu_A^i(q) \dot{q}^A$$

\mathcal{D} distribution on Q , $r = \dim Q - m$.

$$\mathcal{D}^\circ = \langle \mu_A^i dq^A, 1 \leq i \leq m \rangle$$

$L : TQ \longrightarrow \mathbb{R}$ regular, regular distribution \mathcal{D} on Q

$$\left\{ \begin{array}{l} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^A} \right) - \frac{\partial L}{\partial q^A} = \lambda_i \mu_A^i \\ \Phi^i(q, \dot{q}) = \mu_A^i \dot{q}^A = 0 \end{array} \right. \iff \left\{ \begin{array}{ll} (i_X \omega_L - dE_L)|_{\mathcal{M}} & \in S^*(TM^\circ) \\ X|_{\mathcal{M}} & \in TM \end{array} \right.$$

Compatibility condition

$$F^{\perp\omega_L} \cap TM = 0$$

F distribution along M with annihilator is $S^*(TM^0)$ and $F^{\perp\omega_L}$ its symplectic orthogonal

$$T_x TQ = F_x^{\perp\omega_L} \oplus T_x M, \quad x \in M$$

$$\mathcal{P}_x : T_x TQ \longrightarrow T_x M,$$

$$\mathcal{Q}_x : T_x TQ \longrightarrow F_x^{\perp\omega_L}.$$

$$\Gamma_{L,M} = \mathcal{P}(\Gamma_{L|M})$$

where

$$i_{\Gamma_L} \omega_L = dE_L$$

NONHOLONOMIC REDUCTION*

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(Received May 29, 1992 — Revised September 15, 1992)

The geometric structure of Hamiltonian systems with nonholonomic constraints, reproduced under reduction by symmetry, is described.

1. Introduction

The theory of classical mechanical systems with constraints is an old and venerable topic dating back into the last century. In this theory one may view the constraints as a differential system on configuration space, which may or may not be integrable. In the latter case the constraints are said to be nonholonomic. These constraints are linear restrictions on velocities, and naturally occur in such mechanical problems as a penny rolling without sliding on a plane.

For the sake of simplicity, we will only consider such constraints in this paper. There are numerous difficulties involved with nonlinear constraints, not the least of which is deciding what the correct mechanical principles are that give the equations of motion. These difficulties are discussed at length in the book of Neimark and Fufaev [7], and the article by Weber [12]. More importantly for mechanics, there seems to be known mechanical system that necessitates such a theory.

The plan of this paper is to describe the structure of Hamiltonian systems with constraints in a way that facilitates understanding the reduction of the problem in the presence of symmetry. We look only at Hamiltonian systems that come from a Lagrangian system of the form kinetic minus potential where the Legendre transformation is a diffeomorphism, and the Lagrangian and the constraints are independent of time. We do this for the sake of simplicity of exposition. As the generalization to time-dependent constraints, terms linear in the velocities in the Lagrangian, magnetic terms in the symplectic structure etc. involves no new ideas and is only notationally more complex, it is not done here.

The reduction of these systems by symmetry is formulated in such a way that it looks as much as possible like a nonholonomic version of the standard theory of reduction of symplectic manifolds with symmetry. This is one good reason to work with the Hamiltonian formalism, but there is another, more subtle reason, and this is that the constraints in the

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$$T_x TQ = (F_x \cap T_x M) \oplus (F_x \cap T_x M)^{\perp \omega_L} \quad x \in M$$

$$\bar{\mathcal{P}}_x : T_x TQ \longrightarrow (F_x \cap T_x M),$$

$$\bar{\mathcal{Q}}_x : T_x TQ \longrightarrow (F_x \cap T_x M)^{\perp \omega_L}.$$

$$\Gamma_{L,M} = \bar{\mathcal{P}}(\Gamma_{L|M})$$

$\mathcal{H} = F \cap TM$, symplectic distribution

A. Ibort, M. de León, J. C. Marrero and D. Martín de Diego:
Dirac brackets in constrained dynamics, *Fortschr. Phys.* **47**
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$$i_X(\omega_L)|_{\mathcal{H}} = d_{\mathcal{H}}E_L$$

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Reduction of the dynamics of nonholonomic systems

Arch. Rational Mech. Anal. 136 (1996) 21-99. © Springer-Verlag 1996

Nonholonomic Mechanical Systems with Symmetry

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Abstract

This work develops the geometry and dynamics of mechanical systems with nonholonomic constraints and symmetry from the perspective of Lagrangian mechanics and with a view to control-theoretical applications. The basic methodology is that of geometric mechanics applied to the Lagrange-d'Alembert formulation, generalizing the use of connections and momentum maps associated with a given symmetry group to this case. We begin by formulating the mechanics of nonholonomic systems using an Ehresmann connection to model the constraints, and show how the curvature of this connection enters into Lagrange's equations. Unlike the situation with standard configuration-space constraints, the presence of symmetries in the nonholonomic case may or may not lead to conservation laws. However, the momentum map determined by the symmetry group still satisfies a useful differential equation that decouples from the group variables. This momentum equation, which plays an important role in control problems, involves parallel transport operators and is computed explicitly in coordinates. An alternative description using



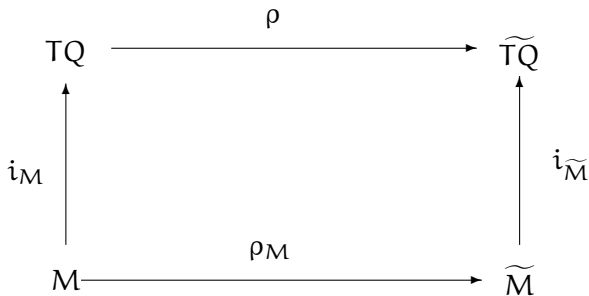
$$G \times Q \longrightarrow Q$$

$$\Phi : G \times TQ \longrightarrow TQ \quad \left\{ \begin{array}{l} L \text{ is } G\text{-invariant} \\ M \text{ is } G\text{-invariant} \end{array} \right.$$

$\alpha_L, \omega_L, E_L, \Gamma_L, F, \Gamma_{L,M}$ are G -invariant.

equivariant momentum map: $J : TQ \longrightarrow \mathfrak{g}^*$

$$\langle J(v_q), \xi \rangle = (\alpha_L(\xi_{TQ}))(v_q)$$



$$\mathcal{V} = \ker T\rho$$

$$\mathbf{u} = (F \cap TM) \cap (\mathcal{V} \cap F)^{\perp \omega_L}, \quad \text{vector subbundle of } TTQ|_M$$

$$\begin{array}{l}
 \mathbf{u} \longrightarrow \widetilde{\mathbf{u}} \\
 \omega_{\widetilde{\mathbf{u}}}, \quad \widetilde{dE} = (d(\widetilde{E}_L)_{\widetilde{M}})|_{\widetilde{\mathbf{u}}}
 \end{array}$$

Proposition

The projection $\widetilde{\Gamma}_{L,M}$ of $\Gamma_{L,M}$ onto \widetilde{M} is a section of \widetilde{U} satisfying the equation

$$i_{\widetilde{\Gamma}_{L,M}} \omega_{\widetilde{U}} = d\widetilde{E}$$

The momentum equation

$$\mathfrak{g}^q = \{\xi \in \mathfrak{g} / \xi_{TQ}(v_q) \in F_{v_q}, \text{ for all } v_q \in T_q Q \cap M\}$$

$$\mathfrak{g}^M = \cup_{q \in Q} \mathfrak{g}^q \longrightarrow Q$$

Nonholonomic momentum map

The nonholonomic momentum map is the mapping $J^{\text{nh}} : TQ \longrightarrow (\mathfrak{g}^M)^*$ defined by

$$\langle J^{\text{nh}}(v_q), \xi \rangle = \alpha_L(\xi_{TQ})(v_q)$$

A global section $\tilde{\xi}$ of the vector bundle $\mathfrak{g}^M \rightarrow Q$ induces a vector field Ξ on Q as follows

$$\Xi(q) = (\tilde{\xi}(q))_Q(q) \in T_q Q$$

Momentum equation

$$\Gamma_{L,M}(J_{\tilde{\xi}}^{nh}) = \Xi^c(L)$$

Classification of nonholonomic systems with symmetry

$$\left\{ \begin{array}{l} \text{Principal or purely kinetical case} : \{0\} = \mathcal{V}_M \cap \mathcal{F} \\ \text{Case of Horizontal symmetries} : \mathcal{V}_M \cap \mathcal{F} = \mathcal{V}_M \\ \text{general case} : \{0\} \subsetneq \mathcal{V}_x \cap \mathcal{F}_x \subsetneq \mathcal{V}_x \end{array} \right.$$

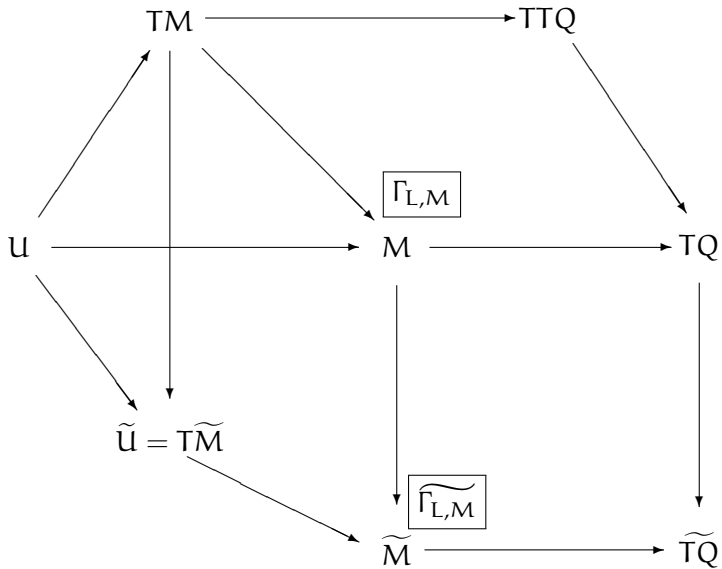
$$\{0\} = \mathcal{V}_M \cap F, \quad T_x M = \mathcal{V}_x + \mathcal{H}_x$$

$$TM = \mathcal{V}_M \oplus \mathcal{H}$$

$$U = \mathcal{H}, \quad \tilde{U} = T\tilde{M}$$

$$i_{\widetilde{\Gamma}_{L,M}} \widetilde{\omega}_L = d\widetilde{E}_L$$

$\widetilde{\omega}_L$ is an almost symplectic 2-form on \widetilde{M} .



Theorem

The reduced equation of motion can be written in the form

$$i_{\widetilde{\Gamma}_{L,M}} \widetilde{\omega} = d\widetilde{E}_L - \widetilde{\alpha}$$

where $\widetilde{\omega} = -d\widetilde{\alpha}_L$ is a symplectic form, $\widetilde{\alpha}_L$ is the projection on \widetilde{M} of $\mathbf{h}^*(i_M^* \alpha_L)$ and $\widetilde{\alpha}$ is the projection of $\alpha = i_{\Gamma_{L,M}}(\mathbf{h}^* d(i_M^* \alpha_L) - d\mathbf{h}^*(i_M^* \alpha_L))$. Moreover, we have

$$i_{\widetilde{\Gamma}_{L,M}} \widetilde{\alpha} = 0.$$

Čaplygin systems

By a Čaplygin system we mean a mechanical system given by a lagrangian function $L : TQ \longrightarrow \mathbb{R}$ with a configuration manifold Q which is a principal G -bundle, say $\pi : Q \longrightarrow Q/G$.

$$TQ = H \oplus \mathcal{W}$$

where \mathcal{W} denotes the vertical bundle.



S.A. Čaplygin (1869–1942)

$$L^* : T(Q/G) \longrightarrow \mathbb{R} \quad L^*(Y_{\bar{q}}) = L((Y_q)^H)$$

$$i_{\tilde{\Gamma}_{L,M}} \omega_{L^*} = dE_{L^*} - \tilde{\alpha}.$$

The case of horizontal symmetries

$$\mathcal{V}|_M \cap F = \mathcal{V}|_M$$

$$J^{nh} = J : TQ \longrightarrow \mathfrak{g}^*$$

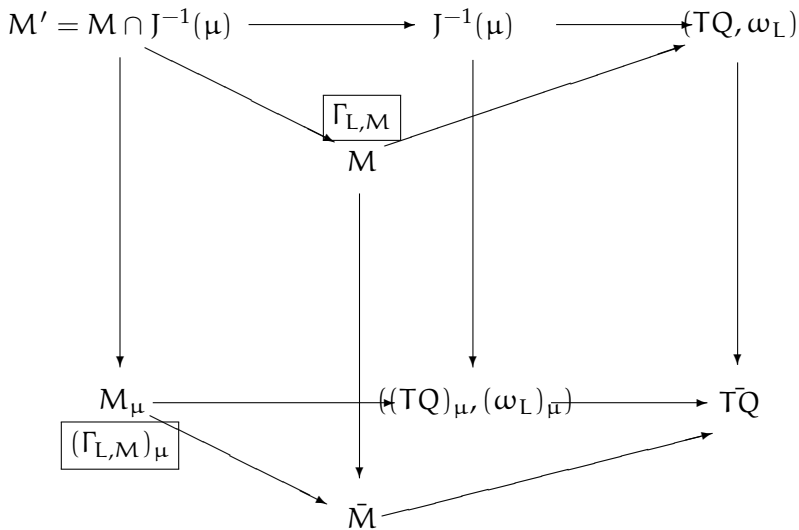
$$\Gamma_{L,M}(J\xi) = 0 \text{ for all } \xi \in \mathfrak{g}$$

Theorem

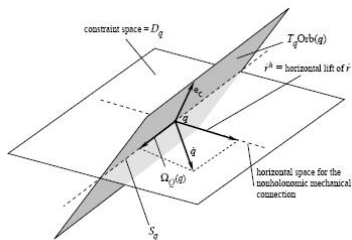
The projection $(\Gamma_{L,M})_\mu$ of the restriction of $\Gamma_{L,M}$ to M' is a solution of the reduced equations of motion

$$\begin{cases} \mathfrak{i}_{(\Gamma_{L,M})_\mu}(\omega_L)_\mu - d(E_L)_\mu \in F_\mu^o \\ (\Gamma_{L,M})_\mu \in TM_\mu \end{cases}$$

where $(E_L)_\mu$ is the reduced energy.



$$\{0\} \subsetneq \mathcal{V}_x \cap \mathcal{F}_x \subsetneq \mathcal{V}_x$$

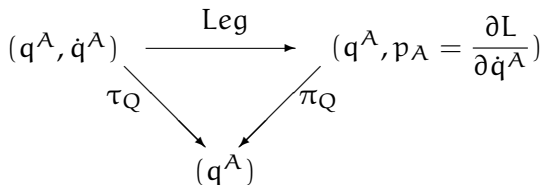
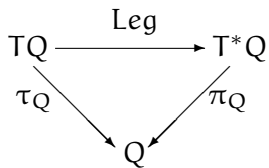


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J. Cortés, M. de León: Reduction and reconstruction of the dynamics of nonholonomic systems, *J. Physics A: Math. and Gen.* **32** (1999), 8615-8645.

Admissibility condition

$$\dim TM^o = \dim S^*(TM^o).$$



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$$\begin{array}{ccc}
 & \text{Leg} & \\
 E_L & \longrightarrow & H \\
 \omega_L & \longrightarrow & \omega_Q \\
 M & \longrightarrow & \bar{M} \\
 \phi^i & \longrightarrow & \psi^i(q^A, p_A) = \phi^i(q^A, \frac{\partial H}{\partial p_A}) \\
 S^*(TM^o) & \longrightarrow & F^o
 \end{array}$$

Compatibility condition

$$(\mathcal{H}^{AB}) = (\partial^2 H / \partial p_A \partial p_B)$$

$$F^\perp \cap T\bar{M} = \{0\} \iff (C^{ij}) = \left(\frac{\partial \psi^i}{\partial p_A} \mathcal{H}_{AB} \frac{\partial \psi^j}{\partial p_B} \right) \text{ is regular}$$

$$\begin{aligned} T\bar{M} \oplus F_{|\bar{M}}^\perp &= T_{\bar{M}}(T^*Q) \longrightarrow P: T_{\bar{M}}(T^*Q) \longrightarrow T\bar{M} \\ Q: T_{\bar{M}}(T^*Q) &\longrightarrow F^\perp. \end{aligned}$$

$$\mu^i = \frac{\partial \psi^i}{\partial p_A} \mathcal{H}_{AB} dq^B, \quad 1 \leq i \leq m$$

F^\perp is generated by $Z^i, 1 \leq i \leq m, i_{Z^i} \omega_Q = \mu^i$

$$Z^i = -\frac{\partial \psi^i}{\partial p_A} \mathcal{H}_{AB} \frac{\partial}{\partial p_B}.$$

$$P = \text{id} + C_{ij} Z^i \otimes d\psi^j \longrightarrow \boxed{P(X_H) = X_{H, \bar{M}}}$$

$$(F \cap T\bar{M}) \oplus (F \cap T\bar{M})^\perp = T_{\bar{M}}(T^*Q) \longrightarrow \begin{array}{l} \mathcal{P}: T_{\bar{M}}(T^*Q) \longrightarrow (F \cap T\bar{M}) \\ \mathcal{Q}: T_{\bar{M}}(T^*Q) \longrightarrow (F \cap T\bar{M})^\perp. \end{array}$$

$$\mathcal{P} = \text{id} - C_{ij} C_{i'j'} \{\psi^j, \psi^{j'}\} Z^i \otimes \mu^{i'} - C_{ij} X_{\psi^i} \otimes \mu^j + C_{ij} Z^i \otimes d\psi^j,$$

$$\text{If } \Delta|_M \in TM \longrightarrow \boxed{\mathcal{P}(X_H) = X_{H, \bar{M}}}.$$

ON THE HAMILTONIAN FORMULATION OF NONHOLONOMIC MECHANICAL SYSTEMS

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A simple procedure is provided to write the equations of motion of mechanical systems with constraints as Hamiltonian equations with respect to a "Poisson" bracket on the constrained state space, which does not necessarily satisfy the Jacobi identity. It is shown that the Jacobi identity is satisfied if and only if the constraints are holonomic.

1. Introduction

The theory of mechanical systems with nonholonomic constraints has a long history in classical mechanics; see e.g. the books by Neimark & Fufaev [14], Edelen [6], Rosenberg [16], Arnold [1] and the references quoted in there. In this literature, nonholonomic mechanical systems are described within the variational framework by Euler-Lagrange equations with extra terms corresponding to the constraint forces.

The present note is largely influenced by a recent paper of Bates & Śniatycki [4], see also Stanchenko [17], where it is shown that the dynamics of mechanical systems with nonholonomic constraints may be alternatively described within a Hamiltonian framework. However, the two-form with respect to which the Hamiltonian equations of motion (on a reduced state space, and without constraint forces) are defined is not necessarily closed, as may be demonstrated on simple examples. As a consequence, the resulting equations of motion, albeit of a Hamiltonian format, need not admit canonical coordinates and thus need not be transformable to the standard Hamiltonian equations: $\dot{q}_i = \frac{\partial H}{\partial p_i}$, $\dot{p}_i = -\frac{\partial H}{\partial q_i}$, $i = 1, \dots, n$. In the present note we will use, instead of the notion of a (not necessarily closed) two-form, the dual object of a "Poisson" bracket not necessarily satisfying the *Jacobi identity*. We will show in a



Alternative constructions of nonholonomic brackets

Let Λ be the bivector induced by ω_Q and η, ν two arbitrary sections of $T_{\bar{M}}T^*Q$

Bivector	bracket
$\Lambda_1(\eta, \nu) = \Lambda(\mathcal{P}^*(\eta), \mathcal{P}^*(\nu)) ,$	$\{f, g\}_1 = \Lambda_1(df, dg) .$
$\Lambda_2(\eta, \nu) = \Lambda(\mathcal{P}^*(\eta), \mathcal{P}^*(\nu)) = \omega_Q(\mathcal{P}(X_\eta), \mathcal{P}(X_\nu))$	$\{f, g\}_2 = \Lambda_2(df, dg) .$

Proposition

Along the constraint submanifold \bar{M} we have that

$$\Lambda_1 = \Lambda_2 \quad (\Lambda_{nh} \text{ y } \{ , \}_{nh}) .$$

Homogeneous case \longrightarrow $\dot{f} = X_{H, \bar{M}}(f) = \{f, H\}_{\text{nh}}$.

Almost Poisson bracket on the constraint submanifold

$$\Lambda_{\bar{M}}(x)(\eta_{\bar{M}}(x), \nu_{\bar{M}}(x)) = \Lambda_{\text{nh}}(x)(\eta(x), \nu(x))$$

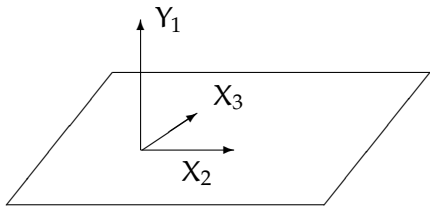
$$\{f, g\}_{\bar{M}} = \Lambda_{\bar{M}}(df, dg) = \Lambda_{\text{nh}}(dF, dG) = \{F, G\}_{\text{nh}}$$

Van der Shaft and Maschke construction

$$L = T - V, \quad \Phi^\alpha := \mu_\Lambda^\alpha(q) \dot{q}^\Lambda = 0, \quad (\Lambda = 1, \dots, m),$$

$$H = \frac{1}{2} g^{AB} p_A p_B + V(q) \quad \psi^i = \mu_\Lambda^i \frac{\partial H}{\partial p_\Lambda}, \quad 1 \leq i \leq m.$$

$$\begin{aligned} \tilde{p}_a &= X_a^A p_A, \quad m+1 \leq a \leq n \\ \tilde{p}_i &= Y_i^A p_A, \quad 1 \leq i \leq m \end{aligned}$$



$$\psi^i = C^{ij} \frac{\partial \tilde{H}}{\partial \tilde{p}_j} = \tilde{p}_i = 0.$$

$$\begin{aligned}
\{q^A, q^B\} &= 0 & \{q^A, \tilde{p}_a\} &= X_a^A & \{q^A, \tilde{p}_i\} &= Y_i^A \\
\{\tilde{p}_a, \tilde{p}_b\} &= X_b^A p_B \frac{\partial X_a^B}{\partial q^A} - X_a^A p_B \frac{\partial X_b^B}{\partial q^A} \\
\{\tilde{p}_i, \tilde{p}_b\} &= X_b^A p_B \frac{\partial Y_i^B}{\partial q^A} - Y_i^A p_B \frac{\partial X_b^B}{\partial q^A} \\
\{\tilde{p}_i, \tilde{p}_j\} &= Y_j^A p_B \frac{\partial Y_i^B}{\partial q^A} - Y_i^A p_B \frac{\partial Y_j^B}{\partial q^A}
\end{aligned}
\longrightarrow
\begin{pmatrix}
\{q^A, q^B\} & \{q^A, \tilde{p}_b\} & \{q^A, \tilde{p}_j\} \\
\{\tilde{p}_a, q^B\} & \{\tilde{p}_a, \tilde{p}_b\} & \{\tilde{p}_a, \tilde{p}_\beta\} \\
\{\tilde{p}_i, q^B\} & \{\tilde{p}_i, \tilde{p}_b\} & \{\tilde{p}_i, \tilde{p}_j\}
\end{pmatrix}$$

$$J_{\bar{M}}(q^i, \tilde{p}_a) = \begin{pmatrix} \{q^A, q^B\} & \{q^A, \tilde{p}_b\} \\ \{\tilde{p}_a, q^B\} & \{\tilde{p}_a, \tilde{p}_b\} \end{pmatrix}$$

$$\{q^A, q^B\}_{nh} = 0 \quad \{q^A, \tilde{p}_a\}_{nh} = X_a^A \quad \{q^A, \tilde{p}_i\}_{nh} = 0$$

$$\{\tilde{p}_a, \tilde{p}_b\}_{nh} = X_b^A p_B \frac{\partial X_a^B}{\partial q^A} - X_a^A p_B \frac{\partial X_b^B}{\partial q^A}, \quad \{\tilde{p}_i, \tilde{p}_b\}_{nh} = 0 \quad \{\tilde{p}_i, \tilde{p}_j\}_{nh} = 0.$$

$$\{f, g\}_{\bar{M}} = \left(\left(\frac{\partial f}{\partial q^A} \right)^T, \left(\frac{\partial f}{\partial \tilde{p}_a} \right)^T \right) J_{\bar{M}} \begin{pmatrix} \frac{\partial g}{\partial q^B} \\ \frac{\partial g}{\partial \tilde{p}_b} \end{pmatrix}$$

General case (no homogeneous)

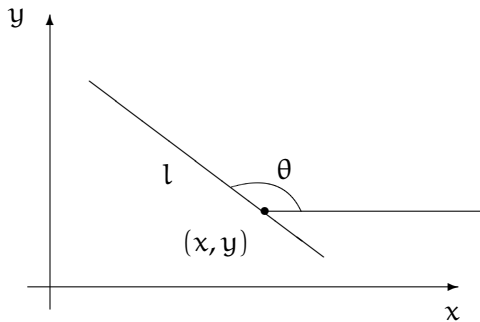
$$\dot{f} = X_{H, \bar{M}}(f) = R_H(f) + \{f, H\}_{n_h}$$

donde $R_H = P(X_H) - \mathcal{P}(X_H)$.

$$\begin{aligned}\dot{f} &= R_H(f) + \{f, H\}_{n_h} \\ &= \left(-C_{ij} Z^i(H) P(X_{\psi^j})(f) \right) + \left(\{f, H\} + C_{ij} Z^i(H) \{f, \psi^j\} \right. \\ &\quad \left. - C_{ij} Z^i(f) \{H, \psi^j\} + C_{ij} C_{i'j'} \{ \psi^j, \psi^{j'} \} Z^i(f) Z^{i'}(H) \right)\end{aligned}$$

$$\dot{f} = R_H(f) + \{f, H_{\bar{M}}\}_{\bar{M}}$$

Example: Chaplygin's skate



$$H = \frac{1}{2} (p_x^2 + p_y^2 + p_\theta^2)$$

$$\mu = d\theta + \sin \theta dx - \cos \theta dy$$

$$\Psi = p_\theta + p_x \sin \theta - p_y \cos \theta$$

Nonholonomic bracket:

$$\{x, y\}_{nh} = 0$$

$$\{x, \theta\}_{nh} = 0$$

$$\{y, \theta\}_{nh} = 0$$

$$\{x, p_x\}_{nh} = 1 - \frac{1}{2} \sin^2 \theta$$

$$\{x, p_y\}_{nh} = \frac{1}{2} \sin \theta \cos \theta$$

$$\{x, p_\theta\}_{nh} = -\frac{1}{2} \sin \theta$$

$$\{y, p_x\}_{nh} = -\frac{1}{2} \sin \theta \cos \theta$$

$$\{y, p_y\}_{nh} = 1 - \frac{1}{2} \cos^2 \theta$$

$$\{y, p_\theta\}_{nh} = \frac{1}{2} \cos \theta$$

$$\{y, p_x\}_{nh} = -\frac{1}{2} \sin \theta$$

$$\{y, p_y\}_{nh} = \frac{1}{2} \cos \theta$$

$$\{\theta, p_\theta\}_{nh} = \frac{1}{2}$$

$$\{p_x, p_y\}_{nh} = 0$$

$$\{p_x, p_\theta\}_{nh} = -\frac{1}{2} \sin \theta (p_x \cos \theta + p_y \sin \theta)$$

$$\{p_y, p_\theta\}_{nh} = \frac{1}{2} \cos \theta (p_x \cos \theta + p_y \sin \theta)$$

Eliminating p_θ with the constraint and quotienting by translations on x and y we obtain a reduced problem on $S^1 \times \mathbb{R}^2$ with coordinates (θ, p_x, p_y) with hamiltonian

$$h = \frac{1}{2} \left((1 + \sin^2 \theta) p_x^2 - 2p_x p_y \sin \theta \cos \theta + (1 + \cos^2 \theta) p_y^2 \right)$$

and bracket relations

$$\{\theta, p_x\} = -\frac{1}{2} \sin \theta, \quad \{\theta, p_y\} = \frac{1}{2} \cos \theta, \quad \{p_x, p_y\} = 0$$

$$\{\{\theta, p_x\}, p_y\} + \{\{p_x, p_y\}, \theta\} + \{\{p_y, \theta\}, p_x\} = -\frac{1}{4}$$

If we reduce by the entire group of Euclidean motions, taking

$$p_l = p_x \cos \theta + p_y \sin \theta$$

we obtain

$$h = \frac{1}{2}(2p_\theta^2 + p_l^2), \quad \{p_l, p_\theta\} = \frac{1}{2}p_\theta$$

The equations of motion are

$$\begin{aligned}\dot{p}_l &= \{p_l, h\} = p_\theta^2 \\ \dot{p}_\theta &= \{p_\theta, h\} = -p_l p_\theta\end{aligned}$$

Construction of nonholonomic integrators

- $L_d : Q \times Q \rightarrow \mathbb{R}$ a *regular discrete Lagrangian*
- *The constraint distribution* \mathcal{D}
- *The discrete constraint embedded submanifold* \mathcal{M}_c
 $i_{\mathcal{M}_c} : \mathcal{M}_c \rightarrow Q \times Q$ is an embedded submanifold of $Q \times Q$

Assumption

$$\dim \mathcal{M}_c = \dim \mathcal{D}$$

$(L_d, \mathcal{M}_c, \mathcal{D}) \equiv$ *a discrete nonholonomic Lagrangian system on* $Q \times Q$

$$(q_0, q_1) \in \mathcal{M}_c$$

$((q_0, q_1), (q_1, q_2))$ is a solution



$$(q_1, q_2) \in \mathcal{M}_c$$

$$D_2 L_d(q_0, q_1) + D_1 L_d(q_1, q_2) = \lambda_\alpha \mu^\alpha(q_1)$$

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- Singular reduction of nonholonomic systems Bates, Sniatycki...
- Integrability of nonholonomic systems Fedorov, Mamaev, Borisov, Kozlov, Zenkov...
- Nonholonomic field theory, Vankerschaver, de León, Cantrijn, DMdD, ...
- Control, Stabilization, Bloch, Zenkov, Leok, Marsden, Koon
- Non-constant rank constraints. Impulsive motion, Cortés, de León, DMdD, Martínez
- Geometric Integrators, Cortés, S. Martínez, Fedorov, Zenkov, McLachlan, Perlmuther, Iglesias, Marrero, DMdD, E. Martínez, Ferraro, Marsden, Kobilarov...
- Lie algebroids techniques, Cortés, de León, Marrero, DMdD, E. Martínez...