Jet Prolongations of Fibered Supermanifolds

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1. Supermanifolds and Their Maps

1.1. Basic Notions

We define supermanifolds as in [Del, Lei, Man] using sheaves of supercommutative rings. Let $(X, \mathscr{R})$, $(Y, \mathscr{S})$ be sheaves of rings over topological spaces X, Y. A morphism of sheaves ψ is a continuous map $\tilde{\psi}\colon X \to Y$ together with a system of homomorphisms

$$\psi_V^*\colon \mathscr{S}(V) \to \mathscr{R}(\tilde{\psi}^{-1}(V)),$$

where $V \subset Y$ are open sets and for open sets $U \subset X$ and $V \subset Y$ $\mathscr{R}(U)$ and $\mathscr{S}(V)$ are supercommutative rings.
Smooth supermanifolds are defined in the same way as ordinary manifolds using local models. To this end, let $U^{p|q}$ be the sheaf $(U, C^\infty_{p|q})$, where $U \subset \mathbb{R}^p$ is an open set and

$$C^\infty_{p|q}\colon V \mapsto C^\infty(V)[\theta^1, \ldots, \theta^q],$$

where $V \subset U$ is open and θ^i, $1 \le i \le q$ are anticommuting variables, i.e. $\theta^i\theta^j = -\theta^j\theta^i$ for $i \neq j$ and $(\theta^i)^2 = 0$. Each element f of $U^{p|q}$ can thus be written as

$$f = \sum_{|I| \le q} f_I \theta^I,$$

where I is an ordered multiindex of length $|I| = q$, $I = (i_1 \cdots i_q)$, $i_1 < \cdots < i_q$, f_I are smooth functions and $\theta^I = \theta^{i_1} \cdots \theta^{i_q}$.
Let X be a Hausdorff, second countable topological space. A ***smooth supermanifold*** is defined by the sheaf $(X, \mathscr{O}_X)$ which is locally isomorphic to $U^{p|q}$, i.e. for each point $x \in X$ there exists a neighborhood V of x, such that $\mathscr{O}_X(V)$ is isomorphic to some $U^{p|q}$ and the morphisms are considered to be morphisms of sheaves of rings. The imbedded ordinary manifold of $(X, \mathscr{O}_X)$ is denoted by $\tilde{X}$. Morphisms of supermanifolds are simply morphisms of the corresponding sheaves of supercommutative rings. One defines the tangent sheaf $\mathscr{T}_X$ as the sheaf of derivations of the sheaf $\mathscr{O}_X$. It is a finitely generated module over $\mathscr{O}_X$ of rank $(p|q)$. The sections of the tangent sheaf are called ***vector fields***.

1.2. Fibered Supermanifolds

The category of smooth supermanifolds admits Cartesian products which are defined categorically. A ***fibered supermanifold*** is defined to be a triple (Y, π, X), where Y, X are supermanifolds of dimension $(m+n|p+q)$, (n, q) and π a surjective submersion $\pi\colon Y \to X$. We define the local maps $\gamma\colon X \supset U \to Y$, $\pi \circ \gamma = \mathrm{id}_U$, all such maps can be given in adapted coordinates $(x^i, y^a; \theta^j, \phi^b)$ on Y corresponding to coordinates $(x^i; \theta^j)$ on X as

$$\begin{aligned} \gamma^* y^a &= \gamma_0^a(x^i, \theta^j) \qquad & \gamma^*\phi^b &= \gamma_1^b(x^i, \theta^j) \\ \gamma^* x^i &= x^i & \gamma^*\theta^j &= \theta^j, \end{aligned}$$

where $1 \le a \le m$, $1 \le b \le p$, $1 \le i \le n$, $1 \le j \le q$ and γ_0^a resp. γ_1^b are even resp. odd sections of $\mathscr{O}_X$.

2. Jets of Smooth Maps of Supermanifolds

2.1. Weil algebras and W-points

We generalize Weil algebras [KMS, MRM] to the supercommutative setting. A finite dimensional, local (super) commutative $\mathbb{R}$-algebra W with unit is called a ***Weil*** algebra; we denote by I its unique maximal proper ideal. Due to the finite dimension of W, there exists a minimal integer h such that $I^{h+1} = 0$, it is called height of W, the width of W is defined to be the vector space dimension of the quotient I/I^2.
Example: Let $W = \mathbb{R}[\theta^1, \theta^2] \otimes_{\mathbb{R}} \mathrm{D}$, D is the commutative algebra generated by $[1, \epsilon]$ subject to the restriction $\epsilon^2 = 0$. The maximal ideal and its nonvanishing powers are

$$\begin{aligned} I &= [\theta^1, \theta^2, \epsilon, \theta^1\epsilon, \theta^2\epsilon, \theta^1\theta^2, \theta^1\theta^2\epsilon], \\ I^2 &= [\theta^1\epsilon, \theta^2\epsilon, \theta^1\theta^2, \theta^1\theta^2\epsilon], \\ I^3 &= [\theta^1\theta^2\epsilon]. \end{aligned}$$

It follows that $\dim W = 8$, $\dim I = 7$, height $W = 3$, width $W = 3$.
A ***W-point***, denoted by p^W, of the supermanifold $(X, \mathscr{O}_X)$ is a sheaf algebra homomorphism $p^W\colon \mathscr{O}_X \to W$ (W is the constant sheaf over $\tilde{X}$). For each section f of $\mathscr{O}_X$ the W-value of p^W is denoted by $f(p^W)$. The set of all W-points of X is denoted by X^W. We say that a W-point p^W is ***regular*** if the homomorphism is surjective. The ***W-jet*** of the W-point p^W is defined as the kernel of the homomorphism $\mathfrak{p}^W = \ker \mathscr{O}_X \to W$. If the W-point p^W is regular, then we also call its W-jet regular. The set of regular W-jets of $(X, \mathscr{O}_X)$ will be denoted by $J^W X$. There exists the obvious natural projection $\ker\colon X^W \to J^W X$.
Let $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ be supermanifolds. Each morphism $\psi\colon X \to Y$ induces a morphism $J^W\psi\colon J^W X \to J^W Y$ by the prescription

$$J^W\psi\colon \mathfrak{p}^W = \ker p^W \mapsto \ker(p^W \circ \psi^*),$$

which doesn't depend on the choice of representative p^W.

2.2. Jets of Fibered Supermanifolds

Consider the fibered manifold (Y, π, X), the map $\pi^*\colon \mathscr{O}_X \to \mathscr{O}_Y$ allows us to interpret $\mathscr{O}_X$ as a ringed subspace of $\mathscr{O}_Y$. Consider the subset of $\mathscr{O}_X$-regular jets of $J^W Y$ and its image by the map $J^W\pi$. It can be proved that the image in $J^W X$ is isomorphic to $\mathscr{O}_X$.

2.3. Applications

Example: Taking $W = \mathrm{D}$ we obtain the tangent sheaf $\mathscr{T}_X$.
Example: A supermanifold analogy to jet prolongations of fibered manifolds. Let

$$W = C^\infty(\mathbb{R}^n)[\theta^1 \cdots \theta^q] / \\ /(x^{i_1} \cdots x^{i_k})(\theta^{j_1} \cdots \theta^{j_\ell}).$$

$(J^W Y, J^W\pi, J^W X = X)$ is the $(k|\ell)$-prolongation of the fibered manifold (Y, π, X). This prolongation is a ringed space in the sense of [Man], it doesn't naturally possess the structure of a supermanifold.

3. Discussion

The notion of a jet prolongation of a fibered manifold is generalized to supermanifolds with the caveat that the resulting space need not be a supermanifold. This definition does't coincide with its usage in [BBH] although it seems to be more natural. Most Lagrangians in physical theories use sections of the tangent sheaf and the base manifolds (superspacetimes) used are of special type, usually superhomogeneous spaces, where such problems don't occur.

References

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