Reduction of Discrete Nonholonomic Mechanical Systems with Symmetry

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Discrete Nonholonomic Mechanical Systems.

A discrete nonholonomic mechanical system consists of a manifold Q, the configuration space, a distribution $\mathcal{D}$ on Q, the constraint space, a submanifold $\mathcal{D}_d \subset Q \times Q$, the discrete constraint space and a differentiable function $L_d : Q \times Q \to \mathbb{R}$, the discrete Lagrangian.

The discrete Lagrange-D'Alembert Principle consists of finding the critical points of of the action sum $S_d := \sum_{j=1}^{n} L_d(q_{j-1}, q_j)$ among the sequences of points (q_j) with fixed endpoints, where the variations $\delta q_j \in \mathcal{D}_{q_j}$ and $(q_j, q_{j+1}) \in \mathcal{D}_d$ with $j \in \{1, ..., n-1\}$. This leads to the equations

$$D_1 L_d(q_j, q_{j+1}) + D_2 L_d(q_{j-1}, q_j) = \sum a_a \, \omega^a(q_j)$$

$$\phi_d^b(q_{j-1}, q_j) = 0 \quad b = 1, ..., dim \mathcal{D}_d$$

where $\mathcal{D} = Ker \omega^a$ and ϕ_d^b are functions whose annihilation defines $\mathcal{D}_d$.

Symmetry.

Let G be an abelian Lie group acting on Q in so that $Q \to Q/G$ is a principal bundle. We suppose that $\mathcal{D}$ is G-invariant under the lifted action on TQ and that L_d and $\mathcal{D}_d$ are G-invariant under the diagonal action.

The discrete vertical space is defined as $\mathcal{V}_d(q) := \{(q, g\, q) : g \in G\}$.

The composition of a vertical element $(q_0, g\, q_0)$ and an arbitrary element $(q_0, q_1) \in Q \times Q$ is given by $(q_0, g\, q_0) \cdot (q_0, q_1) := (q_0, g\, q_1)$.

A discrete affine connection is a G-invariant choice of a subset of $Q \times Q$ called the discrete horizontal space that is complementary to the discrete vertical space.

Chaplygin Systems.

A discrete mechanical system with symmetry is Chaplygin if $TQ = \mathcal{D} \oplus \mathcal{V}$, where $\mathcal{V}_q = T_q(G \cdot q)$, and $\mathcal{D}_d$ defines a discrete affine connection $\mathcal{A}_d^C$.

Horizontal symmetries.

A discrete mechanical system presents H-horizontal symmetry if for all $q \in Q$, $\mathcal{D}_q \cap \mathcal{V}_q = T_q(H \cdot q)$ and $\mathcal{D}_d(q) \cap \mathcal{V}_d(q) = \{(q, h\, q) : h \in H\}$.

We can decompose $TQ = \mathcal{H} \oplus \mathcal{S} \oplus \mathcal{U}$ where $\mathcal{S}_q := \mathcal{D}_q \cap \mathcal{V}_q$, $\mathcal{D}_q = \mathcal{H}_q \oplus \mathcal{S}_q$ and $\mathcal{V}_q = \mathcal{S}_q \oplus \mathcal{U}_q$.

The discrete momentum map $J_d : Q \times Q \to \mathfrak{h}^*$ associated to the H-action is given by $J_d(q_0, q_1) \cdot \xi := D_2 L_d(q_0, q_1) \xi_Q(q_1)$ where $\mathfrak{h}$ is a Lie-algebra of H. It is a conserved quantity over the discrete trajectories of the system.

Transversality condition: given $\mu \in \mathfrak{h}^*$ and $(q_0, q_1) \in Q \times Q$ there is a unique $h \in H$ such that $J_d(q_0, h^{-1} q_1) = \mu$.

Assuming the transversality condition, $J_d^{-1}(\mu)$ defines the discrete affine connection $\mathcal{A}_d^H : Q \times Q \to H$ given by $\mathcal{A}_d^H(q_0, q_1) = h$ if and only if $J_d(q_0, h^{-1} q_1) = \mu$

Reduction Theorems.

In order to describe the reduction of the symmetry we must work on the reduced space $(Q \times Q)/G$.

Proposition Given a discrete affine connection $\mathcal{A}_d$, the map $\alpha_{\mathcal{A}_d} : (Q \times Q)/G \to Q/G \times Q/G \times G$ given by $\alpha_{\mathcal{A}_d}([(q_0, q_1])) := (\pi(q_0), \pi(q_1), \mathcal{A}_d(q_0, q_1))$ is a fiber bundle isomorphism.

Chaplygin Systems.

The G-invariance of the lagrangian induces an application defined on the reduced space, $\widetilde{l}_d : (Q \times Q)/G \to \mathbb{R}$. We define the applications: $I : Q/G \times Q/G \to (Q \times Q)/G$ by $I([q_0], [q_1]) := \alpha_{\mathcal{A}_d^C}^{-1}([q_0], [q_1], e)$ and $\widehat{L}_d : Q/G \times Q/G \to \mathbb{R}$ by $\widehat{L}_d := \widetilde{l}_d \circ I$.

Theorem Given a discrete path $q_.$ in Q that satisfies the constraint $\mathcal{D}_d$ and $r_j := \pi(q_j) \in Q/G$. Then, the following are equivalent:
1. $q_.$ satisfies $dS_d(q_.)(\delta q_.) = 0, \forall\ \delta q_.$ with fixed endpoints such that $\delta q_j \in \mathcal{D}_{q_j}$.
2. $q_.$ satisfies the Lagrange-D'Alembert equations.
3. $r_.$ satisfies the variational principle

$$d\sigma(r_.)(\delta r_.) = \sum_{j=1}^{n-1} (-\beta_0(r_j, r_{j+1}) - \beta_1(r_{j-1}, r_j)) \delta r_j$$

where $\sigma(r_.) := \sum_{j=1}^{n} \widehat{L}_d(r_{j-1}, r_j)$ and $\delta r_j \in \widehat{\mathcal{D}}_{r_j} := \pi_*(\mathcal{D}_{q_j})$.
4. $r_.$ satisfies the equations

$$D_1 \widehat{L}_d(r_j, r_{j+1}) + D_2 \widehat{L}_d(r_{j-1}, r_j) = -\beta_0(r_j, r_{j+1}) - \beta_1(r_{j-1}, r_j)$$

Horizontal symmetries.

The H-invariance of the lagrangian induces an application defined on the reduced space, $\widetilde{l}_d : (Q \times Q)/H \to \mathbb{R}$. We define the applications: $I : Q/H \times Q/H \to (Q \times Q)/H$ by $I([q_0], [q_1]) := \alpha_{\mathcal{A}_d^H}^{-1}([q_0], [q_1], e)$ and $\widehat{L}_d : Q/H \times Q/H \to \mathbb{R}$ by $\widehat{L}_d := \widetilde{l}_d \circ I$.

Theorem Given a discrete path $q_.$ in Q that satisfies the constraint $\mathcal{D}_d$ and $r_j := \pi(q_j) \in Q/H$ and $\mu \in \mathfrak{h}^*$. Then, the following are equivalent:
1. $q_.$ satisfies $dS_d(q_.)(\delta q_.) = 0, \forall\ \delta q_.$ with fixed endpoints such that $\delta q_j \in \mathcal{D}_{q_j}$. Besides, $J_d(q_0, q_1) = \mu$.
2. $q_.$ satisfies the Lagrange-D'Alembert equations.
Besides, $J_d(q_0, q_1) = \mu$.
3. $r_.$ is such that $(r_j, r_{j+1}) \in \widehat{\mathcal{D}}_d = I^{-1}(\mathcal{D}_d/H)$ and satisfies the variational principle

$$d\sigma(r_.)(\delta r_.) = \sum_{j=1}^{n-1} (-\beta_0(r_j, r_{j+1}) - \beta_1(r_{j-1}, r_j)) \delta r_j$$

where $\delta r_j \in \widehat{\mathcal{D}}_{r_j} := \pi_*(\mathcal{D}_{q_j})$. Besides, $J_d(q_j, q_{j+1}) = \mu, \forall j$.
4. $r_.$ is such that $(r_j, r_{j+1}) \in \widehat{\mathcal{D}}_d$ and satisfies the equations

$$D_1 \widehat{L}_d(r_j, r_{j+1}) + D_2 \widehat{L}_d(r_{j-1}, r_j) + \beta_0(r_j, r_{j+1}) + \beta_1(r_{j-1}, r_j) = \sum_a \widehat{\lambda}_a \, \widehat{\omega}^a(r_j)$$

where $\widehat{\mathcal{D}} r_j$ is the annihilator of the differential one-forms $\widehat{\omega}^a$.
Besides, $J_d(q_j, q_{j+1}) = \mu, \forall j$.

Abelian symmetries. We have already obtained a reduction theorem for discrete mechanical systems with abelian symmetries. As consequence, the previous theorems on Chaplygin and H-symmetries becomes special cases of this abelian reduction.