Constraint algorithm for generalized Dirac structures

Introduction

Dirac structures are a unified framework for dealing with nonholonomic mechanical systems and LC electrical circuits (Yoshimura–Marsden 2006). They are maximal isotropic subbundles

$$D \subset TM \oplus T^*M$$

respect to

$$\langle (u,\alpha),(v,\beta)\rangle = \alpha v + \beta u$$

This isotropy condition implies that, through the solutions of equation

$$(x,\dot{x}) \oplus d\mathcal{E} \in D$$

the energy function $\mathcal{E}$ is constant.
We generalize Dirac structures, by taking out this condition, in order to include dissipative sytems. The geometric constraint algorithm for Dirac structures (Cendra–Etchechoury–Marsden 2008) is applied to these generalized structures with the same formalism

Definition

A generalized Dirac structure on a vector space V is any subspace $D \subset V \oplus V^*$. Define

- $D^\flat(v) = \{\alpha \in V^* \colon (v,\alpha) \in D\}$
- $E^D = \{v \in V \colon \alpha(v) = 0 \,\forall \alpha \in D^\flat(E)\} = (D^\flat(E))^\circ$

A generalized Dirac structure over a manifold M is a subbundle $D \subset TM \oplus T^*M$. If there exists a solution of

$$(x,\dot{x}) \oplus d\mathcal{E} \in D \qquad (1)$$

though the point x, then

$$d\mathcal{E}_x \in D^\flat(T_xM) = (T_xM^D)^\circ$$

Let

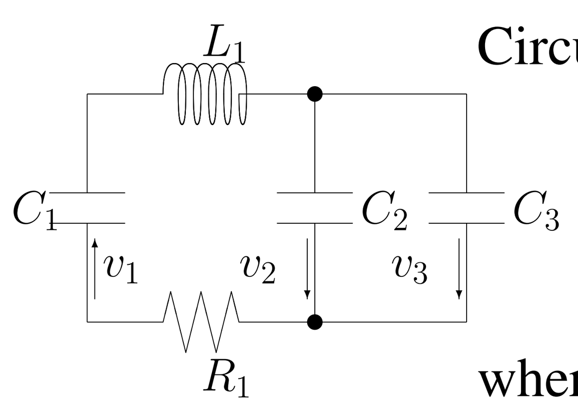
$$M_1 = \{x \in M \colon d\mathcal{E}_x \cdot T_xM = 0\}$$

Solutions of equation (1) stay inside M_1. With a similar argument, we define

$$M_{k+1} = \{x \in M_k \colon d\mathcal{E}_x \cdot T_xM_k = 0\}$$

Supposing that at each step of this algorithm we get a submanifold of the previous one, the algorithm must stop, yielding a submanifold M_c, such that all solutions are constrained to it.

Application to RLC circuits



Circuit evolution is determined by

$$\begin{cases} \dot{q} \in \Delta \\ L\ddot{q} + R\dot{q} + \frac{1}{C}q \in \Delta^\circ \end{cases}$$

where $v = \dot{q}$ represents the current at each branch; L, R and $\frac{1}{C}$ are diagonal matrix and Δ is a constant distribution in TQ corresponding to Kirchoff's laws.
These equations are equivalent to (1) whenever we take

$$\mathcal{E} = pv - Lv^2 + \frac{1}{C}q^2$$
$$D = \{(q,v,p,\dot{q},\dot{v},\dot{p},\alpha,\gamma,\beta) \colon \dot{q} \in \Delta, \beta = \dot{q}, \gamma = 0, \dot{p} + R\dot{q} + \alpha \in \Delta^\circ\}$$

In the circuit of the picture, $\Delta = \{v_1 - v_2 - v_3 = 0\}$. Applying the algorithm we get

$$M_1 = \{v_1 = v_2 + v_3,\ p = Lv\}$$
$$M_2 = \{v_1 = v_2 + v_3,\ p = Lv,\ q_2/C_2 = q_3/C_3\}$$
$$M_3 = M_4 = \{v_1/(C_2 + C_3) = v_2/C_2 = v_3/C_3,\ p = Lv,\ C_2q_3 = C_3q_2\}$$

Taking any three convenient variables as coordinates in the last submanifold (for instance q_1, q_2, and p_1) the system becomes an ordinary differential equation.

Dissipative mechanical systems

Given a lagrangian $L \colon TQ \to \mathbb{R}$ and an energy dissipation function $r \colon TQ \to \mathbb{R}^+$, $r(q,\dot{q}) = \dot{q}R(q,\dot{q})\dot{q}$, with $R(q,\dot{q})$ positive semidefinite; the formulation of the mechanical system is:

$$\begin{cases} \dot{q}\frac{\partial^2 L}{\partial \dot{q}^2}\ddot{q} + \dot{q}\frac{\partial^2 L}{\partial q \partial \dot{q}}\dot{q} - \frac{\partial L}{\partial q}\dot{q} = -r(q,\dot{q}) \\ \delta \int L(q,\dot{q}) = 0 \quad \text{when } \dot{q}R(q,\dot{q})\delta q = 0 \end{cases} \qquad (2)$$

which is equivalent to the implicit equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} + R\dot{q} = 0 \qquad (3)$$

Equations (2) are a second order kinematic constraint and a first order variational constraint respectively, in the sense of (Cendra–Ibort–de León–de Diego 2004).
There are two methods to state these equations as generalized Dirac structures

Method I: D depends on friction force.

$$\mathcal{E} = pv - L(q,v)$$
$$D_R = \{(q,v,p,\dot{q},\dot{v},\dot{p},\alpha,\gamma,\beta) \colon \beta = \dot{q},\ \gamma = 0,\ \dot{p} + \alpha + R\dot{q} = 0\}$$

It can be easily seen that this equation is equivalent to equation (3).

Method II: D depends on kinematic and variational restrictions.

$$\mathcal{E} = pv - L(q,v)$$
$$E_{(q,v)} = \{(\dot{q},\dot{v}) \in T(TQ) \colon v\frac{\partial^2 L}{\partial \dot{q}^2}\dot{v} + v\frac{\partial^2 L}{\partial q \partial \dot{q}}\dot{q} - \frac{\partial L}{\partial q}\dot{q} = -vR\dot{q}\}$$
$$\widetilde{E}_{(q,v)} = \{\delta q \in TQ \colon vR\delta q = 0\}$$
$$D_{(E,\widetilde{E})} = \{(\dot{q},\dot{v}) \in E(q,v),\ \beta = \dot{q},\ \gamma = 0,\ \dot{p} + \alpha \in \widetilde{E}(q,v)^\circ\}$$

Conclusions and prespectives

Generalization of Dirac structures do not alter the geometric constraints algorithm, and allows us to include dissipative systems into the same formalism.
The second method used for mechanical systems with friction, can be adapted to various different second order kinematic constraints and first order variational constraints.
A possible future work consists in looking for conditions for existence and uniqueness of solutions, and desingularization methods for the case when some of the M_k are not submanifolds.

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Germán Zorba germanzorba@mate.unlp.edu.ar – Univ. Nac. de La Plata