Base-controlled mechanical systems and geometric phases

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1. **Setting**
   - Setting
   - Structures
2. **Reconstruction phases**
3. **Conserved momentum case**
   - Unconstrained Deforming bodies
4. **Results for $D-$constraint systems**
   - In the phase space
   - Horizontal symmetries and phases
5. **Aplication**
   - A deformable body subject to internal $D-$constraints
   - Structure
   - Phases

Idea

Mechanical system (Q, L, G, D):

- Q *configuration space* and *principal* $G-$*bundle* $Q \xrightarrow{\pi} Q/G$
- $L : TQ \longrightarrow \mathbb{R}$ *lagrangian*

$$L(q, v) = \frac{1}{2}k_q(v, v) - V(q)$$

  $k_q(\cdot,\cdot)$ metric y V potential $G-$*invariant*
- $D \subseteq TQ$ $G-$invariant non-holonomic constraints.

Controlled base-degrees-of-freedom

If $c(t) \in Q$ gives the system's evolution

$$\pi(c(t)) = \tilde{c}(t)$$

with $\tilde{c}(t) \in Q/G$ a known curve, i.e., given.

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Example(s)

Deformable bodies

Motion of a body with variable shape $\tilde{c}(t)$ (from CM):

- Degrees of freedom corresponding to the shape $\in B$
- Rotation around the center of mass (CM)

Configuration space

$Q \overset{Loc}{\approx} B \times G$

B Shapes

$G = SO(3)$ Rotations

Fiber bundles

In general $Q \xrightarrow{\pi} B$ is a non-trivial principal $G-$bundle over B

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Fiber bundles: Example

from Δ to ∇ : rotation or deformation?

Example(s)

Deformable bodies with controlled shape

Motion of a body with given changing shape $\tilde{c}_0(t)$ (from CM):

- shape degrees of freedom $\in B$: Known
- Rotation around center of mass Unknown

Configuration space

$Q \stackrel{Loc}{\approx} B \times G$
$\tilde{c}_0(t) \in B$ Known
$G = SO(3)$ Unknown

Fiber bundles

Given a *gauge* (lift) $d_0(t) \in Q \xrightarrow{\pi} B \ni \tilde{c}_0(t)$

$$\pi(d_0(t)) = \tilde{c}_0(t)$$

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The problem

if $c(t)$ describes the physical motion in Q,

$$\boxed{c(t) = g(t) \cdot d_0(t)}$$

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Concrete situation (Ratiu)...

... if inside a satellite in orbit an astronaut re-accommodates the furniture (or an antena comes out), how is the rotational motion affected?

Kinematics and dynamics

Kinematics=Geometry

Principal case:

$$D_q + V_q = T_qQ$$

$\forall q \in Q$ with $Ver_q = Ker(\pi_{*q})$

- "There are no constraints on the base motion"

Example: Rotating body with zero angular momentum

$J(\dot{q}) = 0$ defines the *mechanical conection* in $Q \longrightarrow Q/SO(3)$

$$D_q \oplus V_q = T_qQ$$

The body deforms without constraints (purelly kinematical case).

Kinematics and dynamics

Kinematics=Geometry

Base control:

$$c(t) = g(t) \cdot d_0(t)$$

with $d_0(t)$ the gauge y $g(t) \in G$ to be determined.

- "We know the internal part of the system's evolution and want to solve for the vertical one"
- If $D_q \cap V_q \neq 0$ we need additional info about the system to be able to determine $g(t)$

Kinematics and dynamics

Kinematics=Geometry

Moment map $J : TQ \longrightarrow \mathfrak{g}^*$

$$\begin{aligned} J(\frac{d}{dt}c(t)) &= Ad^*_{g(t)} I_{d_0(t)}(g^{-1}\frac{d}{dt}g(t)) + Ad^*_{g(t)} J(\frac{d}{dt}d_0(t)) \\ &=: Ad^*_{g(t)} \Pi(t) \end{aligned}$$

with (generalized) inertia tensor $I_q : \mathfrak{g} \longrightarrow \mathfrak{g}^*$

$$I_q = \sigma_q^* \circ k_q \circ \sigma_q$$

- "J is not conserved quantity in general because of the $D-$constraints"

Cinemática y dinámica

Forces

$D-$forces $\longrightarrow$ D'Alambert's principle

Cinemática y dinámica

Forces

control forces $\longrightarrow$ *properly internal*.

$$F^c_{int}(\delta c) = 0$$

for all *vertical variation* $\delta c = \frac{d}{ds}\Big|_{s=0}(g(t,s)\cdot d_0(t)), \forall d_0(t)$.

Reconstructing dynamics

- $P \longrightarrow P/G$ principal $G-$ bundle
- $c(t) \in P$ ruled by some equation $\dot{c}(t) = X(c(t))$ with X $G-$invariant vector field
- $\tilde{c}(t) \in P/G$ the *reduced dynamics*

$$\begin{array}{ccc} c(t) \in & & P \\ & R \uparrow & \downarrow \pi \\ \tilde{c}(t) \in & & P/G \end{array}$$

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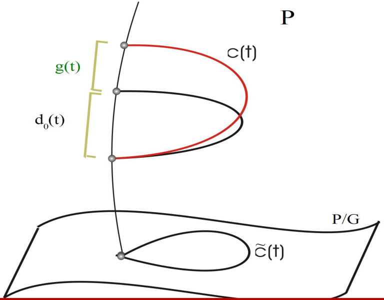
Reconstructing dynamics

- If we fix any *lift* $d_0(t)$ of $\tilde{c}(t)$,

$$c(t) = g(t) \cdot d_0(t)$$

with $g(t)$ to be determined by the equation for $c(t)$



Reconstructing dynamics

The geometric phase concept

If $d_0(t)$ is chosen geometrically w.r.t. $\tilde{c}$ (vg: horizontal lift) and $\tilde{c}(t)$ is closed

$$\tilde{c}(T) = \tilde{c}(0)$$

then

$$d_0(T) = g_G \cdot d_0(0)$$

where g_G is the corresponding THE GEOMETRIC PHASE The restant $g(T)$ is the corresponding DYNAMIC PHASE for the RECONSTRUCTION PHASE FORMULA:

$$c(T) = g(T) g_G \cdot d_0(0)$$

Reconstructing dynamics

The geometric phase

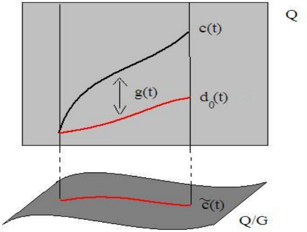
depends only on the *geometry* of the base curve $\tilde{c}(t)$ and not on the speed in which this is followed.

Intuitivelly...

"Is a memory that the system keeps of the path followed by $\tilde{c}(t)$"

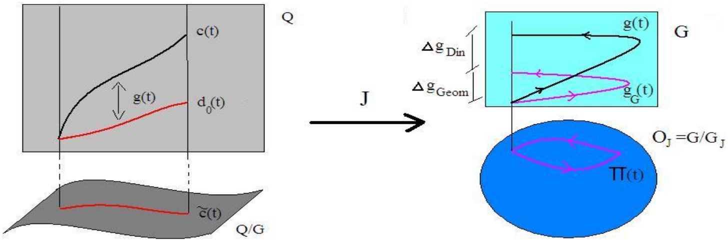
A tale about two reconstructions (Conserved momentum case)

$$
\begin{array}{ccccccccc}
c(t) \in & Q & \longleftarrow & TQ & & * & \hookleftarrow & * & \ni g(t) \\
& \uparrow \pi \;\; \downarrow & & \downarrow & \searrow * & * \downarrow & & \downarrow & \\
\tilde{c}(t) \in & Q/G & \longleftarrow & T(Q/G) & & * & \hookleftarrow & * & *
\end{array}
$$



A tale about two reconstructions (Conserved momentum case)

$$
\begin{array}{ccccccccc}
c(t) \in & Q & \longleftarrow & TQ & & T^*G & \hookleftarrow & L^{-1}(J) \approx G & \ni g(t) \\
(R) \uparrow & \downarrow \pi & & \downarrow & \searrow J & \downarrow L & & \downarrow \quad \uparrow (R) & \\
\tilde{c}(t) \in & Q/G & \longleftarrow & T(Q/G) & & \mathfrak{g}^* & \hookleftarrow & O_J = G/G_J \ni \Pi(t) \ (D) &
\end{array}
$$



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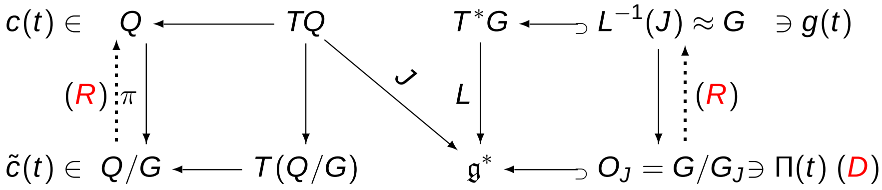

- The eqs of motion for $g(t)$ are the ones for conservation of J
- When $\Pi(T) = \Pi(0)$ we have reconstruction phase formula

$$g(T) = \Delta g_{Din} \Delta g_{Geom} g(0)$$

con $\Delta g_{Din} \in G_J$

A tale about two reconstructions (Conserved momentum case)

The phase formula

Since $\Delta g_{Din} \propto$ Energy, periods, physical inertia tensor.... When $\Pi(T) = \Pi(0)$,

$$c(T) = \Delta g_{Din} \Delta g_{Geom} g(0)\, d_0(T)$$

relating the system's configuration at T with:

- the geometry of the gauge curve d_0
- the geometry of $\Pi(t)$ (GEOMETRIC PHASE)
- system's mechanical magnitudes (DYNAMICAL PHASE)

A generalized Montgomery phase formula*

* A.C. *A generalized Montgomery phase formula for rotating self-deforming bodies.* J. Geom. Phys. 57 (2007) 1405-1420.

- Q deforming body configuration
- $d_0(t) \in Q$ known deforming shape
- $R(t) \in SO(3)$ induced rotation about CM to be determined
- $\Pi(t_2) = \Pi(t_1) \in S^2 \subset so(3) \approx \mathbb{R}^3$ the *body angular momentum*

$$
\begin{aligned}
R(t_2) &= exp(\theta_D \frac{\hat{L}}{\|L\|}) \cdot exp(\theta_G \frac{\hat{L}}{\|L\|}) \cdot R(t_1) \\
&= exp((\overbrace{\theta_D + \theta_G}^{\theta_M}) \frac{\hat{L}}{\|L\|}) \cdot R(t_1),
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$$\theta_M = (\mp)\frac{area(\tilde{D})}{\|L\|^2} + \frac{1}{\|L\|}\int_{t_1}^{t_2} dt\ 2T\left(\tfrac{d}{dt}(c)\right) - 2T\left(\tfrac{d}{dt}d_0\right)$$

A generalized Montgomery phase formula

Relation to quantum Berry phases

The $SO(3) \longrightarrow S^2$ bundle (and the conection) used to reconstruct the *deforming body's rotation about CM* from the body angular momentum is the same as the one underlying BERRY'S PHASE associated to the quantum system consisting of a particle with spin 1 in an external magnetic field that varies adiabatically and periodically.

A.C. *Some geometric features of Berry's phase.* arXiv:0705.2257v2 [math-ph].

Results I

RECONSTRUCTION IN Q:

Non-holonomic Gauge $d_0^{NH}(t)$:= horizontal lift of $\tilde{c}(t) \in Q/G$ in $Q \xrightarrow{\pi} Q/G$ w.r.t. the *non-holonomic conection* ([BKMM])

Geometric phase for purelly kinematic case (generalization of [SW])

When D defines a principal conection in $Q \xrightarrow{\pi} Q/G$, the solution

$$c(t) = d_0^{NH}(t)$$

is geometric w.r.t. $\tilde{c}(t) \in Q/G$. If $\tilde{c}(0) = \tilde{c}(T)$ then

$$c(T) = \Delta g_{Geom} c(0)$$

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Results II

Horizontal symmetries:

$\exists H \subset G$ subgroup compatible with D

1. (infinitesimal generators inside of D) $\xi_Q(q) \in D_q$ $\forall q \in Q$ when $\xi \in \mathfrak{h} := Lie(H) \subset \mathfrak{g}$,
2. (*completness*) $S_q := D_q \cap T_q\left(Orb_G(q)\right) = T_q\left(Orb_H(q)\right)$ $\forall q \in Q$.

Horizontal conservation

the projected momentum to $\mathfrak{h}^*$ is conserved:

$$\frac{d}{dt}(i_{\mathfrak{h}}^* J(\dot{c})) = 0.$$

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non-complete symmetries

If (2) is not satisfied, the $H-$conservations do not exhaust all of the vertical eqs in Q. One has to consider aditional eqs comming from the Lagrangian

Results II

Phases in $g(t)$ for $H-$simetries (completes)*

$$g(t) = h_D(t)\, g_G(t).$$

GEOMETRIC PHASE $g_G(t)$ = *horizontal lift*:

$$\begin{array}{ccc} g_G(t) \in & H \\ \uparrow & \downarrow \pi \\ i_{\mathfrak{h}}^* \Pi(t) \in & O^H_{i_{\mathfrak{h}}^* J(\dot{c})} \end{array}$$

from $g_G(0) = e$, with conection A_P in the bundle $H_{i_{\mathfrak{h}}^* J(\dot{c})}-$principal $H \xrightarrow{\pi} O^H_{i_{\mathfrak{h}}^* J(\dot{c})}$

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DYNAMIC PHASE $h_D(t) \in H_{i_{\mathfrak{h}}^* J(\dot{c})}$

$$\frac{d}{dt} h_D h_D^{-1}(t) = \left(2K(\frac{d}{dt}c(t)) - 2K_{int}(t)\right) \frac{\Psi(i_{\mathfrak{h}}^* J(\dot{c}))}{\left\|\Psi(i_{\mathfrak{h}}^* J(\dot{c}))\right\|^2} +$$

$$+\Sigma_{i=2}^{dim \mathfrak{h}_{i_{\mathfrak{h}}^* J(\dot{c})}} (u_i, (I_{c(t)}^{\mathfrak{h}})^{-1} (i_{\mathfrak{h}}^* J(\dot{c}))) \, u_i$$

K kinetic energy of the system, u_i o.n. base for invariat metric $(,)$ in $\mathfrak{h}$

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$$g(t) = h_D(t)\, g_G(t).$$

$$c(T) = h_D(T)\, g_G(T) \cdot d_0(T) = h_D(T)\, g_G(T) \cdot \Delta g_{GQ} \cdot d_0(0)$$

In the no-complete case, the reconstruction formula is similar but $i_{\mathfrak{h}}^* \Pi(t) \in O^H_{i_{\mathfrak{h}}^* J(\dot{c})}$ follows a different dynamics and the expression for the dynamic phase h_D involves another term (extra vertical variables)

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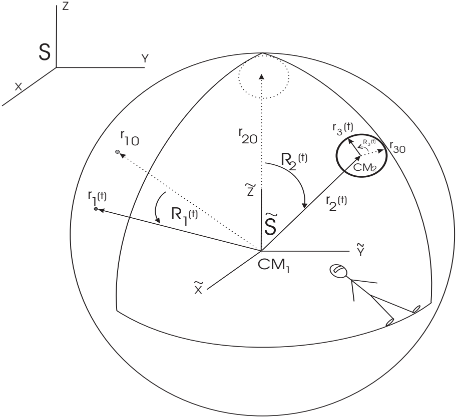
* A.C. *Base-controlled mechanical systems and geometric phases.* J. Geom.Phys. 58 (2008) 334-367.

The satellite and the robot OR the 2 balls

Idea

The small ball models a robot moving without sliding inside the bigger ball, which models an orbiting satellite.



Setting

- $Q = SO(3) \times S^2_r \times SO(3) \ni (R_1, r_2, R_3)$
- $L =$ kinetic energy
- $D-$ constraints= no-sliding $dimD = dimQ - 2 = 6$.
- $G = SO(3)^2 \ni (R, g_3)$

$$(R, g_3) \cdot (R_1, r_2, R_3) = (RR_1, Rr_2, RR_3 g_3^{-1}).$$

- controlled base variables $(Q/G = S^2) \Leftrightarrow$ gauge curve $d_0(t) = (e, r_{2,1}(t), e)$

meaning of d_0

is what the astronaut sees!

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meaning of d_0

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Las dos bolas: el planteo

- SOLUTION
  c(t)=(R$_1(t), R_2(t), R_3(t)) = (R_1(t), R_3^{-1}(t)R_1(t)) \cdot d_0(t) \in Q$
  $dimD - dim(Q/G) = 4$ variables to be determined.
- HORIZONTAL SYMMETRIES not-complete
  $H := \{(R, e),\ R \in SO(3)\} \subset G$
- VERTICAL EQS OF MOTION (4 ecs.):
  $H-$conservation = conservation of total angular momentum (3 ecs.)
  1 ec. vanishing of normal acceleration of small ball

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- HORIZONTAL SYMMETRIES not-complete
  $H := \{(R, e),\ R \in SO(3)\} \subset G$
- VERTICAL EQS OF MOTION (4 ecs.):
  $H-$conservation = conservation of total angular momentum (3 ecs.)
  1 ec. vanishing of normal acceleration of small ball

Results

Factorized solution:

$$c(t) = \left(R\ R_{1,NH}, R\ R_{1,NH}\ R_{2,1}, R\ R_{3,NH}\ g_3^{-1}\right)$$

Phase formula for global rotation $R(t)$

Once solved for the reduced dynamics

$$(i_{\mathfrak{h}}^* \Pi(t), \lambda(t)) \in H/H_{i_{\mathfrak{h}}^* J} \times \mathbb{R}$$

when $i_{\mathfrak{h}}^* \Pi(0) = i_{\mathfrak{h}}^* \Pi(T)$ $(i_{\mathfrak{h}}^* J \neq 0)$

$$R(T) = exp\left(\left(\theta^{Dyn}(T) + \theta^{Geom}\right) \frac{i_{\mathfrak{h}}^* J}{\left\| i_{\mathfrak{h}}^* J \right\|}\right) R(0)$$

Results

Factorized solution:

$$c(t) = \left(R\ R_{1,NH}, R\ R_{1,NH}\ R_{2,1}, R\ R_{3,NH}\ g_3^{-1} \right)$$

Phase formula for global rotation $R(t)$

GEOMETRIC PHASE θ^{Geom} solid angle (oriented) swept by $i_{\mathfrak{h}}^{*}\Pi(t) \in SO(3)/U(1) = S^2_{i_{\mathfrak{h}}^{*}J}$

Results

Factorized solution:

$$c(t) = \left(R\ R_{1,NH}, R\ R_{1,NH}\ R_{2,1}, R\ R_{3,NH}\ g_3^{-1}\right)$$

Phase formula for global rotation $R(t)$

DYNAMIC PHASE:

$$\theta^{Dyn}(t) = \theta_0^{Dyn} + \frac{1}{\left\| i_{\mathfrak{h}}^* J \right\|} \int_{t_1}^{t} ds\ [2K(\frac{d}{dt}c(s)) - 2K_{int}(s)$$

$$+\lambda(s)\left(\frac{2}{5}m_2 a^2\right)\left(\xi_z, \left(I_e^{\mathfrak{h}}\right)^{-1} Ad_{R_2}^{-1}(s) i_{\mathfrak{h}}^* J\right)_{\mathfrak{so}(3)} +$$

$$+\frac{\lambda(s)^2 \left(\frac{2}{5}m_2 a^2\right)^2}{\frac{2}{5}m_1(r+a)^2 + \frac{2}{5}m_2 a^2}]$$

To the audience...

THANK YOU!