Title: Dynamics of vortices on surfaces

Jair Koiller, Fundação Getulio Vargas, Rio de Janeiro

This is joint work with Stefanella Boatto. We consider N point vortices s_j of strengths κ_j moving on a closed (compact, boundaryless, orientable) surface S with riemannian metric g. Green's function $G_g(s, s_o)$ of the Laplace-Beltrami operator $\Delta_g = \mathrm{div}g \circ \mathrm{grad}_g$ is the stream function produced by a unit point vortex at $s_o \in S$ on a background uniform counter vorticity field. It behaves as $\log d(s, s_o)/2\pi$ near s_o. The energy core argument shows that vortex s_o drifts according to the Hamiltonian system (Ω_g, R_g), where Ω_g is the area form of g and R_g is Robins's function $R_g(s_o) = \lim_{s\to s_o} G_g(s, s_o) - \log d(s, s_o)/2\pi$. If S has genus zero, it it known that $R_g = \Delta_g^{-1}K + \mathrm{trace}\Delta_g^{-1}/A(S)$. The collective motion is governed by the Hamiltonian system

$$\Omega_{\text{collective}} = \sum_{j=1}^{N} \kappa_J \Omega(s_j) \ \ , \ \ H = \sum_{1\le i<j\le N} \kappa_i\kappa_j G_g(s_i, s_j) + \sum_{\ell=1}^{N} \frac{1}{2}\kappa_\ell^2 R_g(s_\ell) \ .$$

For Jordan domains $D \subset S$ the structure is the same, using the appropriate hydrodynamical Green function. The extension for vortices with mass is also immediate. Under conformal changes of metrics $\tilde{g} = h^2 g$ the symplectic form changes accordingly to $\Omega_{\tilde{g}} = h^2\Omega_g$ while the new Hamiltonian is given by

$$\tilde{H} = H - \frac{1}{4\pi}\sum_{\ell=1}^{N} \kappa_\ell^2 \log(h(s_\ell)) - \frac{\kappa}{\tilde{A}(S)}\sum_{\ell=1}^{N} \kappa_\ell \Delta^{-1} h^2(s_\ell) \ \ , \ \ \kappa = \sum_{\ell=1}^{N} \kappa_\ell \ .$$

The presence of the total vorticity in the conformal change formula reflects the fact that when the sum of the vorticities vanish, the collective vortex stream function $\psi(s; s_1, ..., s_N)$ is independent of the conformal metric $\tilde{g} = \exp(2\phi)g$. In this case the $\tilde{g}$-regularization of ψ at any of the vortices simplifies: one just subtracts off $\frac{1}{2\pi}\phi(s_j)$ from the g-regularized stream function at s_j. In particular, when S is conformal to the standard sphere, making an artificial puncture at any given $s^* \in S$ allows to easily write vortex motions for any metric on the sphere. We give a simple proof of Kimura's conjecture that a dipole describes geodesic motion, thus searching for integrable vortex pairs systems is in order. The vortex pair system on a triaxial ellipsoid extends Jacobi's geodesics. Is it Arnold-Liouville integrable? Not in our wildest dreams is another possibility: that quantizing a vortex system could relate with a million dollars worth question, Riemann's conjecture.