Geometric Discretization and Motion Planning of Nonholonomic Systems with Symmetries

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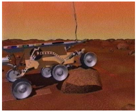
December 16, 2008

A robotic aerial vehicle example

Autonomous helicopter flying among buildings

Motivation: autonomous vehicles in natural environments

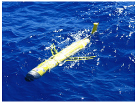
DARPA Challenges



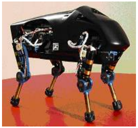
JPL Rover

BigDog

USC RESL Boat

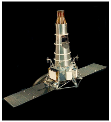


SLOCUM glider

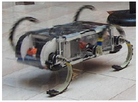


LittleDog

USC RESL Heli



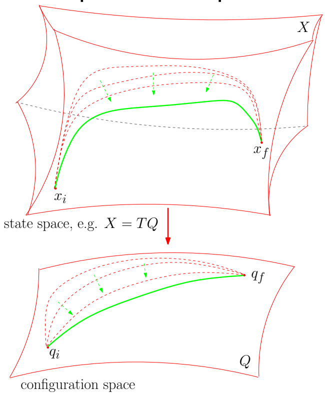
Satellite



RHex Robot

A more abstract view

Optimizing a trajectory in a complex state space



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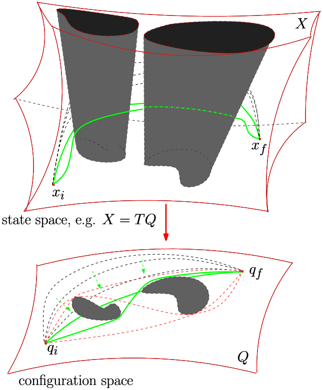

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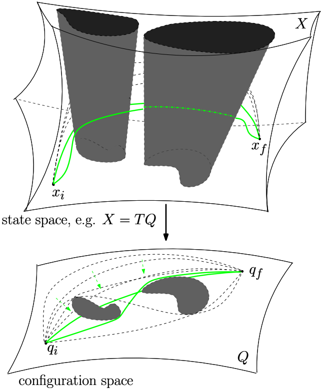

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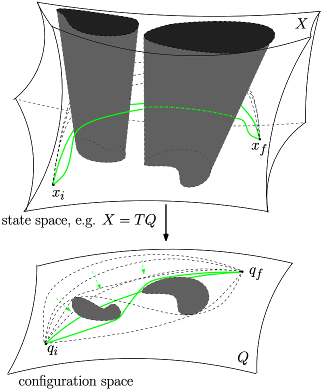
Optimizing a trajectory in a complex state space

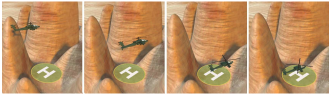


Key Points

- Trajectory numerical representation: accuracy and efficiency
  - geometric discretization
  - variational integrators
- Optimal control
  - discrete necessary conditions
  - local optimality
- Global solution among multiple homotopy classes
  - global state-space exploration
  - optimal motion primitives
  - dynamic programming

Framework for integration and control of vehicles

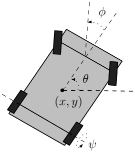
- Preview of some results: examples of computed motions



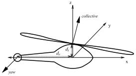
Helicopter - optimal landing

Multiple vehicles in an urban canyon

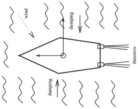
- Example developed models:



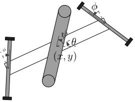
Car



Helicopter



Boat



Snakeboard

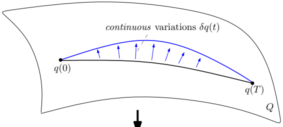
Outline

Outline

Outline

Continuous vs. Discrete Mechanics

Continuous Mechanics



continuous variational principle

$$\delta \int_0^T L(q,\dot{q})dt + \int_0^T f \cdot \delta q dt = 0$$

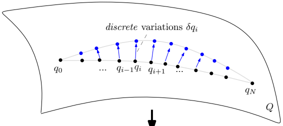
Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = f$$

discretization

finite differences
stadard ODE integrators

Discrete Mechanics



discrete variational principle

$$\delta \sum_{k=0}^{N-1} L_d(q_k, q_{k+1}) + \sum_{k=0}^{N} f_k \cdot \delta q_k dt = 0$$

$$L_d(q_k, q_{k+1}) = hL\left(\frac{q_k + q_{k+1}}{2}\ \frac{q_{k+1} - q_k}{h}\right)$$

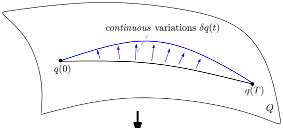
discrete Euler-Lagrange equations

$$\frac{\partial L_d}{\partial q_k}(q_{k-1}, q_k) - \frac{\partial L_d}{\partial q_k}(q_k, q_{k+1}) = h^2 f_k$$

variational integrator
used directly for computation

Continuous vs. Discrete Mechanics

Continuous Mechanics



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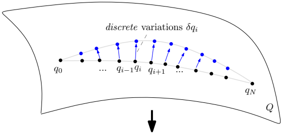

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discrete Euler-Lagrange equations

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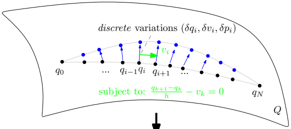
variational integrator
used directly for computation

Discrete Mechanics
(Pontryagin-D'Alembert)



discrete variational principle

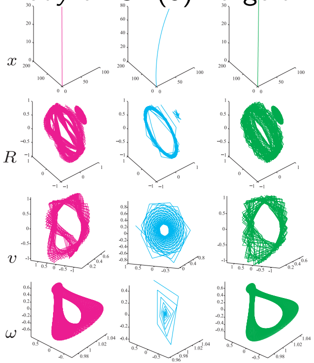
$$\delta \sum_{k=0}^{N-1} L(q_k,v_k) + \left\langle p_k, \frac{q_{k+1}-q_k}{h} - v_k \right\rangle + \sum_{k=0}^{N} f_k\cdot \delta q_k dt = 0$$

discrete Euler-Lagrange equations

$q_{k+1} = q_k + hv_k$	reconstruction
$p_k = \frac{\partial L}{\partial v}(q_k,v_k)$	Legendre transform
$p_k - p_{k-1} = h\frac{\partial L}{\partial q}(q_k,v_k) + hf_k$	

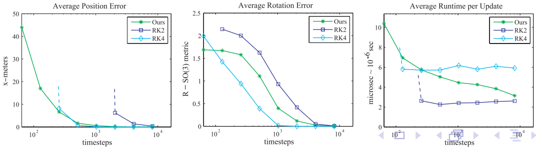
Superior numerics of discrete geometric integrators

- Orbits of the *rigid Body* on *SE*(3)- large time-steps



Ground truth   Runge-Kutta 4   variational integrator

- Accuracy and efficiency vs. resolution



Nonholonimic Integrators

- Comparisons: Accuracy and efficiency vs. resolution

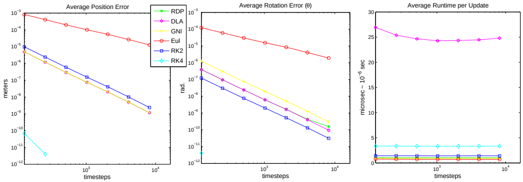


Figure: Snakeboard 10 second trajectories

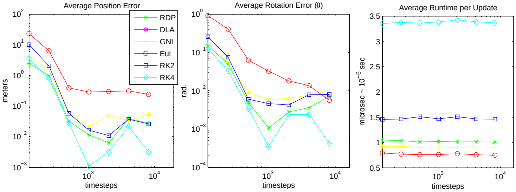


Figure: Snakeboard 10 minute trajectories

- *DLA*: Discrete Lagrange-d'Alembert (Cortez, 2002)
- *GNI*: Geometric Nonholonomic Integrator (Ferraro, Iglesias, De Diego, 2007)
- *RDP*: Reduced d'Alembert-Pontryagin (Kobilarov, 2007)

A typical system setup

- configuration space $Q = M \times G$, configuration $q \in Q$
- M – *shape space*, e.g. joint angles
- G – *Lie group*, e.g. $SE(3)$ denoting the system *pose*
- *nonholonomic constraints* $\dot{q} \in \mathcal{D}$, distribution $\mathcal{D} \subset TQ$
- *symmetries* associated with group transformations
- *external forces*, e.g. gravity, friction

State Space Structure

Principle bundle $\pi : Q \to Q/G$; distribution $\mathcal{D}_q \subset T_qQ$, $q \in Q$.

$$\mathcal{V}_q = T_q \,\mathrm{Orb}(q), \quad \mathcal{S}_q = \mathcal{D}_q \cap \mathcal{V}_q, \quad \mathcal{D}_q = \mathcal{S}_q \oplus \mathcal{H}_q.$$

- $\mathcal{V}_q$: space of tangent vectors parallel to symmetry directions, i.e. the *vertical space*
- $\mathcal{S}_q$: space of symmetry directions that satisfy the constraints (generated by $\mathfrak{s}_q = \{\xi \in \mathfrak{g} \mid \xi_Q(q) \in \mathcal{S}_q\} \subset T_qQ/G$)
- $\mathcal{H}_q$: space of tangent vectors that satisfy the constraints but are not aligned with any directions of symmetry, i.e. the *horizontal space*

Nonholonomic Connection

A principle connection $\mathcal{A} : TQ \to \mathfrak{g}$ with horizontal distribution $\mathcal{H}_q$.

Nonholonomic Connection (*Bloch* 2003; *Cendra, Marsden*, 2001)

Constructed as $\mathcal{A} = \mathcal{A}^{\mathrm{kin}} + \mathcal{A}^{\mathrm{sym}}$,
$\mathcal{A}^{\mathrm{kin}}$ is the kinematic, $\mathcal{A}^{\mathrm{sym}}$ is the mechanical connection

$$g^{-1}\dot{g} + \mathcal{A}(r)\dot{r} = \Omega,$$

defining vertical and horizontal velocity components

$$\dot{q} = \mathrm{ver}_r\, \dot{q} + \mathrm{hor}_r\, \dot{q} \Leftrightarrow (\dot{r}, g^{-1}\dot{g})_r = (0, \Omega) + (\dot{r}, -\mathcal{A}(r)\dot{r}),$$

where $\Omega \in \mathfrak{s}_r$ is the *locked angular velocity*.

Vertical Variations $(\delta r, \delta g)$

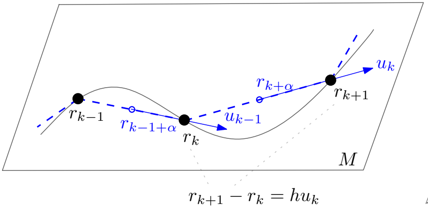
Variations such that $\delta r = 0$ and $\delta g g^{-1} = \mathcal{A}(r, g) \cdot (\delta r, \delta g) \in \mathfrak{s}_r$

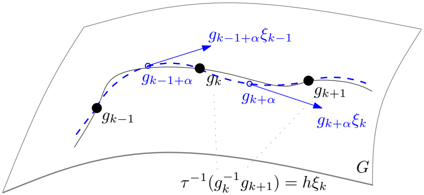
Horizontal Variations $(\delta r, \delta g)$

Variations such that $\mathcal{A}(r, g) \cdot (\delta r, \delta g) = 0$, or $(\delta r, g^{-1}\delta g) = (\delta r, -A(r)\delta r) \in (TM \times \mathfrak{g})_r$

Discrete Trajectory

Pick coordinates $(r, g) \in M \times G$





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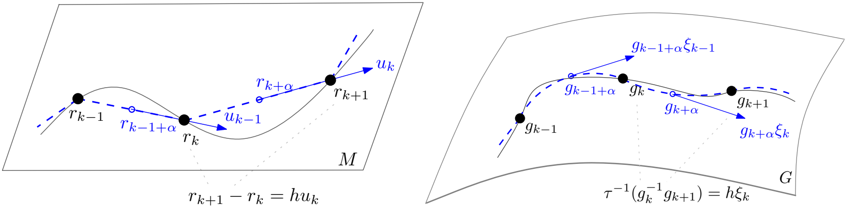

Discrete path
$(r, u, p, g, \Omega, \mu)_d : \{t_k\}_{k=0}^N \longrightarrow (TM \oplus T^*M) \times G \times \mathfrak{s} \times \mathfrak{g}^*$
subject to the constraints

$$r_{k+1} - r_k = hu_k, \qquad \tau^{-1}(g_k^{-1}g_{k+1}) = h\xi_k,$$

where $\xi_k = \Omega_k - \mathcal{A}(r_{k+\alpha})u_k$, with $r_{k+\alpha} := (1-\alpha)r_k + \alpha r_{k+1}$ for a chosen $\alpha \in [0, 1]$ and the map $\tau : \mathfrak{g} \to G$ represents the *difference* between two configurations in the group

Lagrange-D'Alembert-Pontryagin Nonholonomic Principle

Discrete Reduced LDAP Principle

Denoting $\xi_k := \Omega_k - \mathcal{A}(r_{k+\alpha})u_k$:

$$\delta \sum_{k=0}^{N-1} h\,[\ell(r_{k+\alpha}, u_k, \xi_k) + \langle p_k, (r_{k+1} - r_k)/h - u_k\rangle$$

$$+\langle \mu_k, \tau^{-1}(g_k^{-1} g_{k+1})/h - \xi_k\rangle] + \sum_{k=0}^{N-1} [h\langle f_{k+\alpha}, \delta r_{k+\alpha}\rangle] = 0,$$

subject to:

vertical variations $(\delta r_k, g_k^{-1}\delta g_k) = (0, \eta_k), \eta_k \in \mathfrak{s}_{r_k}$

horizontal variations $(\delta r_k, g_k^{-1}\delta g_k) = (\delta r_k, -\mathcal{A}(r_k)\delta r_k)$,

$\ell(r, \dot{r}, \xi) = L(r, \dot{r}, e, g^{-1}\dot{g})$: the *reduced Lagrangian*

Discrete Equations of Motion

$$
\begin{aligned}
&g_k^{-1} g_{k+1} = \tau(h(\Omega_k - \mathcal{A}(r_{k+\alpha})u_k)), \\
&r_{k+1} - r_k = h u_k, \\
&\mu_k = \frac{\partial \ell_{k+\alpha}}{\partial \Omega}, \\
&\langle \mathcal{DEP}_\tau(k), e_b(r_k) \rangle = 0, \\
&\left(\frac{\partial \ell_{k+\alpha}}{\partial u} - \frac{\partial \ell_{k-1+\alpha}}{\partial u} \right) - h \left(\alpha \frac{\partial \ell_{k-1+\alpha}}{\partial r} + (1-\alpha) \frac{\partial \ell_{k+\alpha}}{\partial r} \right) \\
&= \mathcal{A}(r_k)^* \, \mathcal{DEP}_\tau(k) + h \left(\alpha f_{k-1+\alpha} + (1-\alpha) f_{k+\alpha} \right),
\end{aligned}
$$

where the *discrete Euler-Poincaré* operator $\mathcal{DEP}_\tau$ is defined as

$$
\mathcal{DEP}_\tau(k) := (\mathrm{d}\tau^{-1}_{h(\Omega_k - \mathcal{A}(r_{k+\alpha})u_k)})^* \mu_k - (\mathrm{d}\tau^{-1}_{-h(\Omega_{k-1} - \mathcal{A}(r_{k-1+\alpha})u_{k-1})})^* \mu_{k-1}
$$

Disrete Euler-Poincare equations

- the unconstrained case $Q = G$

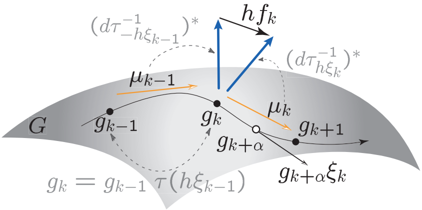

Figure: Tangent maps $d\tau^{-1}$ transforming momenta

Continuous	Discrete
$\dot{\mu} = \mathrm{ad}^*_\xi\,\mu + f$	$(\mathrm{d}\tau^{-1}_{h\xi_k})^*\mu_k - (\mathrm{d}\tau^{-1}_{-h\xi_{k-1}})^*\mu_{k-1} = hf_k$

Implementation

Simple matrix operations. Example: $G = SE(2)$, $\tau = \mathrm{cay}$

$$\mathrm{cay}(\widehat{v}) = \begin{bmatrix} \frac{1}{4+(v^1)^2} \begin{bmatrix} (v^1)^2 - 4 & -4v^1 & -2v^1v^3 + 4v^2 \\ 4v^1 & (v^1)^2 - 4 & 2v^1v^2 + 4v^3 \end{bmatrix} \\ \begin{matrix} 0 & \qquad\quad 0 & \qquad\quad 1 \end{matrix} \end{bmatrix}$$

The maps $[\mathrm{d}\tau_{\xi}^{-1}]$ can be expressed as the 3×3 matrices:

$$[\mathrm{dcay}_{\widehat{v}}^{-1}] = \mathbf{I}_3 - \frac{1}{2}[\mathrm{ad}_v] + \frac{1}{4}\begin{bmatrix} v^1 \cdot v & \mathbf{0}_{3\times 2} \end{bmatrix}$$

where

$$[\mathrm{ad}_v] = \begin{bmatrix} 0 & 0 & 0 \\ v^3 & 0 & -v^1 \\ -v^2 & v^1 & 0 \end{bmatrix}.$$

Note: a general method for any matrix group is also available

Discrete Nonholonomic Momentum Map

- Define the *local* discrete momentum map $\mathsf{J}^{\mathrm{loc}} : TM \times \mathfrak{g} \to \mathfrak{g}^*$

$$\mathsf{J}^{\mathrm{loc}}(r_k, u_k, \xi_k) = (\mathrm{d}\tau^{-1}_{h\xi_k})^* \mu_k, \qquad \text{where } \mu_k = \frac{\partial \ell}{\partial \xi}(r_k + \alpha u_k, u_k, \xi_k),$$

and the *spatial* discrete momentum map $\mathsf{J} : TQ \to \mathfrak{g}^*$ through

$$\mathsf{J}(r_k, u_k, g_k, v_k) := \mathrm{Ad}^*_{g_k^{-1}} \mathsf{J}^{\mathrm{loc}}(r_k, u_k, g_k^{-1} v_k),$$

where $(r_k, u_k) \in TM$ and $(g_k, v_k) \in TG$.

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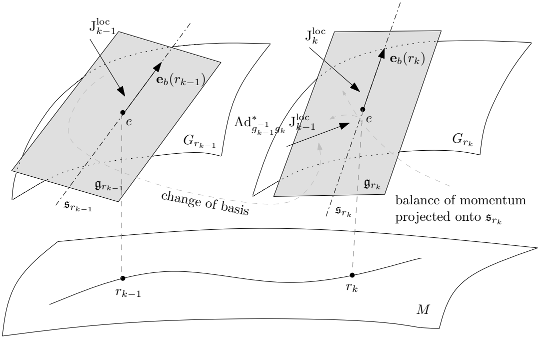
  where $(r_k, u_k) \in TM$ and $(g_k, v_k) \in TG$.
- The *momentum components* $\mathsf{J}^{\mathrm{nh}}_b(r_k, u_k, g_k, v_k)$ at point k along the basis elements $e_b : Q \to \mathfrak{s}$ are

  $$\begin{aligned} \mathsf{J}^{\mathrm{nh}}_b(r_k, u_k, g_k, v_k) &= \langle \mathsf{J}(r_k, u_k, g_k, v_k), e_b(r_k, g_k) \rangle \\ &= \langle \mathsf{J}^{\mathrm{loc}}(r_k, u_k, g_k^{-1} v_k), e_b(r_k) \rangle. \end{aligned}$$

Discrete Momentum Map Evolution



Discrete Momentum Map Change

The momentum components J^{nh}_b evolve along discrete LDAP solution trajectories according to (denote $J(k) := J(r_k, u_k, g_k, v_k)$)

$$J^{nh}_b(k) - J^{nh}_b(k-1) = \langle J(k-1), e_b(r_k, g_k) - e_b(r_{k-1}, g_{k-1}) \rangle.$$

* consistent with previous results, e.g. *Cortes, 2001; Ferraro et. al. 2007*

Outline

Optimal Control

Goal: find an optimal trajectory to a desired state

- Compute the forces $f(t)$ such that the systems moves from $(q(0), \dot{q}(0))$ to $(q(T), \dot{q}(T))$ during a time interval $[0, T]$
- minimizing the cost function

$$J(q, f) = \int_0^T C(q(t), f(t))\mathrm{dt}, \tag{1}$$

e.g. minimum control effort: $C = \frac{1}{2}\|f\|^2$;
min. time: $C = 1$.

- subject to discrete equations of motion
- other constraints such as joint limits, obstacles, etc...

For clarity, consider the simpler case $Q = G$

Nonholonomic distribution $\mathfrak{h} \subset \mathfrak{g}$ (the *sub-Riemannian* case):

$$\text{velocity } \xi \in \mathfrak{h} = \text{span}\{X_1, ..., X_m\}, m < n, \quad \langle\langle X_i, X_j \rangle\rangle = \delta_{ij}$$

The dynamics satisfies

$$\begin{aligned}
&\langle (\mathrm{d}\tau^{-1}_{h\xi_k})^* \mu_k - (\mathrm{d}\tau^{-1}_{-h\xi_{k-1}})^* \mu_{k-1} - hf_k, X_i \rangle = 0, \quad i = 1, ..., m, \\
&\langle\langle \xi_k, X_i \rangle\rangle = 0, \quad\quad i = m+1, ..., n, \\
&\langle \mu_k, X_i \rangle = \begin{cases} \langle \mathbb{I}\, \xi_k, X_i \rangle, & i = 1, ..., m \\ 0, & i = m+1, ..., n \end{cases}, \\
&g_k^{-1} g_{k+1} = \tau(h\xi_k).
\end{aligned}$$

Necessary Conditions for Optimality

Define the Lagrangian multipliers $\eta_k \in \mathfrak{h}$, $\rho_k \in \mathfrak{h}^{\perp *}$, $\lambda_k \in \mathfrak{g}^*$ and and the *Hamiltonian* function

$$H_k := H(\xi_{k-1}, \xi_k, f_k, \eta_k) =$$

$$\langle (\mathrm{d}\tau^{-1}_{h\xi_k})^* \mathbb{I}\,\xi_k - (\mathrm{d}\tau^{-1}_{-h\xi_{k-1}})^* \mathbb{I}\,\xi_{k-1} - hf_k, \eta_k \rangle + \frac{h}{2}\|f_k\|^2,$$

and the *augmented* discrete cost function

$$J'_d(\xi_{0:N-1}, f_{0:N}, \zeta_{0:N}, \rho_{0:N-1}, \lambda_{0:N-1})$$

$$= \sum_{k=0}^{N} H_k + \sum_{k=0}^{N-1} \left(h\langle \rho_k, \xi_k \rangle + \langle \lambda_k, \tau^{-1}(g_k^{-1} g_{k+1}) - h\xi_k \rangle \right),$$

An optimal solution must satisfy

$$(\mathrm{d}\tau^{-1}_{h\xi_k})^* \lambda_k - (\mathrm{d}\tau^{-1}_{-h\xi_{k-1}})^* \lambda_{k-1} = 0,$$

$$\text{where } \lambda_k = \frac{\partial (H_k + H_{k+1})}{\partial \xi_k} + \rho_k = \frac{\partial \widetilde{H}_k}{\partial \xi_k} + \rho_k,$$

$$\widetilde{H}_k := -\langle (\mathrm{d}\tau^{-1}_{h\xi_k})^* \mathbb{I}\,\xi_k, \mathrm{Ad}_{\tau(h\xi_k)} \widetilde{f}^{\sharp}_{k+1} - \widetilde{f}^{\sharp}_k \rangle.$$

Indirect Optimal Control Formulation

An optimal trajectory (minimizing the control effort $\frac{h}{2}\sum_{k=0}^{N}\|\widetilde{f}_k\|^2$) satisfies

$$(\mathrm{d}\tau^{-1}_{h\xi_k})^*\lambda_k - (\mathrm{d}\tau^{-1}_{-h\xi_{k-1}})^*\lambda_{k-1} = 0, \quad k = 1, ..., N-1 \tag{2}$$

$$\tau^{-1}(\tau(h\xi_0)\cdots\tau(h\xi_{N-1})\cdot(g(0)^{-1}g(T))^{-1}) = 0, \tag{3}$$

where $\lambda_k \in \mathfrak{g}^*$is computed through

$$(\lambda_k)_i = \left\langle \mathbb{I}(\mathrm{d}\tau^{-1}_{h\xi_k}(\nu_k)) - h(\mathrm{d}\tau_{h\xi_k})^*\,\mathrm{ad}^*_{\left(\mathrm{Ad}_{\tau(h\xi_k)}\widetilde{f}^{\sharp}_{k+1}\right)}(\mathrm{d}\tau^{-1}_{h\xi_k})^*\,\mathbb{I}(\xi_k) + \rho_k, e^i \right\rangle$$
$$+ \left\langle \mathbb{I}(\xi_k), h\left(\mathrm{D}\,\mathrm{d}\tau^{-1}_{h\xi_k}\cdot e^i\right)(\nu_k)\right\rangle, \text{ where } \{e^i\} \text{ is the basis for } \mathfrak{g}$$

$$\nu_k = \mathrm{Ad}_{\tau(h\xi_k)}\widetilde{f}^{\sharp}_{k+1} - \widetilde{f}_k^{\sharp},$$

$$\xi_k \in \mathfrak{h},\ \rho_k \in \mathfrak{h}^{\perp *}.$$

Nn equations (2)-(3) in the Nn unknowns $\xi_{0:N-1}, \rho_{0:N-1}$ solved with standard root-finding

Outline

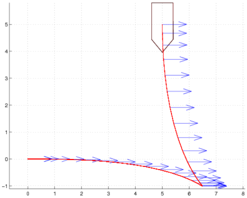
Simple boat model

- Group $G = SE(2)$ with coordinates $q = (\theta, x, y)$
- body fixed velocity $\xi \in \mathfrak{se}(2)$ defined by $\xi = (\omega, v, v^{\perp})$
- forces $f : SE(2) \times \mathfrak{se}(2) \rightarrow \mathfrak{se}(2)^*$ in the form

$$f(g, \xi) = -R(g, \xi)\xi + f_{ext}(g, \xi) + Bu,$$

where R is a damping matrix, f_{ext} are external forces due to wind or current, and $u = (u_r, u_l)$ are the thruster control inputs and B is

$$B = \begin{bmatrix} -c & c \\ 1 & 1 \\ 0 & 0 \end{bmatrix}.$$



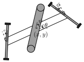
Boat station-keeping

RESL boat

Snakeboard

- $Q = SE(2) \times S \times S$, shape $r = (\psi, \phi)$, $G = SE(2)$ with coordinates (θ, x, y); distance l center-to-wheels, mass m, moments of inertia I and J.
- Constraint distribution:

$$\mathcal{D}_q = \text{span}\left\{\frac{\partial}{\partial\psi}, \frac{\partial}{\partial\phi}, c\frac{\partial}{\partial\theta} + a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right\},$$



where $a = -2l\cos\theta\cos^2\phi$, $b = -2l\sin\theta\cos^2\phi$, $c = \sin 2\phi$.

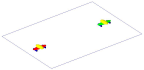
- Vertical space:

$$\mathcal{V}_q = \text{span}\left\{\frac{\partial}{\partial\theta}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\},$$

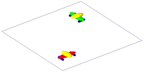
- Constrained symmetry space:

$$\mathcal{S}_q = \mathcal{V}_q \cap \mathcal{D}_q = \text{span}\left\{c\frac{\partial}{\partial\theta} + a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right\}.$$

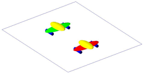
Optimal trajectories:



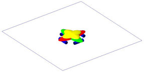
forward motion



90^o turn



parallel parking



axis turn

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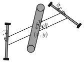

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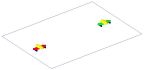

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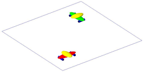

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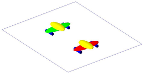

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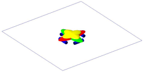

axis turn

Outline

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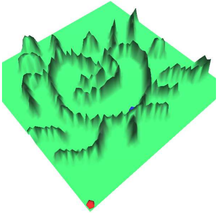
- Goal of this part:
  - extend DMOC to complex state-spaces cluttered with obstacles
  - find near globally optimal solution
  - guarantee efficiency

Outline

Finding global solution

- Example - what is the optimal motion in this complex terrain?



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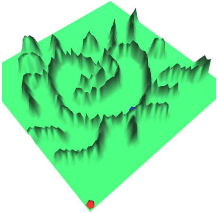

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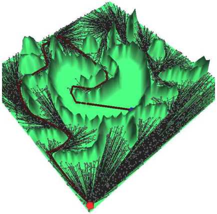

sampling-based roadmap

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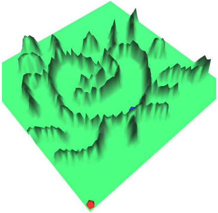

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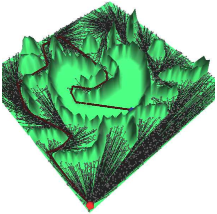

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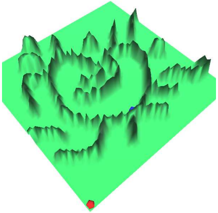

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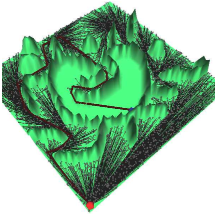

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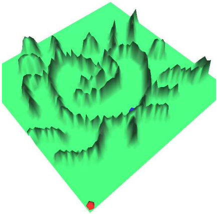

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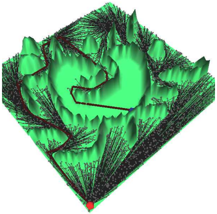

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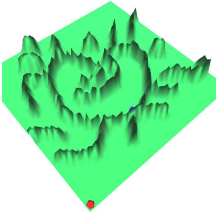

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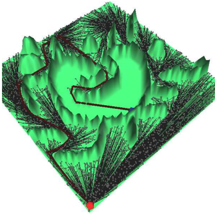

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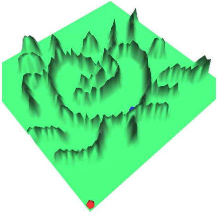

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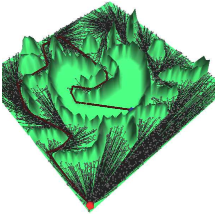

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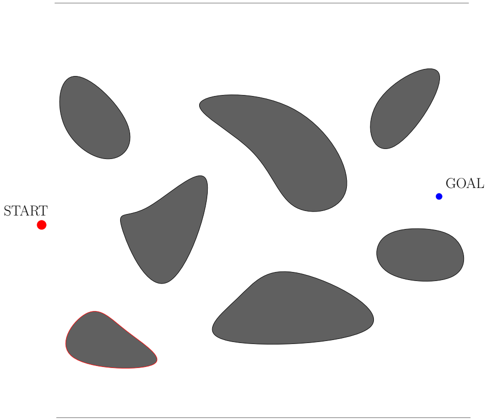
  - *global solution*, optimal with respect to the approximation

Combining DMOC with incremental roadmaps



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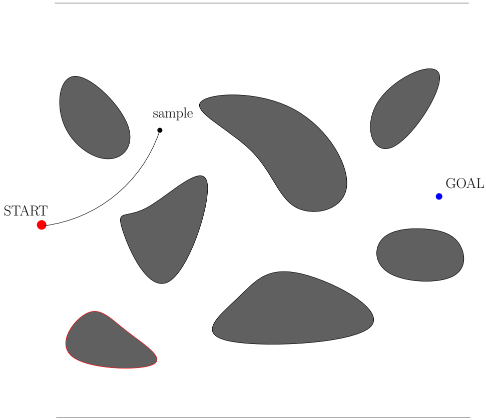

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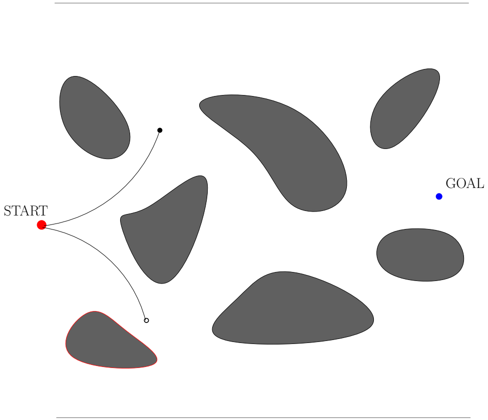

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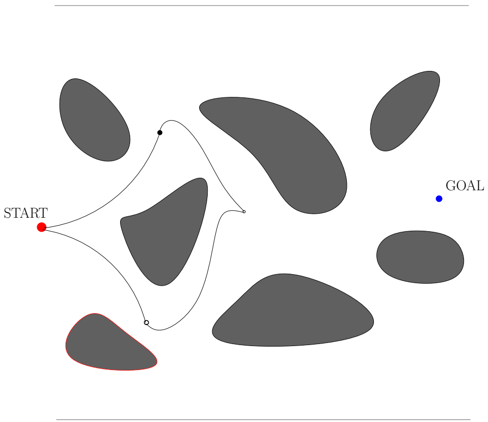

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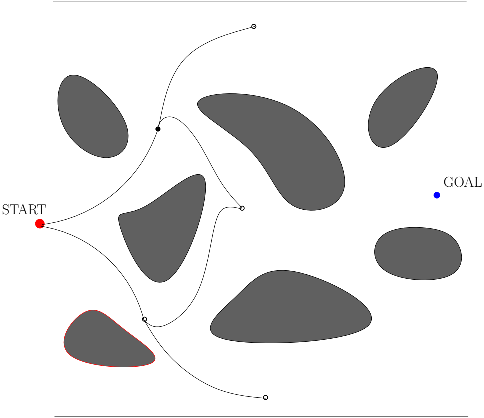

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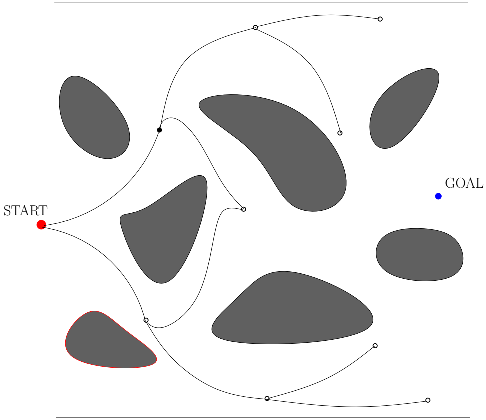

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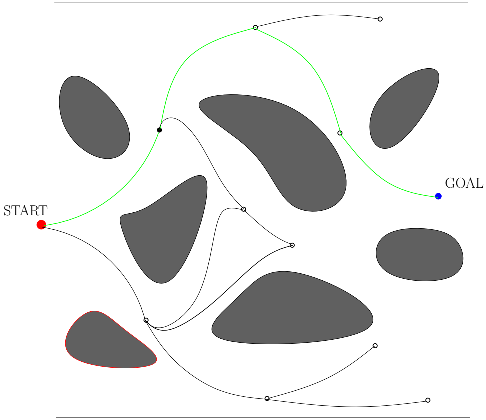

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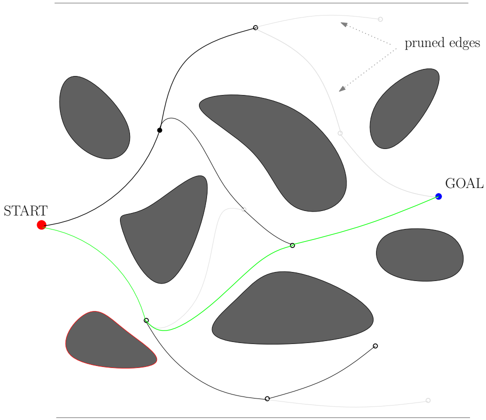

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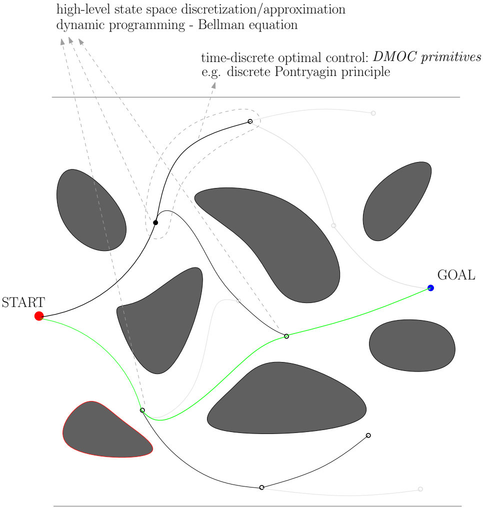

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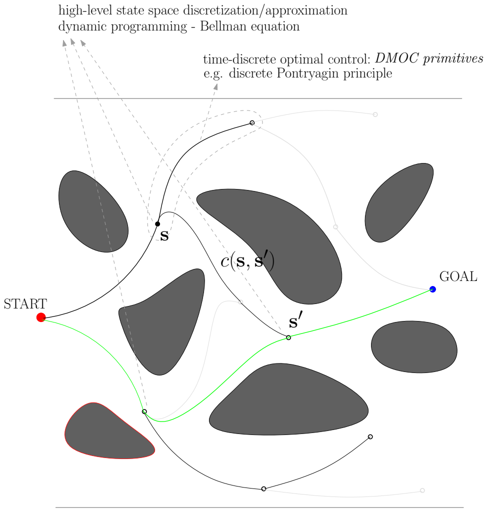
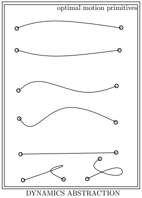
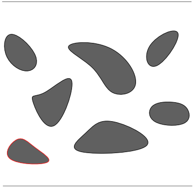


Bellman's principle: optimal cost $J^*(\mathbf{s}) = \min_{\mathbf{s}'}[J^*(\mathbf{s}') + c(\mathbf{s}, \mathbf{s}')]$,
where $J(\mathbf{s})$ - cost-to-go from $\mathbf{s}$ to the goal; $c(\mathbf{s}, \mathbf{s}')$ - cost b/n $\mathbf{s}$ and $\mathbf{s}'$.

Outline

Combining DMOC with incremental roadmaps





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  - simple optimized motions (precomputed)
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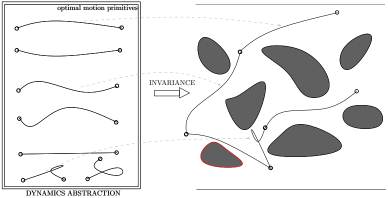

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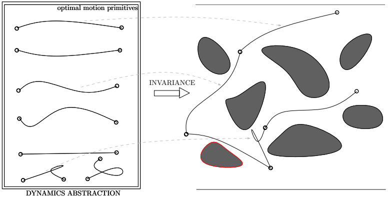

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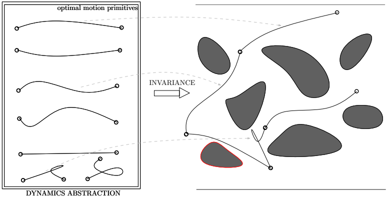

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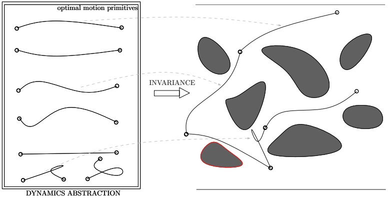

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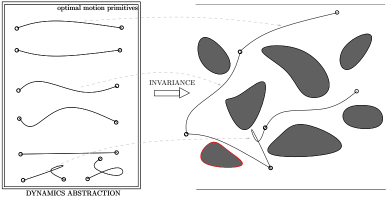

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  - the complex differential control problem reduced to a lower dimensional algebraic one

Frazzoli, Dahleh, Feron, *IEEE Trans. on Robotics and Automation, 2005.*

Primitive Invariance

Denote state-space by X, state $x \in X$, and control set U, $u \in U$, e.g. $X = TQ$, $x = (q, v)$

- The flow $\varphi : X \times \mathbb{R} \to X$ of a primitive is G-*invariant* i.e.

$$\Phi_g(\varphi(x_0, t)) = \varphi(\Phi_g(x_0), t), \quad \Phi_g \text{ is the group action with } g \in G$$

- Two primitives π_1 and π_2 are *equivalent*, if $\exists g, T$ s.t.

$$(x_1(t), u_1(t)) = (\Phi_g(x_2(t - T)), u_2(t - T)), \forall t \in [t_{i,1}, t_{f,1}]$$

- Two primitives π_1 and π_2 are *compatible* if $\exists g_{12} \in G$ s.t.

$$x_1(T_1) = \Phi(g_{12}, x_2(0))$$

For discrete DMOC trajectories use discrete flow $\varphi_d : X \times \mathbb{N} \to X$

$$\varphi_d(x_k, i) \approx \varphi(x(kh), ih), \qquad x_k \approx x(kh)$$

Types of Primitives

- *Trim Primitives*: continuously parametrized steady-state motions

$$\alpha : t \in [0, T] \to (x_\alpha(t), u_a(t))$$

along left invariant vector field $\xi_\alpha \in \mathfrak{g}$ with constant control inputs

$$x_\alpha(t) = \Phi(\exp(t\xi_\alpha), x_\alpha(0)), u_\alpha(t) = u_\alpha, \forall t \in [0, T].$$

The trim primitives are denoted

$$\alpha(\tau) : t \in [0, T] \to (\Phi(\exp(t\xi_\alpha), x_\alpha(0)), u_\alpha),$$

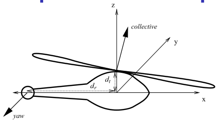
where τ is called coasting time, and the set of all such primitives defined by $\mathcal{T}_\alpha = \{\alpha(\tau), \tau \geq 0\}$.
The displacement of a trim primitive α with coasting time τ is simply $g_\alpha = \exp(\tau\xi_\alpha)$.

- *Maneuvers*: switches b/n steady-state motions. Therefore they are defined to be compatible form left and right with trim primitives. The set of maneuvers is denoted $\mathcal{M}(\mathcal{S}, G) \subseteq \mathcal{P}(\mathcal{S}, G)$. Formally, a maneuver π satisfies

$$\pi \in \mathcal{M}(\mathcal{S}, G) \Leftrightarrow \exists \alpha, \beta \in \mathcal{T}(\mathcal{S}, G) : \alpha\pi\beta \in \mathcal{P}(\mathcal{S}, G).$$

The displacement as a result of executing the maneuver is denoted g_π.

Example: helicopter



- Model – underactuated rigid body
  - *State*: orientatation $R \in SO(3)$, position $x \in \mathbb{R}^3$, angular velocity $\omega \in \mathbb{R}^3$, linear velocity $v \in \mathbb{R}^3$
  - *Controls*: collective u_c, yaw u_ψ, rotor forward pitch γ_p, rotor sideways roll γ_r with control input covectors

$$f^1(\gamma) = (d_t \sin\gamma_r, d_t \sin\gamma_p \cos\gamma_r, 0, \sin\gamma_p, \cos\gamma_p \cos\gamma_r),$$
$$f^2(\gamma) = (0, 0, d_r, 0, -1, 0).$$

  - gravity; bounds on velocity and controls
- Primitives: invariant under $G' = \mathrm{SO}(2) \times \mathbb{R}^3$

trim vector $\xi_\alpha \in \mathfrak{se}(3)$:

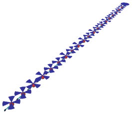
$$\xi_\alpha = \begin{bmatrix} 0 & -\omega_z & 0 & v_x \\ \omega_z & 0 & 0 & v_y \\ 0 & 0 & 0 & v_z \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

invariance conditions ($\dot{\xi} = 0$), with θ-pitch, ϕ-roll:

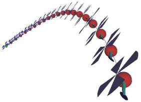
$$\gamma_p = 0, \gamma_r = 0, u_\psi = 0,$$
$$v_y \omega_z = -g \sin\theta,$$
$$-v_x \omega_z = g \cos\theta \sin\phi,$$
$$u_c = g \cos\theta \cos\phi.$$

Example: helicopter (cont.)

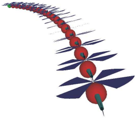
Trim Primitives:



α_1



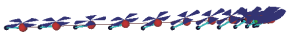
α_2



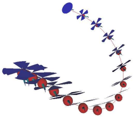
α_3

Maneuvers:

π_1



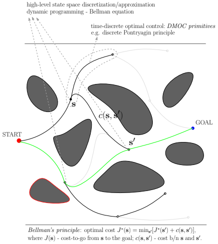
π_2



π_3

Outline

State space exploration

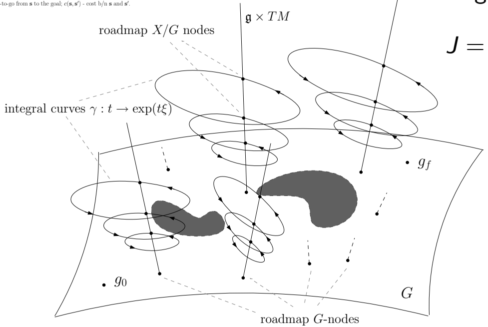


- State space approximation/discretization: sets of orbit of invariant vector fields $X \approx \{x_\alpha(t) \mid \Phi_g(\varphi(x_\alpha, t)) = \varphi(\Phi_g(x_\alpha), t)\}$
- Each graph node is a set of orbits attached at some $g \in G$.
- The Control Problem: Find $\{\pi_i, \alpha_i, \tau_i\}$ such that

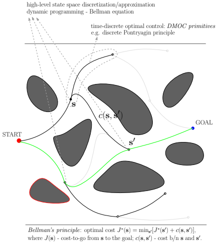
$$g_0^{-1} g_f = [\Pi_i \exp(\tau_i \xi_{\alpha_i}) g_{\pi_i}] \exp(\tau_N \xi_{\alpha_N})\}$$

minimizing total cost, for cost $c : \mathcal{C}_d \to \mathbb{R}$

$$J = \sum_i [(\tau_i c(\alpha_i) + c(\pi_i)] + \tau_N c(\alpha_N)$$



State space exploration



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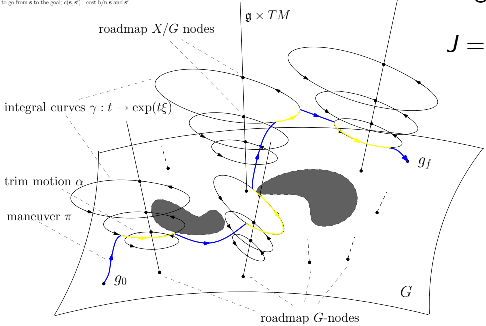
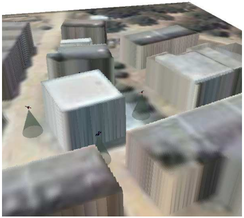


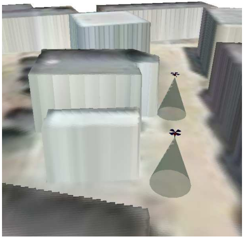
Example: helicopter

Part of the roadmap

topview

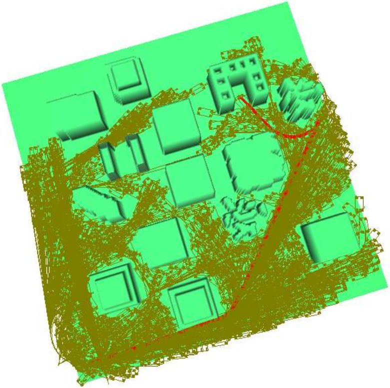


sideview



closeview

Example: car

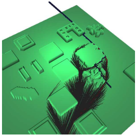


roadmap construction

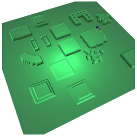
Outline

Roadmap extensions

- Goal with *time-dependent* dynamics



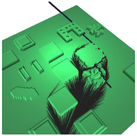
roadmap construction



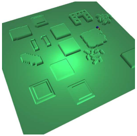
motion

Roadmap extensions

- Goal with *time-dependent* dynamics

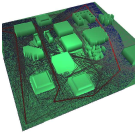


roadmap construction



motion

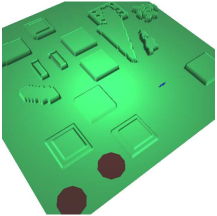
- Maximizing *coverage*



roadmap construction

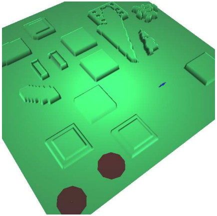
Goal with *uncertain dynamics*

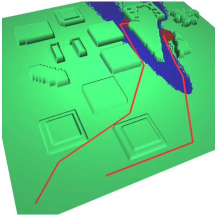
- goal distribution as particles
- goal heading north with uncertainty
- two vehicles with circular sensing radius



Goal with *uncertain dynamics*

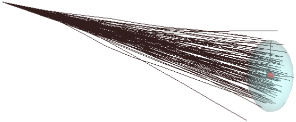
- goal distribution as particles
- goal heading north with uncertainty
- two vehicles with circular sensing radius
- vehicle controlled to gain maximum information about goal position



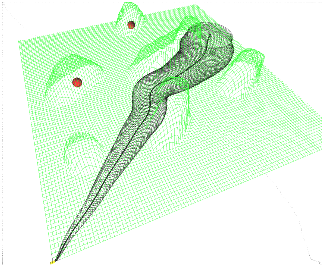


Control under uncertainty

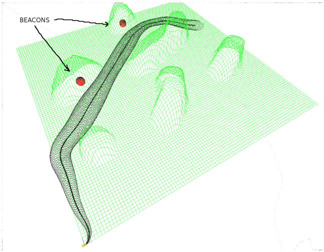
- Propagation of uncertainty



- Planning with Uncertainty: distance vs. uncertainty



Shortest Distance



Minimum Uncertainty

Issues / Directions

- Improve motion planning control in complex state spaces
- More insight into the structure / numerics of optimal control problems?
- Propagation of uncertainty / robustness to noise