ABNORMAL TRAJECTORIES IN AFFINE CONNECTION CONTROL SYSTEMS

M. Barbero-Liñán, M-C. Muñoz-Lecanda   *Dept. of Applied Mathematics IV, UPC*

SUMMARY

In optimal control theory, Pontryagin's Maximum Principle [5] gives necessary conditions for optimality. Due to the work of Montgomery [3] and Liu and Sussmann [2], the existence of abnormal optimal trajectories in subRiemannian geometry is known. We study the existence of abnormal and strict abnormal extremals in a particular class of mechanical systems assuming accessibility [4] and using tools as the symmetric product [1] and the vector-valued quadratic forms [1].

AFFINE CONNECTION CONTROL SYSTEMS, ACCS

Let Q be a smooth n-dimensional manifold, ∇ be an affine connection on Q and $\gamma\colon I\subset\mathbb{R}\to Q$.

$$\text{Dynamics equations of the control system}:\quad \nabla_{\dot\gamma(t)}\dot\gamma(t)=u^k(t)Y_k(\gamma(t)), \qquad (1)$$

the *controls* $u^k\colon I\to U\subset\mathbb{R}^m$ are locally integrable, U is an open set, Y_k are *input vector fields* on Q.

The tuple $(Q,\nabla,\mathscr{Y},U)$ is called an Affine Connection Control System, where $\mathscr{Y}=\{Y_1,\dots,Y_m\}$.

The geodesic spray Z associated to ∇ is a vector field on TQ whose integral curves are the velocity curves of the geodesics for ∇.

In local coordinates (x,v) for TQ, $\quad Z=v^i\frac{\partial}{\partial x^i}-\Gamma^i_{jk}(x)v^jv^k\frac{\partial}{\partial v^i}, \quad \Gamma^i_{jk}$ Christoffel symbols for ∇ .

Equation (1) is rewritten as a first-order nonlinear control-affine system on TQ,

$$\dot\Upsilon(t)=Z(\Upsilon(t))+u^k(t)Y_k^V(\Upsilon(t))$$

where $\Upsilon\colon I\to TQ$ is a curve such that $\Upsilon=\dot\gamma$ and Y_k^V denotes the vertical lift of the vector field Y_k.

As usual we assume that the system is accessible [4] to have curves between any initial conditions.

OPTIMAL CONTROL PROBLEM FOR ACCS

In optimal control problem for mechanical systems we look for a curve γ in Q and controls u such that $(\dot\gamma,u)$ minimizes the integral of a cost function $F\colon TQ\times U\to\mathbb{R}$.

Statement 1. *(Fixed-Time Optimal Control Problem for ACCS without velocity initial conditions)*
Given $(Q,\nabla,\mathscr{Y},U)$, $I=[a,b]$, $x_a,x_b\in Q$.
Find *$(\gamma,u)\colon I\to Q\times U$ such that for a curve $\Upsilon\colon I\to TQ$ satisfying $\tau_Q\circ\Upsilon=\gamma$,*
(1) $\gamma(a)=x_a$, $\gamma(b)=x_b$;
(2) $\dot\Upsilon(t)=Z(\Upsilon(t))+u^k(t)Y_k^V(\Upsilon(t))$, and
(3) $\mathcal{S}[\Upsilon,u]=\int_I F(\Upsilon(t),u(t))dt$ is minimum over all curves on $TQ\times U$ satisfying (1) and (2).

Notation: $\Sigma=(Q,\nabla,\mathscr{Y},U,F,[a,b],x_a,x_b)$.

Assumption: Y_k^V and F are continuous on $TQ\times\overline{U}$ and continuously differentiable with respect to TQ.

Pontryagin's Hamiltonian for Σ is $H\colon T^*(TQ)\times U\to\mathbb{R}$,

$$H(\Lambda_{v_x},u)=\langle\Lambda_{v_x},Z(v_x)+u^kY_k^V(v_x)\rangle+p_0F(v_x,u),\quad \Lambda_{v_x}\in T^*_{v_x}(TQ)\text{ and }p_0\in\{-1,0\}\ .$$

Theorem 1. *(Pontryagin's Maximum Principle)*
Let $(\Upsilon,u)\colon[a,b]\to TQ\times U$ be a solution of the optimal control problem with initial conditions x_a, x_b.
Then there exist $\Lambda\colon[a,b]\to T^*(TQ)$ along Υ and $p_0\in\{-1,0\}$ such that:
1. Λ satisfies Hamilton's equations, in local coordinates (x^i,v^i,p_i,q_i) for $T^*(TQ)$

$$\dot{\widehat{(x^i\circ\Lambda)}}=\frac{\partial H}{\partial p_i}(\Lambda,u),\quad \dot{\widehat{(v^i\circ\Lambda)}}=\frac{\partial H}{\partial q_i}(\Lambda,u),\quad \dot{\widehat{(p_i\circ\Lambda)}}=-\frac{\partial H}{\partial x^i}(\Lambda,u),\quad \dot{\widehat{(q_i\circ\Lambda)}}=-\frac{\partial H}{\partial v^i}(\Lambda,u);$$

2. $\Upsilon=\pi_{TQ}\circ\Lambda$, $\gamma=\tau_Q\circ\Upsilon$, where $\pi_{TQ}\colon T^*(TQ)\to TQ$ and $\tau_Q\colon TQ\to Q$ are the natural projections;
3. γ satisfies the initial conditions in Q;
4(a) $H(\Lambda(t),u(t))=\max_{\widetilde u\in U}H(\Lambda(t),\widetilde u)$ almost everywhere;
(b) $\max_{\widetilde u\in U}H(\Lambda(t),\widetilde u)$ is constant everywhere;
(c) $(p_0,\Lambda(t))\neq 0$ for each $t\in[a,b]$.

The necessary conditions 1-4 of Theorem 1 determine different kinds of extremals.

Definition 1. *A curve $(\Upsilon,u)\colon[a,b]\to TQ\times U$ is*
1. an extremal for Σ if there exists $\Lambda\colon[a,b]\to T^*(TQ)$ such that $\Upsilon=\pi_{TQ}\circ\Lambda$ and (Λ,u) satisfies the necessary conditions of Pontryagin's Maximum Principle;
2. a normal extremal for Σ if it is an extremal and $p_0=-1$;
3. an abnormal extremal for Σ if it is an extremal and $p_0=0$;
4. a strict abnormal extremal for Σ if it is not a normal extremal, but it is an abnormal extremal.
The covector curve $\Lambda\colon[a,b]\to T^*(TQ)$ is called biextremal.

Remark: The abnormal extremals do not depend on the cost function, but the concept of strict abnormality depends on the cost function as we will see in the example.

CONSTRAINT ALGORITHM FOR ABNORMALITY

We look for a submanifold of $T^*(TQ)$ containing the covectors Λ along the abnormal extremals Υ.

For abnormal biextremals, the Hamiltonian is $H(\Lambda,u)=\langle\Lambda,Z(\Upsilon)+u^kY_k^V(\Upsilon)\rangle=(H_Z+H_{u^kY_k^V})(\Lambda)$ and the Hamiltonian vector field is $X=\overrightarrow{H}=\overrightarrow{H_Z}+u^k\overrightarrow{H_{Y_k^V}}$.

A necessary condition for maximizing the Hamiltonian with respect to the controls is

$$\frac{\partial H}{\partial u^k}(\Lambda,u)=\langle\Lambda,Y_k^V(\pi_{TQ}(\Lambda))\rangle=H_{Y_k^V}(\Lambda)=0\quad\text{for}\quad k=1,\dots,m\ \to\ \textit{primary constraints}.$$

Primary submanifold

$$N_0=\{(\Lambda,u)\in T^*(TQ)\times U\,|\,H_{Y_k^V}(\Lambda)=0\quad\text{for}\ \ k=1,\dots,m\}=\operatorname{ann}\mathscr{Y}^V.$$

First consistency step: $\quad N_1=\{(\Lambda,u)\in N_0\,|\,X(\Lambda)\in T_\Lambda N_0\}\subseteq N_0.$

The tangency condition is satisfied if $dH_{Y_k^V}(X)=0$, then

$$N_1=\{(\Lambda,u)\in T^*(TQ)\times U\,|\,H_{Y_k^V}(\Lambda)=0,\ H_{[Z,Y_k^V]}(\Lambda)=0\quad\text{for}\ \ k=1,\dots,m\}\subseteq N_0.$$

Second consistency step: $\quad N_2=\{(\Lambda,u)\in N_1\,|\,X(\Lambda)\in T_\Lambda N_1\}\subseteq N_1.$

The tangency condition is satisfied if $dH_{[Z,Y_k^V]}(X)=0$, then

$$N_2=\{(\Lambda,u)\in N_1\,|\,H_{[Z,[Z,Y_k^V]]+u^l[Y_l^V,[Z,Y_k^V]]}(\Lambda)=0\quad\text{for}\ \ k=1,\dots,m\}\subseteq N_1\subseteq N_0.$$

If there exists $i\in\mathbb{N}$ such that $N_i=N_{i-1}$, then $N_f=N_{i-1}$ is the final constraint submanifold.

Observe that $[Y_l^V,[Z,Y_k^V]]=(\nabla_{Y_l}Y_k+\nabla_{Y_k}Y_l)^V$, where $\nabla_{Y_l}Y_k+\nabla_{Y_k}Y_l$ is known as the symmetric product and denoted by $\langle Y_l\colon Y_k\rangle$.

With this in mind, in [1] the following vector-valued quadratic forms associated to an ACCS are defined and used to characterize some aspects of the controllability of the system

$$B_q\colon\mathscr{Y}_q\times\mathscr{Y}_q\longrightarrow T_qQ/\mathscr{Y}_q,\qquad (w_1,w_2)\longmapsto\pi_{\mathscr{Y}_q}(\langle W_1\colon W_2\rangle)$$

where W_1 and W_2 are in the submodule generated by $Y_1,\dots,Y_m$ and extend $w_1,w_2\in\mathscr{Y}_q$, $\pi_{\mathscr{Y}_q}\colon T_qQ\to T_qQ/\mathscr{Y}_q$ is the natural projection.

For any $\lambda\in(T_qQ/\mathscr{Y}_q)^*=\operatorname{ann}\mathscr{Y}_q$, we have a real quadratic form $\lambda B_q\colon\mathscr{Y}_q\times\mathscr{Y}_q\to\mathbb{R}$, $(w_1,w_2)\mapsto\langle\lambda,B_q(w_1,w_2)\rangle$. According to [1], any vector-valued quadratic form B_q is

- either strongly semidefinite, i.e., there exists a nonzero $\lambda\in\operatorname{ann}\mathscr{Y}_q$ such that the real quadratic form $\lambda\,B_q$ is nonzero and semidefinite positive;
- or essentially indefinite, i.e., for all $\lambda\in\operatorname{ann}\mathscr{Y}_q$, $\lambda\,B_q$ is zero or indefinite.

"If $\lambda\,B_q$ is indefinite or definite, the controls are determined in N_2." *This is under research!*

AN EXAMPLE OF STRICT ABNORMALITY

Let $Q=\mathbb{R}^3$, ∇ be the zero connection, $\mathscr{Y}=\operatorname{span}\{Y_1,Y_2\}$ where the vector fields are defined below. The control set U is open in $\mathbb{R}^2$ and contains the zero. The initial conditions are $A=(2,0,0)$ and $B=(2,1,0)$. We have an ACCS. In coordinates (x,y,z,v_x,v_y,v_z) for TQ,

$$Z=v_x\frac{\partial}{\partial x}+v_y\frac{\partial}{\partial y}+v_z\frac{\partial}{\partial z},\quad Y_1=\frac{\partial}{\partial x},\quad Y_2=(1-x)\frac{\partial}{\partial y}+x^2\frac{\partial}{\partial z},\quad [Z,Y_1^V]=-\frac{\partial}{\partial x},\quad \langle Y_1\colon Y_1\rangle^V=0,$$
$$[Z,Y_2^V]=(-1+x)\frac{\partial}{\partial y}-x^2\frac{\partial}{\partial z}-v_x\frac{\partial}{\partial v_y}+2xv_x\frac{\partial}{\partial v_z},\quad \langle Y_1\colon\langle Y_1\colon Y_2\rangle\rangle^V=2\frac{\partial}{\partial v_z},\quad \langle Y_2\colon Y_2\rangle^V=0,$$
$$[Z,[Z,Y_1^V]]=0,\quad [Z,[Z,Y_2^V]]=2v_x\frac{\partial}{\partial y}-4xv_x\frac{\partial}{\partial z}+2v_x^2\frac{\partial}{\partial v_z},\quad \langle Y_1\colon Y_2\rangle^V=-\frac{\partial}{\partial v_y}+2x\frac{\partial}{\partial v_z}.$$

The system is accessible because

$$\mathfrak{X}(TQ)=\operatorname{span}\{[Z,Y_1^V],[Z,Y_2^V],[Z,\langle Y_1\colon Y_2\rangle^V],Y_1^V,Y_2^V,\langle Y_1\colon Y_2\rangle^V,[Z,\langle Y_1\colon\langle Y_1\colon Y_2\rangle\rangle^V],\langle Y_1\colon\langle Y_1\colon Y_2\rangle\rangle^V\}\ .$$

If $x=0$ or $x=2$, we do not need the last two brackets.

The Hamiltonian for abnormal extremals is $H=p_xv_x+p_yv_y+p_zv_z+u^1q_x+u^2(1-x)q_y+u^2x^2q_z$

with Hamilton's equations in coordinates $(p_x,p_y,p_z,q_x,q_y,q_z)$ for the π_{TQ}-fibers in $T^*(TQ)$

$$\begin{array}{llll}\dot x=v_x & \dot v_x=u^1 & \dot p_x=q_yu^2-2q_zu^2x & \dot q_x=-p_x\\ \dot y=v_y & \dot v_y=u^2(1-x) & \dot p_y=0 & \dot q_y=-p_y\\ \dot z=v_z & \dot v_z=u^2x^2 & \dot p_z=0 & \dot q_z=-p_z\end{array}$$

Primary submanifold $\quad N_0=\{(\Lambda,u)\in T^*(TQ)\times U\,|\,q_x=0,\ q_y(1-x)+q_zx^2=0\}$

$$N_1=\{(\Lambda,u)\in N_0\,|\,p_x=0,\ (-1+x)p_y-x^2p_z-v_xq_y+2xv_xq_z=0\}$$

$$N_2=\{(\Lambda,u)\in N_1\,|\,(-q_y+2xq_z)u^2=0,\ (-q_y+2xq_z)u^1=-(2p_yv_x-4xv_xp_z+2v_x^2q_z)\}\ .$$

If $x\neq 1$, $q_y=\dfrac{q_zx^2}{x-1}$. The condition $(-q_y+2xq_z)u^2=0$ in N_2 becomes

$$\frac{xq_zu^2(2-x)}{x-1}=0\Rightarrow\begin{cases}x=0 & \text{No.} \qquad\qquad q_z=0\quad \text{No abnormal.}\\ u^2=0 & \text{No strict abnormal.}\quad x=2\quad OK!\end{cases}$$

$$\begin{aligned}N_0&=\{(\Lambda,u)\in T^*(TQ)\times U-\{xq_zu^2=0\}\,|\,q_x=0,\ -q_y+4q_z=0\}\\ N_1&=\{(\Lambda,u)\in N_0-\{xq_zu^2=0\}\,|\,p_x=0,\ p_y-4p_z=0\}\\ N_2&=\{(\Lambda,u)\in N_1-\{xq_zu^2=0\}\,|\,x=2,\ v_x=0\}\\ N_3&=\{(\Lambda,u)\in N_2-\{xq_zu^2=0\}\,|\,v_x=0,\ u^1=0\}=N_4\ .\end{aligned}$$

Integrating Hamilton's equations on the interval $I=[0,1]$ we have

$$\Lambda(t)=(0,4p_z,p_z,0,-4p_zt+4q_z^0,-p_zt+q_z^0)$$
$$\textit{Abnormal extremal}\to(\Upsilon(t),u(t))=(2,-u^2\tfrac{t^2}{2}+v_y^0t,2u^2t^2+v_z^0t,0,-u^2t+v_y^0,4u^2t+v_z^0,0,u^2(t))$$
$\Big\}$ *Abnormal biextremal* .

The initial condition $B=(2,1,0)$ at $t=1$ is satisfied if $u^2=-\frac{v_z^0}{2}=2(v_y^0-1)$.

Then the initial conditions for the velocities, although are not given, must be $(0,v_y^0,4(1-v_y^0))$.

WHEN IS THE CURVE (Υ,u) STRICT ABNORMAL? Let $F=\frac{{u^1}^2+{u^2}^2}{2}$ be the cost function, $p_0=-1$

$$H_F=p_xv_x+p_yv_y+p_zv_z+q_xu^1+q_yu^2(1-x)+q_zu^2x^2-\frac{{u^1}^2+{u^2}^2}{2}\ ,$$

with the same Hamilton's equations as before. From the maximization of the Hamiltonian,

$$\frac{\partial H_F}{\partial u^1}=q_x-u^1=0\ ,\quad \frac{\partial H_F}{\partial u^2}=q_y(1-x)+q_zx^2-u^2=0\ . \qquad (2)$$

The curve Υ in Hamilton's equation gives $u^1=0$ then $q_x=0$ because of Equation (2) and $p_x=0$ because $\dot q_x=-p_x$. From Equation (2) $u^2=-q_y+4q_z$. Hence $0=\dot p_x=-(u^2)^2$.

We also have $u^2=2(v_y^0-1)$, then $0=-2(1-v_y^0)^2$. But this is satisfied if and only if $v_y^0=1$. If $v_y^0=1$, then $u^2=0$ and (Υ,u) is a normal extremal.

Therefore, we have found an extremal for an optimal control problem for ACCS without initial velocity conditions and this extremal is **strict abnormal** if $v_y^0\neq 1$.

Acknowledgements: MEC Projects: MTM2005-04947, Network MTM2006-27467-E/ and FPU grant.

We thank Professor Andrew D. Lewis for his useful ideas and help to develop this work.

References

[1] F. BULLO, A. D. LEWIS, *Geometric Control of Mechanical Systems. Modeling, analysis and design for simple mechanical control*, Texts in Applied Mathematics 49, Springer-Verlag, New York-Heidelberg-Berlin 2004.

[2] W. LIU, H. J. SUSSMANN, "Shortest paths for sub-Riemannian metrics on rank-two distributions", *Mem. Amer. Math. Soc.* 564, Jan. 1996.

[3] R. MONTGOMERY, "Abnormal Minimizers", *SIAM J. Control Optim.*, **32**(6)(1994), 1605-1620.

[4] H. NIJMEIJER, A. J. VAN DER SCHAFT, Nonlinear Dynamical Control Systems, Springer-Verlag, New York 1990.

[5] L. S. PONTRYAGIN, V. G. BOLTYANSKI, R. V. GAMKRELIDZE, E. F. MISCHENKO, *The Mathematical Theory of Optimal Processes*, Interscience Publishers, Inc., New York 1962.