Lagrange-d'Alembert-Poincaré equations by stages

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1 Introduction

In the case of nonholonomic systems, the Lagrange-d'Alembert principle, that gives the equation of motion, can be reduced using covariant derivatives. The resulting equations are called the Lagrange-d'Alembert-Poincaré equations.

If the symmetry group has a normal subgroup, then reducing by the whole group is shown to be isomorphic to what one gets by reducing by stages. That is, the reduction process of a Lagrangian system with symmetry can be repeated and we have the theory of Lagrangian reduction by stages.

In this work we study the Lagrangian reduction by stages for the case of a nonholonomic system, following the theory developed by Cendra, Marsden and Ratiu in [1, 2]. We obtain a formula for the Lie bracket for a given chain of normal subgroups. Using this, we write the Euler-d'Alembert-Poincaré equations, and also local Lagrange-d'Alembert-Poincaré equations, by stages.

2 The Lie bracket by stages

2.1 An explicit formula for the Lie bracket for two stages

We know that for any principal bundle $\pi : Q \to Q/G$ with structure group G there exists an invariant Riemannian metric. Let N be a normal subgroup of G and choose a G-invariant metric on Q. For each $q \in Q$ we have a direct sum orthogonal decomposition $T_qQ = \mathrm{Ver}^N(T_qQ) + H_N(q)$, where $H_N(q)$ is the orthogonal complement of $Ver^N(T_qQ)$. The collection of all $H_N(q)$ defines a G-invariant connection over the principal bundle Q with structure group N. Let $\mathcal{A}_N$ be the corresponding 1-form connection. Also we have the following proposition (see [2]):

Proposition: Consider a Lie group G, a normal subgroup N of G and $K = G/N$. Let $\mathcal{G}$, $\mathcal{N}$ and $\mathcal{K}$ be the Lie algebras of G, N and K respectively. We choose an identification $\mathcal{G} \equiv \mathcal{K} \times \mathcal{N}$ as linear spaces. Let $\mathcal{A}_N$ be the principal connection on the principal bundle G with structure group N having the property that $\mathcal{A}_N(gv_q) = \mathrm{Ad}_g\mathcal{A}_N(v_q)$, for every $g, q \in G,\ v_q \in T_qG$. Then $\mathcal{G} \equiv \mathcal{K}^{\mathcal{A}_N} \oplus \mathcal{N}$ where $\mathcal{K}^{\mathcal{A}_N}$ is the horizontal lift of $\mathcal{K}$ in the bundle

$G \to G/N$. In this context, the bracket in the Lie algebra $\mathcal{G}$ can be written in terms of the brackets of the Lie algebra $\mathcal{N}$ and the Lie algebra $\mathcal{K}$, and also in terms of $\nabla^{(\mathcal{A}_N,V)}$ and $\widetilde{B}^{\mathcal{A}_N}$ (see [2]) as follows:

$$[\kappa_1 \oplus \eta_1, \kappa_2 \oplus \eta_2] = [\kappa_1, \kappa_2] \oplus [\nabla^{(\mathcal{A}_N,V)}]_{G/N,\kappa_1}\eta_2 - [\nabla^{(\mathcal{A}_N,V)}]_{G/N,\kappa_2}\eta_1 - [\widetilde{B}^{\mathcal{A}_N}]_{G/N}(\kappa_1,\kappa_2) + [\eta_1,\eta_2]\,,$$

where

$$[\widetilde{B}^{\mathcal{A}_N}]_{G/N}(\kappa_1,\kappa_2) = \Big[\Big[e_G, -\mathcal{A}_N(e_G)\left([\kappa_1^{\mathcal{A}_N}, \kappa_2^{\mathcal{A}_N}]\right)\Big]_N\Big]_{G/N},$$

with $\kappa_i^{\mathcal{A}_N}$ is the horizontal lift of κ_i in e_G to G for $i = 1, 2$.

Here we have identified $\mathcal{K}$ with its adjoint bundle $\widetilde{\mathcal{K}}$ given by $\kappa \equiv [e_K, \kappa]_K$. Likewise, we identify $\mathcal{N}$ with the space of sections of the adjoint bundle $\widetilde{\mathcal{N}}$ (as vector bundle over $K = G/N$) given by $\eta \equiv \sigma_\eta$, where $\sigma_\eta : G/N \to \widetilde{\mathcal{N}}$ is defined by $\sigma_\eta([g]_N) := [g, \mathrm{Ad}_g \eta]_N$. We have that

$$[\nabla^{(\mathcal{A}_N,V)}]_{G/N,\ \kappa}\ \eta = \big[[e, [\kappa^{\mathcal{A}_N}, \eta]]_N\big]_K$$

where $\kappa^{\mathcal{A}_N}$ is the horizontal lift of κ in e_G to G.

Using the previous formulas we obtain an explicit formula for the Lie bracket on $\mathcal{G} = \mathcal{K}^{\mathcal{A}_N} \oplus \mathcal{N}$:

$$[\kappa \oplus \eta, \overline{\kappa} \oplus \overline{\eta}] = [\kappa, \overline{\kappa}] \oplus [\nabla^{(\mathcal{A}_N,V)}]_{K,\kappa}\ \overline{\eta} - [\nabla^{(\mathcal{A}_N,V)}]_{K,\overline{\kappa}}\ \eta - [\widetilde{B}^{\mathcal{A}_N}]_K(\kappa,\overline{\kappa}) + [\eta,\overline{\eta}] =$$

$$= [\kappa,\overline{\kappa}] \oplus \big[[e,[\kappa^{\mathcal{A}_N},\overline{\eta}]]_N\big]_K - \big[[e,[\overline{\kappa}^{\mathcal{A}_N},\eta]]_N\big]_K - \Big[\Big[e_G, -\mathcal{A}_N(e_G)\left([\kappa_1^{\mathcal{A}_N}, \kappa_2^{\mathcal{A}_N}]\right)\Big]_N\Big]_{G/N} + [\eta,\overline{\eta}].$$

Defining the bilinear forms $b_N : \mathcal{K} \times \mathcal{N} \to \widetilde{\mathcal{N}}/K \equiv \mathcal{N}$ and $a_N : \mathcal{K} \times \mathcal{K} \to \widetilde{\mathcal{N}}/K \equiv \mathcal{N}$ by $b_N(\kappa,\eta) := \big[[e,[\kappa^{\mathcal{A}_N},\eta]]_N\big]_K$ and $a_N(\kappa,\overline{\kappa}) := \big[[e, -\mathcal{A}_N(e)\left([\kappa^{\mathcal{A}_N},\overline{\kappa}^{\mathcal{A}_N}]\right)]_N\big]_K$, respectively, we can write the Lie bracket as follows:

$$[\kappa \oplus \eta, \overline{\kappa} \oplus \overline{\eta}] = [\kappa,\overline{\kappa}] \oplus b_N(\kappa,\overline{\eta}) - b_N(\overline{\kappa},\eta) - a_N(\kappa,\overline{\kappa}) + [\eta,\overline{\eta}].$$

2.2 The Lie bracket for n stages.

Theorem 2.1 *Let G be a Lie group and a chain of n normal subgroups $N_j \lhd N_{j-1}$ for $1 \le j \le n$, where $N_0 = G$. Consider $N_{n+1} = (e)$ and denote $N_{(j-1,j)}$ to the groups N_{j-1}/N_j. We will denote $\mathcal{N}_j$ to the Lie algebra of the subgroup N_j and $\mathcal{N}_{(j-1,j)}$ to the Lie algebra of $N_{(j-1,j)}$.*

Consider also a linear isomorphism

$$\mathcal{G} \equiv \mathcal{K} \oplus \mathcal{N}_{(1,2)} \oplus ... \oplus \mathcal{N}_{(n-1,n)} \oplus \mathcal{N}_{(n,n+1)}$$

as an identification, being $\mathcal{K} = \mathcal{N}_{(0,1)}$ the Lie algebra of $G/N_1 = N_{(0,1)}$.

Let $\kappa \oplus \left(\bigoplus_{i=1}^{n} \eta^{(i,i+1)}\right)$ *and* $\overline{\kappa} \oplus \left(\bigoplus_{j=1}^{n} \overline{\eta}^{(j,j+1)}\right)$ *be two elements of* $\mathcal{G}$ *with* $\kappa, \overline{\kappa} \in \mathcal{K}$ *and* $\eta^{(i,i+1)} \in \mathcal{N}_{(i,i+1)}$, $\overline{\eta}^{(j,j+1)} \in \mathcal{N}_{(j,j+1)}$.

Consider the maps $b_{(N_{j-1},N_j)} : \mathcal{N}_{(j-1,j)} \times \mathcal{N}_j \to \widetilde{\mathcal{N}}_j / N_{(j-1,j)} \simeq \mathcal{N}_j$, *where* $\widetilde{\mathcal{N}}_j$ *is the adjoint bundle of the principal bundle* $N_{j-1} \to N_{(j-1,j)}$ *with structure group* N_j *with left action, defined by* $b_{(N_{j-1},N_j)}(\eta, \overline{\eta}) := \left[\left[e, [\eta^{\mathcal{A}_{N_j}}, \overline{\eta}]\right]_{N_j}\right]_{N_{(j-1,j)}}$*; and*

$a_{(N_j,N_{j+1})} : \mathcal{N}_{(j,j+1)} \times \mathcal{N}_{(j,j+1)} \to \widetilde{\mathcal{N}}_{j+1}/N_{(j,j+1)} \simeq \mathcal{N}_{j+1}$ *defined by*

$a_{(N_j,N_{j+1})}(\eta, \overline{\eta}) := \left[\left[e, -\mathcal{A}_{N_{j+1}}(e)\left([\eta^{\mathcal{A}_{N_{j+1}}}, \overline{\eta}^{\mathcal{A}_{N_{j+1}}}]\right)\right]_{N_{j+1}}\right]_{N_{(j,j+1)}}$.

The map $b_{(N_{j-1},N_j)}$ *is identified with a quotient vertical connection because* $\left[\left[e, [\eta^{\mathcal{A}_{N_j}}, \overline{\eta}]\right]_{N_j}\right]_{N_{(j-1,j)}} \equiv [\nabla^{(\mathcal{A}_{N_j},V)}]_{N_{(j-1,j)},\eta}\, \overline{\eta}$ *being* $\eta^{\mathcal{A}_{N_j}}$ *the horizontal lift of* η *in the bundle* $N_{j-1} \to N_{j-1}/N_j$ *with connection* $\mathcal{A}_{N_j}$.

The map $a_{(N_j,N_{j+1})}$ *is related directly with the curvature of the connection* $\mathcal{A}_{N_{j+1}}$ *due to* $-\mathcal{A}_{N_{j+1}}(e)\left([\eta^{\mathcal{A}_{N_{j+1}}}, \overline{\eta}^{\mathcal{A}_{N_{j+1}}}]\right) = B^{\mathcal{A}_{N_{j+1}}}(e)(\eta^{\mathcal{A}_{N_{j+1}}}(e), \overline{\eta}^{\mathcal{A}_{N_{j+1}}}(e))$.

Then we have the following formula for the Lie bracket over $\mathcal{G}$:

$$\left[\kappa \oplus \left(\bigoplus_{i=1}^{n} \eta^{(i,i+1)}\right), \overline{\kappa} \oplus \left(\bigoplus_{j=1}^{n} \overline{\eta}^{(j,j+1)}\right)\right] =$$

$$= [\kappa, \overline{\kappa}] \oplus \left(\bigoplus_{i=1}^{n}\left([\eta^{(i,i+1)}, \overline{\eta}^{(i,i+1)}] + \sum_{k=1}^{n}\left(b^{(i,i+1)}_{(N_0,N_1)}(\kappa, \overline{\eta}^{(k,k+1)}) - b^{(i,i+1)}_{(N_0,N_1)}(\overline{\kappa}, \eta^{(k,k+1)})\right) + \right.\right.$$
$$\sum_{j=2}^{i}\left(b^{(i,i+1)}_{(N_{j-1},N_j)}\left(\eta^{(j-1,j)}, \sum_{l=j}^{n} \overline{\eta}^{(l,l+1)}\right) - b^{(i,i+1)}_{(N_{j-1},N_j)}\left(\overline{\eta}^{(j-1,j)}, \sum_{p=j}^{n} \eta^{(p,p+1)}\right)\right) - a^{(i,i+1)}_{(N_0,N_1)}(\kappa, \overline{\kappa})$$
$$\left.\left. - \sum_{m=1}^{i-1} a^{(i,i+1)}_{(N_m,N_{m+1})}(\eta^{(m,m+1)}, \overline{\eta}^{(m,m+1)})\right)\right).$$

The notation $b^{(i,i+1)}_{(N_{j-1},N_j)}$ *indicates the component of* $b_{(N_{j-1},N_j)}$ *in* $\mathcal{N}_{(i,i+1)}$.

Corollary 2.2 *In particular, if* $n = 2$ *we have the expression*

$$\left[\kappa \oplus \eta^{(1,2)} \oplus \eta^{(2,3)}, \overline{\kappa} \oplus \overline{\eta}^{(1,2)} \oplus \overline{\eta}^{(2,3)}\right] = [\kappa, \overline{\kappa}] \oplus [\eta^{(1,2)}, \overline{\eta}^{(1,2)}] + b^{(1,2)}_{(N_0,N_1)}(\kappa, \overline{\eta}^{(1,2)}) + b^{(1,2)}_{(N_0,N_1)}(\kappa, \overline{\eta}^{(2,3)})$$

$-b^{(1,2)}_{(N_0,N_1)}(\overline{\kappa},\eta^{(1,2)}) - b^{(1,2)}_{(N_0,N_1)}(\overline{\kappa},\eta^{(2,3)}) - a^{(1,2)}_{(N_0,N_1)}(\kappa,\overline{\kappa}) \oplus [\eta^{(2,3)},\overline{\eta}^{(2,3)}] + b^{(2,3)}_{(N_0,N_1)}(\kappa,\overline{\eta}^{(1,2)})+$
$b^{(2,3)}_{(N_0,N_1)}(\kappa,\overline{\eta}^{(2,3)}) - b^{(2,3)}_{(N_0,N_1)}(\overline{\kappa},\eta^{(1,2)}) - b^{(2,3)}_{(N_0,N_1)}(\overline{\kappa},\eta^{(2,3)}) - a^{(2,3)}_{(N_0,N_1)}(\kappa,\overline{\kappa}) + b^{(2,3)}_{(N_1,N_2)}(\eta^{(1,2)},\overline{\eta}^{(2,3)})$

$$-b^{(2,3)}_{(N_1,N_2)}(\overline{\eta}^{(1,2)},\eta^{(2,3)}) - a^{(2,3)}_{(N_1,N_2)}(\eta^{(1,2)},\overline{\eta}^{(1,2)}).$$

3 Equations of motion by stages.

3.1 Euler-d'Alembert-Poincaré equations by stages.

Consider $\mathcal{G} \equiv \mathcal{K} \oplus \mathcal{N}_{(1,2)} \oplus ... \oplus \mathcal{N}_{(n-1,n)} \oplus \mathcal{N}_{(n,n+1)}$ and let $\mathcal{D}$ be a distribution over G that describes the kinematic restriction. Since $\mathcal{D}_e \subset \mathcal{G}$,

$$\mathcal{D} := \mathcal{D}_e \equiv \mathcal{D}_K \oplus \mathcal{D}_{(1,2)} \oplus ... \oplus \mathcal{D}_{(n-1,n)} \oplus \mathcal{D}_{(n,n+1)}$$

where $\mathcal{D}_K = \mathcal{D} \cap \mathcal{K}$ and $\mathcal{D}_{(i,i+1)} = \mathcal{D} \cap \mathcal{N}_{(i,i+1)}$, with $1 \leq i \leq n$.

Consider $\mathcal{S}_g = \mathcal{D}_g \cap \mathcal{V}_g$, being $\mathcal{V}$ the vertical distribution. In this case, $\mathcal{V}_g = T_gG$ and $\mathcal{S} = \mathcal{D}$. Suppose

$$\mathcal{S} := \mathcal{S}_e \equiv \mathcal{S}_K \oplus \mathcal{S}_{(1,2)} \oplus ... \oplus \mathcal{S}_{(n-1,n)} \oplus \mathcal{S}_{(n,n+1)}$$

where $\mathcal{S}_K = \mathcal{S} \cap \mathcal{K} = \mathcal{D}_K$ and $\mathcal{S}_{(i,i+1)} = \mathcal{S} \cap \mathcal{N}_{(i,i+1)} = \mathcal{D}_{(i,i+1)}$, for $1 \leq i \leq n$.

If $\mu = \dfrac{\partial l}{\partial v} = \alpha \oplus \left(\bigoplus_{j=1}^{n} \beta_{(j,j+1)} \right) \in \mathcal{S}^*$, using the formula for the Lie bracket by stages,

we obtain the Euler-d'Alembert-Poincaré equations as follows

$$\left\{ \begin{array}{rcl} \dot{\alpha} \mid_{\mathcal{S}_K} &=& \mathrm{ad}^*_{\kappa}\alpha, \\ \dot{\beta}_{(i,i+1)} \mid_{\mathcal{S}_{(i,i+1)}} &=& \mathrm{ad}^*_{\eta^{(i,i+1)}}\beta_{(i,i+1)} + \beta_{(i,i+1)}\left(b^{(i,i+1)}_{(N_0,N_1)}(\kappa,\ .\ )\right) + \beta_{(i,i+1)}\left(\displaystyle\sum_{j=2}^{i} b^{(i,i+1)}_{(N_{j-1},N_j)}(\eta^{(j-1,j)},\ .\ )\right), \\ 0 &=& \beta_{(i,i+1)}\left(b^{(i,i+1)}_{(N_0,N_1)}\left(\ .\ ,\displaystyle\sum_{k=1}^{n}\eta^{(k,k+1)}\right) + a^{(i,i+1)}_{(N_0,N_1)}(\kappa,\ .\ )\right) \mid_{\mathcal{S}_K}, \\ \alpha &=& \dfrac{\partial l}{\partial \kappa}, \\ \beta_{(i,i+1)} &=& \dfrac{\partial l}{\partial \eta^{(i,i+1)}}. \end{array} \right.$$

3.2 Local Lagrange-d'Alembert-Poincaré equations by stages.

Let $Q = X \times G$ be a trivial principal bundle with a left action. Consider $\mathcal{G} \equiv \mathcal{K} \oplus \mathcal{N}_{(1,2)} \oplus ... \oplus \mathcal{N}_{(n,n+1)}$ and $\widetilde{\mathcal{G}} \equiv X \times \mathcal{G}$. Let $\mathcal{D}$ be a distribution over G.

Suppose that $\mathcal{D} := \mathcal{D}_e \equiv \mathcal{D}_K \oplus \mathcal{D}_{(1,2)} \oplus ... \oplus \mathcal{D}_{(n,n+1)}$ where $\mathcal{D}_K = \mathcal{D} \cap \mathcal{K}$ and $\mathcal{D}_{(i,i+1)} = \mathcal{D} \cap \mathcal{N}_{(i,i+1)}$, with $1 \leq i \leq n$.

Consider also $\mathcal{S}_g = \mathcal{D}_g \cap \mathcal{V}_g$ where $\mathcal{V}$ is the vertical distribution, and suppose $\mathcal{S} := \mathcal{S}_e \equiv \mathcal{S}_K \oplus \mathcal{S}_{(1,2)} \oplus ... \oplus \mathcal{S}_{(n,n+1)}$ where $\mathcal{S}_K = \mathcal{S} \cap \mathcal{K}$ and $\mathcal{S}_{(i,i+1)} = \mathcal{S} \cap \mathcal{N}_{(i,i+1)}$, for $1 \leq i \leq n$.

Let $\xi = \kappa \oplus \left(\bigoplus_{i=1}^{n} \eta^{(i,i+1)} \right) \in \mathcal{G}$, then $\dfrac{\partial l}{\partial \overline{v}} \equiv \dfrac{\partial l}{\partial \xi} \in \mathcal{G}^*$ and suppose that $\dfrac{\partial l}{\partial \xi} = \dfrac{\partial l}{\partial \kappa} \oplus \left(\bigoplus_{j=1}^{n} \dfrac{\partial l}{\partial \eta^{(j,j+1)}} \right) = \alpha \oplus \left(\bigoplus_{j=1}^{n} \beta_{(j,j+1)} \right)$, where $\alpha \in \mathcal{G}^*$ and $\beta_{(j,j+1)} \in \mathcal{N}^*_{(j,j+1)}$.

Since $\mathcal{A}(x)\dot{x} \in \mathcal{G}$, it has a decomposition $\mathcal{A}(x)\dot{x} = \mathcal{A}^K(x, \dot{x}) \oplus \left(\bigoplus_{i=1}^{n} \mathcal{A}^{(i,i+1)}(x, \dot{x}) \right)$ with $\mathcal{A}^K(x, \dot{x}) \in \mathcal{K}$ and $\mathcal{A}^{(i,i+1)}(x, \dot{x}) \in \mathcal{N}_{(i,i+1)}$.

Also consider $B(x, e)((\dot{x}, 0), (\delta x, 0)) = \psi \oplus \left(\bigoplus_{k=1}^{n} \varphi^{(k,k+1)} \right) \in \mathcal{G}$.

Then, the local Lagrange-d'Alembert-Poincaré equations

$$\begin{cases} \dfrac{d}{dt}\dfrac{\partial l}{\partial \xi} = \mathrm{ad}^*_{\xi}\dfrac{\partial l}{\partial \xi} - \mathrm{ad}^*_{\mathcal{A}(x)\dot{x}}\dfrac{\partial l}{\partial \xi} \\ \dfrac{\partial l}{\partial x}\ \delta x - \dfrac{d}{dt}\dfrac{\partial l}{\partial \dot{x}}\ \delta x = \dfrac{\partial l}{\partial \xi}\widetilde{B}(x)(\dot{x}, \delta x) - \dfrac{\partial l}{\partial \xi}[\mathcal{A}(x)\delta x, \xi] \end{cases},$$

by stages, and in the case of a nonholonomic system are

$$
\begin{aligned}
\dot\alpha \mid_{\mathcal{S}_K} &= \mathrm{ad}^*_{\kappa-\mathcal{A}^K(x,\dot x)}\alpha = \langle \alpha, [\kappa - \mathcal{A}^K(x,\dot x),\ .\ ]\rangle; \\
\dot\beta_{(i,i+1)} \mid_{\mathcal{S}_(i,i+1)} &= \Big\langle \beta_{(i,i+1)}, [\eta^{(i,i+1)} - \mathcal{A}^{(i,i+1)}(x,\dot x),\ .\ ] + b^{(i,i+1)}_{(N_0,N_1)}(\kappa - \mathcal{A}^K(x,\dot x),\ .\ ) \\
&\quad - \sum_{j=2}^{i} b^{(i,i+1)}_{(N_{j-1},N_j)}(\eta^{(j-1,j)} - \mathcal{A}^{(j-1,j)}(x,\dot x),\ .\ )\Big\rangle; \\
\frac{\partial l}{\partial x}\,\delta x - \frac{d}{dt}\frac{\partial l}{\partial \dot x}\,\delta x &= \langle \alpha, \psi - [\mathcal{A}^K(x,\delta x),\kappa]\rangle \oplus \left(\bigoplus_{i=1}^{n}\Big\langle \beta_{(i,i+1)}, \varphi^{(i,i+1)} - [\mathcal{A}^{(i,i+1)}(x,\delta x), \eta^{(i,i+1)}]\right. \\
&\quad - \sum_{k=1}^{n}(b^{(i,i+1)}_{(N_0,N_1)}(\mathcal{A}^K(x,\delta x),\eta^{(k,k+1)}) - b^{(i,i+1)}_{(N_0,N_1)}(\kappa, \mathcal{A}^{(k,k+1)}(x,\delta x))) \\
&\quad - \sum_{j=2}^{i}\left(b^{(i,i+1)}_{(N_{j-1},N_j)}\left(\mathcal{A}^{(j-1,j)}(x,\delta x), \sum_{l=j}^{n}\eta^{(l,l+1)}\right)\right. \\
&\quad \left.-b^{(i,i+1)}_{(N_{j-1},N_j)}\left(\eta^{(j-1,j)}, \sum_{p=j}^{n}\mathcal{A}^{(p,p+1)}(x,\delta x)\right)\right) + \\
&\quad \left. a^{(i,i+1)}_{(N_0,N_1)}(\mathcal{A}^K(x,\delta x),\kappa) + \sum_{m=1}^{i-1} a^{(i,i+1)}_{(N_m,N_{m+1})}(\mathcal{A}^{(m,m+1)}(x,\delta x),\eta^{(m,m+1)})\right)\Big\rangle; \\
0 &= \Big\langle \beta_{(i,i+1)}, -b^{(i,i+1)}_{(N_0,N_1)}\left(\ .\ , \sum_{k=1}^{n}\left(\eta^{(k,k+1)} - \mathcal{A}^{(k,k+1)}(x,\dot x)\right)\right) \\
&\quad - a^{(i,i+1)}_{(N_0,N_1)}(\kappa - \mathcal{A}^K(x,\dot x),\ .\ )\Big\rangle \mid_{\mathcal{S}_K};
\end{aligned}
$$

where $\alpha = \dfrac{\partial l}{\partial \kappa}$ and $\beta_{(i,i+1)} = \dfrac{\partial l}{\partial \eta^{(i,i+1)}}$.

References

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