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NO LINEALES HAMILTONIANOS
CON N GRADOS DE LIBERTAD

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CERTIFICA que la Tesis Doctoral

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De lo cual doy constancia a los efectos oportunos, en Burgos a 9 de junio de 2009.

Fdo.: Ángel Ballesteros Castañeda

Dedicado a mis padres y a Amanda

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Resumen

Presentamos a continuación un resumen de cada uno de los Capítulos que constituyen esta Memoria de Tesis Doctoral. Por motivos de simplicidad en la presentación, prescindiremos de las referencias bibliográficas correspondientes, que se detallarán adecuadamente en la versión inglesa.

Capítulo 1. Introducción

Tanto la Mecánica Clásica como la Mecánica Cuántica, encuentran su marco más natural dentro del formalismo hamiltoniano. De hecho, el enfoque más usual de un sistema desde el punto de vista físico consiste en formular la función hamiltoniana para el caso clásico o su correspondiente operador para el caso cuántico. Desde una perspectiva estrictamente matemática, el formalismo hamiltoniano está ligado a la estructura de álgebra de Lie, dando una idea unitaria de la simetrías subyacentes en Mecánica Clásica y en Mecánica Cuántica. De hecho, aunque en esta Memoria nos concentraremos en el estudio de sistemas clásicos, el enfoque algebraico de este trabajo permitiría la formulación cuántica de los resultados aquí presentados.

Consideremos un sistema de partículas clásicas con N grados de libertad. A partir del hamiltoniano del sistema, $\mathcal{H}^{(N)}(q_1, \dots, q_N, p_1, \dots, p_N)$ escrito en función de las variables canónicas, las ecuaciones de Hamilton:

$$\dot{q}_i = \frac{\partial \mathcal{H}^{(N)}}{\partial p_i} \quad \dot{p}_i = -\frac{\partial \mathcal{H}^{(N)}}{\partial q_i} \quad i = 1, \dots, N \quad (1)$$

nos suministran toda la información dinámica del sistema. El problema fundamental es, por tanto, encontrar la solución de ese sistema de ecuaciones diferenciales. En general, esto no es sencillo y en ocasiones ni siquiera posible, ya que los sistemas físicamente interesantes normalmente dan origen a un sistema de ecuaciones no lineales acopladas.

Por ello, de entre todos los sistemas hamiltonianos, podemos extraer un conjunto muy interesante denominado ‘sistemas exactamente o completamente resolubles’: es decir, aquéllos para los cuales las ecuaciones de Hamilton pueden ser resueltas de manera exacta. El problema es que el número de sistemas que verifican esta propiedad es muy pequeño y, normalmente, la existencia de soluciones exactas está profundamente ligada a existencia de simetrías subyacentes del modelo en estudio.

Esta Memoria de Tesis Doctoral se ocupa de la construcción de nuevos sistemas hamiltonianos exactamente resolubles. La piedra angular de nuestro enfoque radica en la siguiente afirmación: si para un sistema hamiltoniano dado tenemos un número ‘suficiente’ de constantes del movimiento independientes, entonces las ecuaciones de Hamilton pueden ser integradas mediante ‘cuadraturas’. Esto significa que las soluciones del sistema pueden ser obtenidas mediante un número finito de operaciones algebraicas (incluyendo inversión de funciones) y cuadraturas, es decir, la evaluación de integrales de ciertas funciones dadas. Este resultado, reformulado en un contexto más técnico quiere decir:

Sea $\mathcal{R}^{2N} = (q_1, \dots, q_N, p_1, \dots, p_N)$ el espacio de fases de un sistema hamiltoniano descrito mediante la función $\mathcal{H}^{(N)}(q_1, \dots, q_N, p_1, \dots, p_N)$, junto con la definición habitual del corchete de Poisson

$$\{f, g\} = \sum_{i=1}^N \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i} \frac{\partial f}{\partial p_i} \right). \quad (2)$$

Si el sistema anterior tiene $(N - 1)$ integrales del movimiento funcionalmente independientes $(\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_{N-1})$ y en involución,

$$\{\mathcal{H}, \mathcal{F}_j\} = 0 \quad \{\mathcal{F}_i, \mathcal{F}_j\} = 0 \quad i, j = 1, \dots, (N - 1) \quad (3)$$

entonces las ecuaciones de Hamilton del sistema pueden resolverse mediante cuadraturas.

De hecho, un sistema hamiltoniano $\mathcal{H}^{(N)}$ con un número arbitrario, pero finito, de grados de libertad se dice que es *completamente integrable* (en el sentido de Liouville, anteriormente descrito), si existe ese conjunto de $(N - 1)$ integrales $(\mathcal{F}_1, \dots, \mathcal{F}_{N-1})$ funcionalmente independientes. Por tanto, la integrabilidad completa de un sistema supone que podemos ser capaces de encontrar la solución explícita de su comportamiento dinámico. De hecho, es muy importante enfatizar que los sistemas integrables son el paradigma del movimiento regular, y pueden ser interpretados desde ese punto de vista como el extremo opuesto a los sistemas caóticos. Pero llegados a este punto, es muy importante subrayar que la integrabilidad de un hamiltoniano no es la regla, sino claramente la excepción, dentro de los sistemas hamiltonianos.

Existen sistemas que tienen incluso más de $(N - 1)$ integrales del movimiento funcionalmente independientes en involución con $\mathcal{H}^{(N)}$ son los denominados sistemas *superintegrables*. Es importante señalar que aunque existan más de $(N - 1)$ integrales sólo $(N - 1)$ de ellas pueden conmutar mutuamente entre sí bajo la acción del corchete de Poisson. Así, un sistema N -dimensional hamiltoniano $\mathcal{H}^{(N)}$ se dice que es *maximalmente superintegrable* (MS), si existe un conjunto máximo posible de $(2N - 2)$ integrales del movimiento globalmente definidas y funcionalmente independientes que conmutan con el hamiltoniano. A partir de la afirmación anterior se deduce que para estos sistemas, es posible extraer al menos dos conjuntos diferentes de tamaño $(N - 1)$ de integrales que conmutan entre sí. Finalmente, un sistema hamiltoniano N -dimensional se dice *cuasi-maximalmente superintegrable*, si tiene $(2N - 3)$ integrales independientes verificando las propiedades anteriormente reseñadas.

Solamente se conocen un pequeño número de sistemas que son MS para cualquier número de grados de libertad. Los dos ejemplos más significativos lo constituyen el oscilador armónico y el potencial de Kepler. La superintegrabilidad es una propiedad que tiene profundas implicaciones geométricas, ya que todas las trayectorias acotadas de un sistema MS son necesariamente periódicas. De esta forma, los sistemas superintegrables serían el ejemplo paradigmático del comportamiento absolutamente no caótico.

Debido a todas estas propiedades, la construcción de sistemas hamiltonianos completamente integrables (en dimensión finita e infinita) ha merecido atención destacada desde hace más de un siglo. De hecho, las técnicas y enfoques que se han utilizado para su tratamiento han sido de lo más variado. A pesar de todo ello, uno de los principales problemas que persisten es la ausencia de un criterio suficiente y/o métodos sistemáticos para decidir si un sistema hamiltoniano dado es o no integrable. De hecho se han desarrollado algunas técnicas para la caracterización de ciertas condiciones de suficiencia como por ejemplo *el test de Painlevé* o el denominado *test de Yoshida-Ziglin*, aunque, a pesar de ello, el acercamiento actual a los sistemas integrables es de tipo fundamentalmente constructivo.

Además, es importante añadir que incluso para los sistemas hamiltonianos con tan sólo dos o tres grados de libertad, la clasificación de aquéllos que son integrables no es en absoluto trivial. Obviamente, las dificultades crecen de manera exponencial cuando el estudio se realiza en dimensiones más altas. De hecho, en la mayoría de los casos N -dimensionales conocidos, las propiedades de integrabilidad están basadas en la existencia de ciertas simetrías que permiten una adecuada extensión de la integrabilidad desde dimensiones bajas.

El objetivo principal de esta Memoria de Tesis Doctoral es la construcción de nuevos sistemas hamiltonianos integrables con N -grados de libertad que incluyan interacciones de tipo no lineal, para los cuales se pretende también la obtención explícita de sus in-

tegrales del movimiento. El enfoque que hemos adoptado para la construcción de estos sistemas es la aplicación de la denominada *simetría de coálgebra*. Este método, cuando es aplicable, proporciona explícitamente las constantes del movimiento y permite igualmente la introducción de deformaciones integrables de los sistemas originales. En este contexto las funciones de Casimir de las álgebras utilizadas serán objetos esenciales ya que de ellas obtendremos las integrales de movimiento del sistema.

A continuación se resume el contenido de cada uno de los capítulos en que se divide esta Memoria, enfatizando las aportaciones originales realizadas. Debemos añadir finalmente que una parte significativa de dichos resultados originales aquí presentados, se han recogido en los trabajos que figuran como referencias con números [23], [27], [28], [30], [31] y [32]. Sin embargo, esta Memoria contiene a su vez, algunos otros resultados no incluidos en estos artículos, dando además una visión unificada de la aplicación de la simetría de coálgebra a la formulación de nuevos sistemas hamiltonianos ND integrables.

Capítulo 2. Sistemas integrables con simetría de coálgebra

2.1 Sistemas hamiltonianos definidos sobre coálgebras de Poisson

En esta Sección se recapitula brevemente la construcción de sistemas hamiltonianos integrables con simetría de coálgebra. Partimos de un álgebra A , que junto con un homomorfismo llamado coproducto Δ y definido como

$$\Delta : A \rightarrow A \otimes A, \quad (4)$$

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta \quad (5)$$

hace que la construcción sea coasociativa. Es decir, se verifica el siguiente diagrama conmutativo

$$\begin{array}{ccccc}
 & & A \otimes A & & \\
 & \nearrow \Delta & & \searrow \Delta \otimes id & \\
 A & & & & A \otimes A \otimes A \\
 & \searrow \Delta & & \nearrow id \otimes \Delta & \\
 & & A \otimes A & &
 \end{array}$$

El coproducto Δ puede iterarse hasta un número arbitrario N de copias del álgebra.

Un hamiltoniano con N grados de libertad se dirá que tiene simetría de coálgebra si se construye como una función cualquiera del coproducto N -ésimo de los generadores de la coálgebra, es decir,

$$\mathcal{H}^{(N)} := \Delta^{(N)}(\mathcal{H}(X_1, \dots, X_l)) = \mathcal{H}(\Delta^{(N)}(X_1), \dots, \Delta^{(N)}(X_l)). \quad (6)$$

La integrabilidad completa de este tipo de sistema proviene de que, por construcción, la aplicación del coproducto a los (r) operadores de Casimir (\mathcal{C}_j) del álgebra utilizada

$$\mathcal{C}_j^{(m)} := \Delta^{(m)}(\mathcal{C}_j(X_1, \dots, X_l)) = \mathcal{C}_j(\Delta^{(m)}(X_1), \dots, \Delta^{(m)}(X_l)) \quad (7)$$

proporciona siempre constantes del movimiento en involución con $\mathcal{H}^{(N)}$:

$$\{\mathcal{C}_j^{(m)}, \mathcal{H}^{(N)}\}_{A \otimes A \otimes \dots \otimes A} = 0 \quad m = 1, \dots, N. \quad (8)$$

Además las constantes así obtenidas están en involución entre ellas

$$\{\mathcal{C}_i^{(m)}, \mathcal{C}_j^{(n)}\}_{A \otimes A \otimes \dots \otimes A} = 0, \quad m, n = 1, \dots, N, \quad i, j = 1, \dots, r. \quad (9)$$

Esta Memoria está basada en el hecho de que toda álgebra de Lie–Poisson g^* con generadores X_i ($i = 1, \dots, l$) y funciones de Casimir $\mathcal{C}_j(X_1, \dots, X_l)$ ($j = 1, \dots, r$), tiene la estructura de coálgebra siguiente

$$\Delta(X_i) = X_i \otimes 1 + 1 \otimes X_i \quad (10)$$

y puede dar origen a hamiltonianos completamente integrables.

El método de la simetría de coálgebra ha sido aplicado anteriormente para la construcción de diferentes sistemas integrables basados en álgebras de dimensión 3 y una única función de Casimir no lineal, tomando siempre realizaciones simplécticas de los generadores en función de un único par de variables canónicas. En esta Sección se realiza un estudio sistemático de las condiciones que deben cumplir las realizaciones simplécticas arbitrarias

$$D : x \rightarrow x(q_1, p_1, q_2, p_2, \dots, q_s, p_s) \quad (11)$$

de un álgebra cualquiera, para poder generar la integrabilidad completa del hamiltoniano $\mathcal{H}^{(N)}$.

Teniendo en cuenta que los invariantes o integrales del sistema serán los operadores de Casimir no lineales, para una coálgebra con dimensión l y R casimires no lineales

con s pares de variables canónicas en la realización simpléctica elegida, encontramos que debe cumplirse la siguiente condición necesaria de integrabilidad

$$s \leq R - \frac{R-1}{N}. \quad (12)$$

La expresión (12) implica que $s = 1$ para coálgebras con $R = 1$ y que $s < R$ para coálgebras con $R > 1$. Nuestro posterior estudio sistemático de las álgebras de dimensiones 3, 4, 5, 6 (nilpotentes), se centrará en aquéllas que verifiquen este criterio. A partir de la fórmula anterior queda claro, que aquellas álgebras con un operador de Casimir no lineal ($R = 1$) deberán ser parametrizadas mediante una única pareja de variables canónicas para dar lugar a sistemas integrables. Ésta va a ser nuestra elección en todos los casos propuestos en este Capítulo y en el siguiente.

Considerando además la realización simpléctica genérica, cuya dimensión viene dada por

$$s_m = \frac{l-r}{2} \quad (13)$$

encontramos que la condición de integrabilidad que debe cumplirse es

$$l \leq (2R+r) - \frac{2}{N}(R-1). \quad (14)$$

Esta relación original, liga los parámetros l, r, R y N , y será básica en todo el desarrollo posterior. En ella vemos que cualquier álgebra con representación $s_m = 1$ y que tenga $R \geq 1$ verificará la condición de integrabilidad.

A continuación, también recordamos que la construcción de coálgebra puede dar lugar, en principio, a sistemas superintegrables, ya que el formalismo puede proporcionar hasta $(2N-3)$ integrales del movimiento, cuya independencia funcional debe de ser comprobada *a posteriori*. En el caso más favorable, nuestros sistemas estarían a una sola integral del alcanzar el número máximo de integrales posibles (caso maximalmente superintegrable), integral que de existir, no puede ser aportada por este formalismo y debe obtenerse aparte.

2.3 La coálgebra de Lie-Poisson $sl(2, \mathbb{R})$

En este apartado, se muestra como ejemplo concreto del funcionamiento del esquema de simetría de coálgebra, la construcción de sistemas completamente integrables basados en $sl(2, \mathbb{R})$. Este tipo de sistemas han sido ya descritos ampliamente en la literatura. A partir de los corchetes de Poisson:

$$\{J_3, J_+\} = 2J_+ \quad \{J_3, J_-\} = -2J_- \quad \{J_-, J_+\} = 4J_3 \quad (15)$$

la función de Casimir es

$$\mathcal{C} = J_- J_+ - J_3^2 \quad (16)$$

y la realización simpléctica genérica viene dada por

$$D(J_-) = q_1^2 \quad D(J_+) = p_1^2 + \frac{b_1}{q_1^2} \quad D(J_3) = q_1 p_1. \quad (17)$$

Con estos ingredientes, se obtienen los sistemas integrables definidos mediante

$$\mathcal{H}^{(N)} = \mathcal{H} \left(\sum_{i=1}^N \left(p_i^2 + \frac{b_i}{q_i^2} \right), \sum_{i=1}^N q_i^2, \sum_{i=1}^N q_i p_i \right) \quad b_i \in \mathbb{R} \quad (18)$$

donde \mathcal{H} es una función genérica con tres variables. Esta construcción será retomada en el Capítulo final, como parte integrante de un esquema algebraico más amplio, que permitirá la generalización axial ND de sistemas integrables 2D.

2.4 Sistemas ND integrables a partir de coálgebras de Lie en 3D

En esta Sección, se procede a la construcción exhaustiva de las realizaciones simplécticas genéricas de las 9 álgebras de Lie reales de dimensión 3 (ver Tabla 3). Igualmente, se presentan los sistemas hamiltonianos integrables a los que dan lugar estas álgebras junto con sus integrales del movimiento. En algunos casos, se aporta más de una posible parametrización para los generadores del álgebra. Como ejemplo de los nuevos hamiltonianos integrables encontrados en este Capítulo mostramos la siguiente construcción que proviene del álgebra $A_{3,3}$. A partir de los corchetes de Poisson no nulos

$$\{J_1, J_3\} = J_1 \quad \{J_2, J_3\} = J_2 \quad (19)$$

se puede formular el siguiente hamiltoniano,

$$\mathcal{H}^{(N)} = \mathcal{H} \left(\sum_{i=1}^N \frac{p_i^2}{2k_i}, \sum_{i=1}^N \frac{p_i^2}{2}, - \sum_{i=1}^N \frac{p_i q_i}{2} \right) \quad (20)$$

cuyas constantes del movimiento son

$$\mathcal{C}^{(m)} = \frac{\left(\sum_{i=1}^m p_i^2 \right)}{\left(\sum_{i=1}^m \frac{p_i^2}{k_i} \right)} \quad m = 2, \dots, N. \quad (21)$$

Nótese que si tomamos la misma realización simpléctica en todas las copias del álgebra (esto es, si $k_1 = k_2 = \dots = k_N$) todas las constantes del movimiento colapsan. Este hecho resalta la importancia de las realizaciones simplécticas elegidas.

2.5 Deformaciones integrables a partir de q -coálgebras de Poisson

En esta Sección final, se recuerda cómo el formalismo de coálgebra puede ser aplicado para introducir de forma natural deformaciones integrables de sistemas hamiltonianos. Para ello, se utilizan las denominadas q -coálgebras de Poisson. Se presentan así los sistemas integrables obtenidos mediante la deformación $sl_z(2, \mathbb{R})$, que modifica la coálgebra $sl(2, \mathbb{R})$ de la siguiente manera:

$$\{J_3, J_+\} = 2J_+ \cosh zJ_- \quad \{J_3, J_-\} = -2 \frac{\sinh zJ_-}{z} \quad \{J_-, J_+\} = 4J_3. \quad (22)$$

El coproducto deformado es

$$\begin{aligned} \Delta_z(1) &= 1 \otimes 1 & \Delta_z(J_-) &= J_- \otimes 1 + 1 \otimes J_- \\ \Delta_z(J_i) &= J_i \otimes e^{zJ_-} + e^{-zJ_-} \otimes J_i & (i = +, 3). \end{aligned} \quad (23)$$

La función de Casimir deformada se debe tomar como

$$C_z = \frac{\sinh zJ_-}{z} J_+ - J_3^2 \quad (24)$$

y la realización simpléctica correspondiente sería

$$D_z(J_-) = q_1^2 \quad D_z(J_+) = \frac{\sinh zq_1^2}{zq_1^2} p_1^2 + \frac{zb_1}{\sinh zq_1^2} \quad D_z(J_3) = \frac{\sinh zq_1^2}{zq_1^2} q_1 p_1. \quad (25)$$

A partir de todo ello, se obtienen sistemas completamente integrables que deforman en términos del parámetro z los ya obtenidos para $sl(2, \mathbb{R})$. Finalmente, se mencionan algunas de las aplicaciones físicas de los mismos.

Capítulo 3. Coálgebras en dimensiones más altas

En este Capítulo se procede al estudio sistemático de todas las álgebras de Lie reales de dimensiones 4,5 y las nilpotentes de grado 6. Se presentan todas aquellas que cumplen la condición de integrabilidad descrita en el Capítulo anterior (12 y 14), y para ellas, se construyen las familias de sistemas hamiltonianos integrables con simetría de coálgebra a los que dan lugar.

3.1 Sistemas integrables a partir de coálgebras 4D

Se encuentran las realizaciones simplécticas de las 12 álgebras de Lie reales de dimensión 4 y se construyen los sistemas dinámicos que surgen de ellos, así como sus invariantes. El material se presenta de acuerdo con el número R de casimires no lineales de cada una de las álgebras.

Álgebras con $R = 1$

En la Tabla 4, se presentan dentro de esta Sección las álgebras $A_{4,1}$, $A_{4,3}$, $A_{4,8}$ y $A_{4,10}$. Como ejemplo de sistema dinámico interesante presentamos

$$\mathcal{H}^{(N)} = \sum_{i=1}^N \frac{p_i^2 + k_{1,i}^2 q_i^2 - k_{2,i}}{2k_{1,i}} + \mathcal{F} \left(\sum_{i=1}^N k_{1,i} q_i \right) \quad (26)$$

obtenido desde el álgebra $A_{4,10}$.

Álgebras con $R = 2$

Siguiendo con el esquema anterior, la Tabla 5, muestra las álgebras 4D ($A_{4,2}^\alpha$, $A_{4,4}$, $A_{4,5}^{a,b}$ y $A_{4,6}^{a,b}$) que cuentan con dos funciones de Casimir no lineales. Desde un punto de vista dinámico los sistemas aquí encontrados son menos interesantes. Como muestra de uno de ellos, presentamos el ejemplo de sistema formulado a partir de $A_{4,2}^\alpha$

$$\mathcal{H}^{(N)} = \mathcal{H} \left(\sum_{i=1}^N \frac{k_{1,i}^\alpha}{k_{2,i}} e^{p_i}, \sum_{i=1}^N k_{1,i} e^{p_i}, \sum_{i=1}^N k_{1,i} p_i e^{p_i}, - \sum_{i=1}^N q_i \right) \quad (27)$$

donde, de nuevo, \mathcal{H} es una función genérica de cuatro variables.

3.2 Sistemas integrables a partir de coálgebras 5D

En el mismo estilo que la Sección anterior, ésta presenta los sistemas que se formulan a partir de las 40 álgebras de Lie reales de dimensión 5, ordenados según el número de Casimires no lineales que aparecen en ellas. Una vez más, en la Memoria sólo se presentan los resultados que verifican la condición necesaria de integrabilidad (12).

Álgebras con $R = 1$

La Tabla 6 muestra las parametrizaciones simplécticas de las álgebras $A_{5,1}$, $A_{5,3}$. Como ejemplo más interesante podemos citar

$$\mathcal{H}^{(N)} = \sum_{i=1}^N \frac{p_i^2}{2k_{2,i}} + \sum_{i=1}^N k_{1,i} q_i + \mathcal{G} \left(\sum_{i=1}^N k_{2,i} q_i + \sum_{i=1}^N \frac{k_{3,i}}{k_{1,i}} \right) \quad (28)$$

obtenido para el álgebra $A_{5,3}$. Nótese el papel destacado que en toda la formulación de la coálgebra desempeñan las constantes $k_{j,i}$, que indexan la realización simpléctica tomada en la copia i -ésima del álgebra.

Álgebras con $R = 2$

La Tabla 7 presenta las realizaciones simplécticas de las álgebras $A_{5,2}$, $A_{5,8}^c$, $A_{5,10}$ y $A_{5,14}^\alpha$. Un ejemplo de sistema hamiltoniano integrable sería

$$\mathcal{H}^{(N)} = \mathcal{H} \left(\sum_{i=1}^N k_{1,i}, \sum_{i=1}^N k_{1,i} p_i, \sum_{i=1}^N \left(\frac{k_{1,i}}{2} p_i^2 - \frac{k_{2,i}}{2k_{1,i}} \right), \sum_{i=1}^N k_{3,i} e^{p_i}, - \sum_{i=1}^N q_i \right). \quad (29)$$

Finalmente el caso $R = 3$ se presenta en el Suplemento 3, ya que su interés dinámico es limitado. Sin embargo, sirve para ejemplificar la complejidad de las realizaciones simplécticas conforme se va subiendo de dimensión.

3.3 Sistemas integrables a partir de coálgebras 6D nilpotentes

Se han encontrado las realizaciones simplécticas de las 22 álgebras nilpotentes de dimensión 6, así como de las 12 solubles que contienen un nilradical abeliano. Del primer grupo, solamente tres de todas ellas verifican la condición necesaria de integrabilidad (12). Por este motivo, se presentan únicamente las álgebras $A_{6,3}$, $A_{6,1}$ y $A_{6,2}$, junto con los sistemas integrables a que dan lugar.

Del resto de álgebras de ese grupo, no se presentan explícitamente las parametrizaciones en esta Memoria, ya que aunque algebraicamente importantes, no dan lugar a sistemas dinámicos interesantes desde un punto de vista físico.

3.4 Una familia de álgebras con dimensión arbitraria

En la parte final del Capítulo, se presenta como resultado original la caracterización de la familia de álgebras de Lie denominada \mathfrak{g}_M y definida por los corchetes de Poisson no

nulos

$$\{J_i, J_M\} = J_{i-1} \quad \forall i = 2, \dots, M-1. \quad (30)$$

Esta familia incluye como casos particulares las álgebras $A_{3,1}$, $A_{4,1}$, $A_{5,2}$, y $A_{6,2}$.

Hemos encontrado la fórmula genérica para sus $(M-2)$ funciones de Casimir, que vienen dadas por

$$\mathcal{C}_{\mathfrak{g}_M} \begin{cases} \mathcal{C}_1 = J_1 \\ \mathcal{C}_2 = \frac{-J_2^2 + 2J_1J_3}{2J_1} \\ \vdots \\ \mathcal{C}_i = \frac{(-1)^{i+1}J_2^i + \sum_{j=1}^{i-2} a_{j,i}J_1^jJ_2^{(i-1-j)}J_{j+2} + i(i-2)!J_1^{i-1}J_{i+1}}{i(i-2)!J_1^{i-1}} \end{cases} \quad i = 3, \dots, M-2 \quad (31)$$

donde a_j son ciertas constantes. Por tanto, esta familia cumple la condición de integrabilidad para todo $M > 3$. La realización simpléctica de \mathfrak{g}_M genérica tiene las siguientes expresiones para los generadores:

$$\begin{aligned} J_1 &= k_1 \\ J_2 &= k_1 p \\ \vdots & \quad \quad \quad \vdots \\ J_j &= k_1 \frac{p^{j-1}}{(j-1)!} + \sum_{i=1}^{j-2} k_{i+1} \frac{p^{j-2-i}}{(j-2-i)!} \quad j = 3, 4, \dots, (M-1) \\ J_M &= -q \end{aligned} \quad (32)$$

a partir de la cual, pueden escribirse los correspondientes hamiltonianos completamente integrables con un número arbitrario de grados de libertad dados por

$$\mathcal{H}^{(N)} = \mathcal{H}(J_1, J_2, \dots, J_M). \quad (33)$$

Capítulo 4.

Integrabilidad ND a partir de la coálgebra ‘two-photon’

En este Capítulo se aplica el formalismo de la simetría de coálgebra para una cierta realización simpléctica (no genérica) de un álgebra de Lie de dimensión 6 denominada álgebra ‘two-photon’. Los nuevos sistemas hamiltonianos clásicos integrables que se

obtienen dependen de N parámetros y de varias funciones arbitrarias. Igualmente dentro de este marco, se presentan todas sus integrales del movimiento. Como se verá en la serie de ejemplos presentados, el álgebra h_6 dará lugar a numerosos sistemas completamente integrables de gran interés físico.

4.1 El álgebra h_6 o ‘two-photon’

Los corchetes de Poisson de este álgebra son

$$\begin{aligned} \{K, A_+\} &= A_+ & \{K, A_-\} &= -A_- & \{A_-, A_+\} &= M \\ \{K, B_+\} &= 2B_+ & \{K, B_-\} &= -2B_- & \{B_-, B_+\} &= 4K + 2M \\ \{A_+, B_-\} &= -2A_- & \{A_+, B_+\} &= 0 & \{M, \cdot\} &= 0 \\ \{A_-, B_+\} &= 2A_+ & \{A_-, B_-\} &= 0. \end{aligned} \quad (34)$$

Sus dos casimires son el generador central M y la función no lineal

$$\mathcal{C} = (MB_+ - A_+^2)(MB_- - A_-^2) - (MK - A_-A_+ + M^2/2)^2. \quad (35)$$

Una realización simpléctica a una partícula de este álgebra, viene dada por

$$\begin{aligned} A_+ &= \lambda_1 p_1 & A_- &= \lambda_1 q_1 & K &= q_1 p_1 - \frac{\lambda_1^2}{2} \\ B_+ &= p_1^2 & B_- &= q_1^2 & M &= \lambda_1^2. \end{aligned} \quad (36)$$

A pesar de que este álgebra no verifica la condición de integrabilidad (12), es posible formular a partir de ella sistemas que están a falta de una única constante del movimiento para alcanzar la integrabilidad completa, tal y como analizamos en la Sección 4.2.

Desde un punto de vista físico, este álgebra contiene a $sl(2, \mathbb{R})$ como subálgebra, lo cual supone que h_6 permite romper la simetría esférica a partir de la inclusión del término $A_- = \lambda_1 q$. Igualmente, enfatizamos que la realización simpléctica (38) que se obtiene del coproducto, contiene N parámetros libres λ_i . Este hecho va a proporcionar una gran abanico de posibilidades para los nuevos hamiltonianos presentados.

4.2 Una familia infinita de sistemas cuasi-integrables

El resultado fundamental de este capítulo es que cualquier hamiltoniano construido como una función

$$\mathcal{H} = \mathcal{H}(K, B_+, B_-, A_+, A_-, M) \quad (37)$$

donde

$$A_+ = \sum_{i=1}^N \lambda_i p_i \quad A_- = \sum_{i=1}^N \lambda_i q_i \quad K = \sum_{i=1}^N \left(q_i p_i - \frac{\lambda_i^2}{2} \right)$$

$$B_+ = \sum_{i=1}^N p_i^2 \quad B_- = \sum_{i=1}^N q_i^2 \quad M = \sum_{i=1}^N \lambda_i^2 \quad (38)$$

es *cuasi integrable*, ya que posee $(N - 2)$ integrales del movimiento dadas por

$$\mathcal{C}^{(m)} = \sum_{1 \leq i < j < k}^m (\lambda_i(p_j q_k - p_k q_j) + \lambda_j(p_k q_i - p_i q_k) + \lambda_k(p_i q_j - p_j q_i))^2 \quad (39)$$

donde $m = 3, \dots, N$. Nótese que el coproducto para $m = 2$ da lugar a una integral trivial. Este hecho genera la ausencia de una integral del movimiento con respecto a todos los casos tratados hasta ahora, y refleja que la realización simpléctica elegida no cumple la condición de integrabilidad.

Se recapitula a continuación la forma general de algunos sistemas hamiltonianos cuasi-integrables que pueden formularse dentro de este marco, y cuyo estudio particular se desarrollará más adelante. De esta manera, se demuestra la cuasi-integrabilidad para cualquier N de:

- *Sistemas naturales.*

$$\mathcal{H}^{(N)} = \sum_{i=1}^N \frac{p_i^2}{2} + \mathcal{F}\left(\sum_{i=1}^N \lambda_i q_i, \sum_{i=1}^N q_i^2\right). \quad (40)$$

- *Hamiltonianos electromagnéticos.* Los sistemas h_6 que contienen términos lineales en los momentos tienen la forma general siguiente:

$$\begin{aligned} \mathcal{H}^{(N)} &= \sum_{i=1}^N \frac{p_i^2}{2} + \left(\sum_{i=1}^N \left(q_i p_i - \frac{\lambda_i^2}{2}\right)\right) \mathcal{F}\left(\sum_{i=1}^N \lambda_i q_i, \sum_{i=1}^N q_i^2\right) \\ &+ \left(\sum_{i=1}^N \lambda_i p_i\right) \mathcal{G}\left(\sum_{i=1}^N \lambda_i q_i, \sum_{i=1}^N q_i^2\right) + \mathcal{R}\left(\sum_{i=1}^N \lambda_i q_i, \sum_{i=1}^N q_i^2\right). \end{aligned} \quad (41)$$

Comparando (para $N = 3$) con un hamiltoniano electromagnético estático genérico

$$\mathcal{H}_{em} = \frac{1}{2} (\vec{p} - e\vec{A})^2 + e\psi \quad (42)$$

donde e es la carga eléctrica, \vec{A} es el potencial vector y ψ el escalar, obtendríamos

$$A_i = -\frac{q_i}{e} \mathcal{F}\left(\sum_{i=1}^N \lambda_i q_i, \sum_{i=1}^N q_i^2\right) - \frac{\lambda_i}{e} \mathcal{G}\left(\sum_{i=1}^N \lambda_i q_i, \sum_{i=1}^N q_i^2\right) \quad i = 1, 2, 3 \quad (43)$$

$$\begin{aligned}
\psi &= \frac{1}{e} \mathcal{R} \left(\sum_{i=1}^N \lambda_i q_i, \sum_{i=1}^N q_i^2 \right) - \frac{1}{2e} \left(\sum_{i=1}^N \lambda_i^2 \right) \mathcal{F} \left(\sum_{i=1}^N \lambda_i q_i, \sum_{i=1}^N q_i^2 \right) \\
&\quad - \frac{1}{2e} \left(\sum_{i=1}^N q_i^2 \right) \left\{ \mathcal{F} \left(\sum_{i=1}^N \lambda_i q_i, \sum_{i=1}^N q_i^2 \right) \right\}^2 \\
&\quad - \frac{1}{e} \left(\sum_{i=1}^N \lambda_i q_i \right) \mathcal{F} \left(\sum_{i=1}^N \lambda_i q_i, \sum_{i=1}^N q_i^2 \right) \mathcal{G} \left(\sum_{i=1}^N \lambda_i q_i, \sum_{i=1}^N q_i^2 \right) \\
&\quad - \frac{1}{2e} \left(\sum_{i=1}^N \lambda_i^2 \right) \left\{ \mathcal{G} \left(\sum_{i=1}^N \lambda_i q_i, \sum_{i=1}^N q_i^2 \right) \right\}^2
\end{aligned} \tag{44}$$

siendo $\mathcal{F}, \mathcal{G}, \mathcal{R}$ funciones arbitrarias.

- *Flujos geodésicos hamiltonianos.* Recordemos la forma general de estos sistemas

$$\mathcal{H}^{(N)} = \sum_{i,j=1}^N g^{ij}(q_1, \dots, q_N) p_i p_j \tag{45}$$

donde la métrica del espacio subyacente va a venir descrita por $g^{ij}(q_1, \dots, q_N)$. Se prueba que el hamiltoniano h_6 invariante (y por tanto, cuasi-integrable) más general de este tipo que podemos escribir es

$$\begin{aligned}
\mathcal{H}^{(N)} &= \left(\sum_{i=1}^N \frac{p_i^2}{2} \right) \mathcal{F} \left(\sum_{i=1}^N \lambda_i q_i, \sum_{i=1}^N q_i^2 \right) + \left(\sum_{i=1}^N \lambda_i p_i \right)^2 \mathcal{G} \left(\sum_{i=1}^N \lambda_i q_i, \sum_{i=1}^N q_i^2 \right) \\
&\quad + \left(\sum_{i=1}^N q_i p_i \right)^2 \mathcal{R} \left(\sum_{i=1}^N \lambda_i q_i, \sum_{i=1}^N q_i^2 \right) \\
&\quad + \left(\sum_{i=1}^N q_i p_i \right) \left(\sum_{i=1}^N \lambda_i p_i \right) \mathcal{S} \left(\sum_{i=1}^N \lambda_i q_i, \sum_{i=1}^N q_i^2 \right).
\end{aligned} \tag{46}$$

4.3 Integrabilidad completa

En todos los ejemplos descritos en la Sección anterior disponemos de $(N-2)$ integrales funcionalmente independientes y en involución. Por ello, es necesario encontrar para qué formas concretas del hamiltoniano existe una integral adicional independiente que pueda escribirse en función de los generadores de h_6 . A continuación se presentan tres posibilidades distintas para abordar este problema.

4.4 Integrabilidad a partir de las subálgebras

Un aspecto importante del álgebra h_6 es su rica estructura de subálgebras. Teniendo en cuenta este hecho, es inmediato probar que si un hamiltoniano se define en función de los generadores de una de las subálgebras de h_6 , podemos entonces completar el conjunto de integrales del movimiento con el coproducto N -ésimo del Casimir de la subálgebra correspondiente. Por tanto, se consigue demostrar la integrabilidad completa de los siguientes hamiltonianos:

$$\begin{aligned}
\mathcal{H}_{\mathcal{D}_+} &= \mathcal{H}_{\mathcal{D}_+}(K, A_+, B_+) \\
\mathcal{H}_{\mathcal{D}_-} &= \mathcal{H}_{\mathcal{D}_-}(K, A_-, B_-) \\
\mathcal{H}_{h_4} &= \mathcal{H}_{h_4}(K, A_-, A_+, M) \\
\mathcal{H}_{\bar{g}_+} &= \mathcal{H}_{\bar{g}_+}(B_+, A_-, A_+, M) \\
\mathcal{H}_{\bar{g}_-} &= \mathcal{H}_{\bar{g}_-}(B_-, A_-, A_+, M) \\
\mathcal{H}_{\bar{e}} &= \mathcal{H}_{\bar{e}}(\mu B_+ + \nu B_-, A_-, A_+, M) \\
\mathcal{H}_{gl(2)} &= \mathcal{H}_{gl(2)}(K, B_-, B_+, M).
\end{aligned} \tag{47}$$

La subálgebra a que corresponde cada caso se describe detalladamente en esta Sección, así como la integral adicional que proporciona.

4.5 Integrabilidad a través de un generador

Proponemos aquí una segunda estrategia para conseguir sistemas completamente integrables: elegimos un generador X de h_6 y, posteriormente, buscamos todos aquellos generadores X_j ($j = 1, \dots, l$) que conmutan con X y todas las subálgebras g_k ($k = 1, \dots, t$) que contienen a X como generador. De esta manera, el hamiltoniano construido como sigue

$$\mathcal{H}_X = \mathcal{H}_X(\mathcal{C}_{g_1}, \dots, \mathcal{C}_{g_t}, X, X_1, \dots, X_l), \tag{48}$$

donde \mathcal{C}_{g_k} es el operador de Casimir de la subálgebra g_k , verifica

$$\{\mathcal{H}_X, X\} = 0. \tag{49}$$

Utilizando el esquema anterior, vamos a poder construir sistemas de diferentes tipos cuya integral adicional será $X^{(N)}$. La integrabilidad completa de este tipo de hamiltonianos se alcanza mediante el conjunto de integrales

$$\{\mathcal{C}_{h_6}^{(3)}, \dots, \mathcal{C}_{h_6}^{(N)}, X^{(N)}, \mathcal{H}\}. \tag{50}$$

Algunos ejemplos significativos de esta construcción son los siguientes:

- *Hamiltonianos en involución con K* . La forma general de estos sistemas será la siguiente:

$$\mathcal{H}_K = \mathcal{H}_K (\mathcal{C}_{D_+}, \mathcal{C}_{D_-}, \mathcal{C}_{h_4}, \mathcal{C}_{gl(2)}, K, M). \quad (51)$$

En este caso, sólo pueden formularse flujos geodésicos hamiltonianos. La expresión más general de estos sistemas integrables es

$$\begin{aligned} \mathcal{H}_K^{(N)} = & \left(\sum_{1 \leq i < j}^N (q_j p_i - q_i p_j)^2 \right) \mathcal{F} \left(\frac{\left(\sum_{i=1}^N \lambda_i q_i \right)^2}{\sum_{i=1}^N q_i^2} \right) + \left(\sum_{i=1}^N q_i p_i \right)^2 \mathcal{G} \left(\frac{\left(\sum_{i=1}^N \lambda_i q_i \right)^2}{\sum_{i=1}^N q_i^2} \right) \\ & + \left(\sum_{1 \leq i < j}^N (\lambda_j p_i - \lambda_i p_j) (\lambda_j q_i - \lambda_i q_j) \right)^2 \mathcal{R} \left(\frac{\left(\sum_{i=1}^N \lambda_i q_i \right)^2}{\sum_{i=1}^N q_i^2} \right) \\ & + \left(\sum_{1 \leq i < j}^N (\lambda_j p_i - \lambda_i p_j) (\lambda_j q_i - \lambda_i q_j) \right) \left(\sum_{i=1}^N q_i p_i \right) \mathcal{S} \left(\frac{\left(\sum_{i=1}^N \lambda_i q_i \right)^2}{\sum_{i=1}^N q_i^2} \right) \quad (52) \end{aligned}$$

donde $\mathcal{F}, \mathcal{G}, \mathcal{R}, \mathcal{S}$ son funciones arbitrarias.

- *Hamiltonianos en involución con A_+* . Ésta es la familia de hamiltonianos que admite mayor variedad dentro de este esquema. Los sistemas tienen como invariante al generador $A_+^{(N)} = \sum_{i=1}^N \lambda_i p_i$, que en esencia representa una generalización de la simetría traslacional. Entre la variedad de sistemas que pueden así construirse presentamos hamiltonianos naturales

$$\mathcal{H}_{A_+}^{(N)} = \sum_{i=1}^N \frac{p_i^2}{2} + \mathcal{F} \left(\sum_{1 \leq i < j}^N (\lambda_j q_i - \lambda_i q_j)^2 \right) \quad (53)$$

sistemas electromagnéticos de la forma

$$\begin{aligned} \mathcal{H}_{A_+}^{(N)} &= \sum_{i=1}^N \frac{p_i^2}{2} + \left(\sum_{1 \leq i < j}^N (\lambda_j p_i - \lambda_i p_j) (\lambda_j q_i - \lambda_i q_j) \right) \mathcal{G} \left(\sum_{1 \leq i < j}^N (\lambda_j q_i - \lambda_i q_j)^2 \right) \\ &+ \left(\sum_{i=1}^N \lambda_i p_i \right) \mathcal{R} \left(\sum_{1 \leq i < j}^N (\lambda_j q_i - \lambda_i q_j)^2 \right) + \mathcal{F} \left(\sum_{1 \leq i < j}^N (\lambda_j q_i - \lambda_i q_j)^2 \right) \end{aligned} \quad (54)$$

y, finalmente, flujos geodésicos del tipo

$$\begin{aligned} \mathcal{H}_{A_+}^{(N)} &= \left(\sum_{1 \leq i < j}^N (\lambda_j p_i - \lambda_i p_j) (\lambda_j q_i - \lambda_i q_j) \right)^2 \mathcal{F} \left(\sum_{1 \leq i < j}^N (\lambda_j q_i - \lambda_i q_j)^2 \right) \\ &+ \left(\sum_{1 \leq i < j}^N (\lambda_j p_i - \lambda_i p_j)^2 \right) \mathcal{G} \left(\sum_{1 \leq i < j}^N (\lambda_j q_i - \lambda_i q_j)^2 \right) \\ &+ \left(\sum_{i=1}^N p_i^2 \right) \mathcal{R} \left(\sum_{1 \leq i < j}^N (\lambda_j q_i - \lambda_i q_j)^2 \right) + \left(\sum_{i=1}^N \lambda_i p_i \right)^2 \mathcal{S} \left(\sum_{1 \leq i < j}^N (\lambda_j q_i - \lambda_i q_j)^2 \right) \\ &+ \left(\sum_{i=1}^N \lambda_i p_i \right) \left(\sum_{1 \leq i < j}^N (\lambda_j p_i - \lambda_i p_j) (\lambda_j q_i - \lambda_i q_j) \right) \mathcal{T} \left(\sum_{1 \leq i < j}^N (\lambda_j q_i - \lambda_i q_j)^2 \right). \end{aligned} \quad (55)$$

- *Hamiltonianos en involución con A_- , B_- ó B_+ .* En este apartado final dedicado a la integrabilidad conseguida a partir de un generador se muestran los sistemas que conmutan con cada uno de los restantes generadores no triviales de h_6 . Los sistemas que se generan son todos ellos flujos geodésicos completamente integrables que se describen explícitamente en la Memoria.

4.6 Búsqueda directa de hamiltonianos integrables

La tercera posibilidad para conseguir la integral adicional (\mathcal{T}) de un sistema cuasi-integrable definido sobre h_6 , consiste en realizar su búsqueda mediante cálculo directo, una vez que se realiza una cierta hipótesis bastante restrictiva sobre la forma del hamiltoniano. En esta Sección se presentan los resultados originales obtenidos mediante este tipo de búsqueda. Recogemos a continuación algunos de los más significativos.

Sistemas naturales: el hamiltoniano de Garnier

Si consideramos el siguiente hamiltoniano

$$\mathcal{H} = \frac{1}{2}B_+ + \delta_1 B_- + \delta_2 A_-^2 + \delta_3 B_-^2 \quad (56)$$

donde $\delta_1, \delta_2, \delta_3$ son constantes arbitrarias, probamos que la función

$$\mathcal{I} = \delta_3 C_{gl(2)} - \delta_2 (A_+^2 + 2A_-^2 (M \delta_2 + \delta_1 + \delta_3 B_-)) \quad (57)$$

conmuta con \mathcal{H} . Como caso particular de este sistema, se obtiene justamente el hamiltoniano de Garnier con dos frecuencias de oscilador distintas. En 2D, si tomamos $\lambda_1 = 1, \lambda_2 = 0$, tenemos

$$\mathcal{H}^{(2)} = \frac{1}{2}(p_1^2 + p_2^2) + Aq_1^2 + Bq_2^2 + \delta_3 (q_1^2 + q_2^2)^2 \quad (58)$$

donde $A = \delta_1 + \delta_2$, y $B = \delta_1$. En tres dimensiones tomando $\lambda_1 = 1, \lambda_2 = \lambda_3 = 0$ obtenemos

$$\mathcal{H}^{(3)} = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + (\delta_1 + \delta_2)q_1^2 + \delta_1 q_2^2 + \delta_1 q_3^2 + \delta_3 (q_1^2 + q_2^2 + q_3^2)^2 \quad (59)$$

que sería la generalización axialmente simétrica de (58).

Hamiltonianos electromagnéticos.

Un ejemplo de sistema integrable tridimensional y lineal en los momentos con campo magnético no nulo y variable, encontrado mediante búsqueda directa, es el siguiente:

$$\mathcal{H} = \frac{1}{2} \sum_{i=1}^3 p_i^2 + \delta_5 \left\{ \sum_{i=1}^3 q_i p_i - \frac{\lambda_i^2}{2} \right\} \left(\sum_{i=1}^3 \lambda_i q_i \right) + \delta_6 \sum_{i=1}^3 \lambda_i p_i. \quad (60)$$

Flujos geodésicos hamiltonianos

Son varios los flujos geodésicos interesantes cuya integrabilidad completa puede demostrarse mediante este método. En particular, el hamiltoniano

$$\mathcal{H} = B_+ (\alpha_1 A_- + \alpha_2 B_- + \alpha_3) \quad (61)$$

o, lo que es lo mismo

$$\mathcal{H}^{(N)} = \left(\sum_{i=1}^N p_i^2 \right) \left\{ \alpha_1 \sum_{i=1}^N \lambda_i q_i + \alpha_2 \sum_{i=1}^N q_i^2 + \alpha_3 \right\} \quad (62)$$

donde $\alpha_1, \alpha_2, \alpha_3$ son constantes, es uno de ellos, ya que la siguiente integral, en involución con (61), puede encontrarse por cálculo directo:

$$\mathcal{I} = 4\alpha_1\alpha_2 A_+ \left(K + \frac{M}{2} \right) + 4\alpha_2\alpha_3 B_+ + 4\alpha_2^2 K(K + M) - \alpha_1^2 \mathcal{C}_{\bar{\mathcal{G}}_+}. \quad (63)$$

La curvatura gaussiana del espacio 2D definido por este flujo es

$$\mathcal{K} = \frac{4\alpha_2\alpha_3 - \alpha_1^2(\lambda_1^2 + \lambda_2^2)}{2[\alpha_2(q_1^2 + q_2^2) + \alpha_3 + \alpha_1(\lambda_1 q_1 + \lambda_2 q_2)]} \quad (64)$$

lo que significa que estamos ante variedades integrables con curvatura no constante. El resto de nuevos flujos geodésicos encontrados presentan esta misma propiedad.

Capítulo 5. Perturbaciones integrables del oscilador armónico en ND

En este Capítulo se presentan dos familias de perturbaciones ND integrables del sistema hamiltoniano integrable por antonomasia: el oscilador armónico. Al igual que en el Capítulo anterior, la construcción algebraica está basada en h_6 . Nótese que, aunque las perturbaciones integrables 2D y 3D del oscilador armónico han sido estudiadas con cierto detalle, el número de aquellas que son integrables en ND es realmente escaso.

5.1 Un nuevo hamiltoniano ND integrable

En esta Sección introducimos la siguiente familia de hamiltonianos

$$\mathcal{H}^{(N)} = \frac{1}{2} \sum_{i=1}^N p_i^2 + \delta_1 \sum_{i=1}^N q_i^2 + \mathcal{F} \left(\sum_{i<j}^N (\lambda_j q_i - \lambda_i q_j)^2 \right) + \mathcal{G} \left(\sum_{i=1}^N \lambda_i q_i \right) \quad (65)$$

y probamos su integrabilidad completa. Es interesante recalcar que sus invariantes son las siguientes expresiones

$$\mathcal{C}^{(m)} = \sum_{1 \leq i < j < k}^m (\lambda_i(p_j q_k - p_k q_j) + \lambda_j(p_k q_i - p_i q_k) + \lambda_k(p_i q_j - p_j q_i))^2 \quad m = 3, \dots, N \quad (66)$$

junto con

$$\mathcal{I}^{(N)} = \left(\sum_{i=1}^N \lambda_i p_i \right)^2 + 2\delta_1 \left(\sum_{i=1}^N \lambda_i q_i \right)^2 + 2 \left(\sum_{i=1}^N \lambda_i^2 \right) \mathcal{G} \left(\sum_{i=1}^N \lambda_i q_i \right). \quad (67)$$

El aspecto más llamativo de este resultado radica en el hecho de que la integral (67) no depende de la función \mathcal{F} .

Como caso particular, eligiendo $\mathcal{F}(x) = \alpha/x$, llegamos al siguiente hamiltoniano

$$\mathcal{H}^{(N)} = \frac{1}{2} \sum_{i=1}^N p_i^2 + \delta_1 \sum_{i=1}^N q_i^2 + \frac{\alpha}{\sum_{1 \leq i < j}^N (\lambda_j q_i - \lambda_i q_j)^2} + \mathcal{G} \left(\sum_{i=1}^N \lambda_i q_i \right). \quad (68)$$

Este sistema, sólo en dos dimensiones reproduce el sistema de Calogero-Moser

$$\mathcal{H}^{(2)} = \frac{1}{2} (p_1^2 + p_2^2) + \delta_1 (q_1^2 + q_2^2) + \frac{\alpha}{(\lambda_2 q_1 - \lambda_1 q_2)^2} + \mathcal{G}(\lambda_1 q_1 + \lambda_2 q_2).$$

Por el contrario, el hamiltoniano anterior, aun no siendo el modelo de Calogero-Moser en dimensión arbitraria, sigue manteniendo su integrabilidad completa.

El problema de la isotropía

A la vista de la familia de hamiltonianos anteriormente introducida (65), se estudia el problema de la posible anisotropía en las frecuencias que podría surgir a partir de la elección de las funciones arbitrarias \mathcal{F} y \mathcal{G} .

Con el fin de aclarar este aspecto, partiendo del siguiente hamiltoniano

$$\begin{aligned} \mathcal{H}^{(N)} = & \frac{1}{2} \sum_{i=1}^N p_i^2 + \delta_1 \sum_{i=1}^N q_i^2 + \alpha_1 \left(\sum_{i < j}^N (\lambda_j q_i - \lambda_i q_j)^2 \right) + \beta_1 \left(\sum_{i=1}^N \lambda_i q_i \right)^2 \\ & + f \left(\sum_{i < j}^N (\lambda_j q_i - \lambda_i q_j)^2 \right) + g \left(\sum_{i=1}^N \lambda_i q_i \right) \end{aligned} \quad (69)$$

encontramos sólo dos posibilidades de anulación de los términos cruzados de tipo $q_i q_j$:

- A) $\alpha_1 = \beta_1$. Que conduce al caso isótropo dado por

$$\mathcal{H}^{(N)} = \frac{1}{2} \sum_{i=1}^N p_i^2 + \omega^2 \sum_{i=1}^N q_i^2 + f \left(\sum_{i < j}^N (\lambda_j q_i - \lambda_i q_j)^2 \right) + g \left(\sum_{i=1}^N \lambda_i q_i \right) \quad (70)$$

con $\omega^2 = \delta_1 + \alpha_1 \sum_{i=1}^N \lambda_i^2$.

• B) Para un i fijado, otra solución emerge de $\lambda_i = \lambda$ y $\lambda_j = 0$ para $j \neq i$. Tomando por ejemplo $\lambda_1 = \lambda$ el sistema hamiltoniano integrable deviene en la siguiente forma

$$\mathcal{H}^{(N)} = \frac{1}{2} p_1^2 + \omega_1^2 q_1^2 + g(\lambda q_1) + \frac{1}{2} \sum_{i=2}^N p_i^2 + \omega_2^2 \sum_{i=2}^N q_i^2 + f\left(\lambda^2 \sum_{i=2}^N q_i^2\right) \quad (71)$$

con $\omega_1^2 = \lambda^2 \beta_1 + \delta_1$ y $\omega_2^2 = \lambda^2 \alpha_1 + \delta_1$. Por tanto, esta solución anisótropa resulta trivialmente separable, y en lo que sigue se considera tan sólo el caso isótropo (70).

5.2 Osciladores no lineales acoplados en ND

Como casos particulares de la familia de perturbaciones anteriormente introducida, consideramos las funciones \mathcal{F} y \mathcal{G} , que dan lugar a osciladores acoplados de grados cuatro y seis.

Perturbaciones cuánticas homogéneas

La perturbación más general de este tipo viene dada por:

$$\mathcal{H}^{(N)} = \frac{1}{2} \sum_{i=1}^N p_i^2 + \delta_1 \sum_{i=1}^N q_i^2 + \alpha_2 \left(\sum_{i<j}^N (\lambda_j q_i - \lambda_i q_j)^2 \right)^2 + \beta_3 \left(\sum_{i=1}^N \lambda_i q_i \right)^4. \quad (72)$$

Si elegimos para el caso bidimensional $\lambda_1 = 1, \lambda_2 = 0$ obtenemos

$$\mathcal{H}^{(2)} = \frac{1}{2} (p_1^2 + p_2^2) + \delta_1 (q_1^2 + q_2^2) + \beta_3 q_1^4 + \alpha_2 q_2^4 \quad (73)$$

que es un sistema separable, siendo su integral la siguiente expresión:

$$\mathcal{I}^{(2)} = p_1^2 + 2(\delta_1 q_1^2 + \beta_3 q_1^4). \quad (74)$$

En 3 dimensiones tomando $\lambda_1 = 1, \lambda_2 = \lambda_3 = 0$ obtenemos

$$\mathcal{H}^{(3)} = \frac{1}{2} (p_1^2 + p_2^2 + p_3^2) + \delta_1 (q_1^2 + q_2^2 + q_3^2) + \alpha_2 (q_2^2 + q_3^2)^2 + \beta_3 q_1^4. \quad (75)$$

Por otra parte, una elección diferente de las λ como por ejemplo $\lambda_1 = 1, \lambda_2 = 1, \alpha_2 = \beta_3 = \epsilon/2$ nos hace recuperar un sistema integrable ya conocido

$$\mathcal{H}^{(2)} = \frac{1}{2} (p_1^2 + p_2^2) + \delta_1 (q_1^2 + q_2^2) + \epsilon (q_1^4 + q_2^4 + 6q_1^2 q_2^2). \quad (76)$$

xxx

En tres dimensiones, si elegimos $\lambda_1 = \lambda_2 = \lambda_3 = 1$ se llega a

$$\begin{aligned} \mathcal{H}^{(3)} = & \frac{1}{2} (p_1^2 + p_2^2 + p_3^2) + \delta_1 (q_1^2 + q_2^2 + q_3^2) \\ & + 2\alpha_2 ((q_2 - q_1)^4 + (q_3 - q_1)^4 + (q_3 - q_2)^4) + \beta_3 (q_1 + q_2 + q_3)^4 \end{aligned} \quad (77)$$

sistema que generaliza el potencial de Chudnovsky.

Perturbaciones séxticas homogéneas

Estas perturbaciones se generan a partir de la elección $f(x) = \alpha_3 x^3$ y $g(y) = \beta_5 y^6$, lo cual conduce a una perturbación del tipo:

$$f + g = \alpha_3 \left(\sum_{i < j}^N (\lambda_j q_i - \lambda_i q_j)^2 \right)^3 + \beta_5 \left(\sum_{i=1}^N \lambda_i q_i \right)^6. \quad (78)$$

Para el caso 3D con, $\lambda_1 = \lambda_2 = 1$ y $\alpha_2 = \beta_5 = \frac{b}{2}$ se recobra un conocido sistema integrable de osciladores de grado seis acoplados:

$$\mathcal{H}^{(2)} = \frac{1}{2} (p_1^2 + p_2^2) + \delta_1 (q_1^2 + q_2^2) + b (q_1^6 + q_2^6 + 15 q_1^2 q_2^2 (q_1^2 + q_2^2)). \quad (79)$$

El cálculo del invariante adicional en todos estos casos se basa en la sustitución directa de la fórmula general (67).

5.3 Otras perturbaciones integrables en ND

Como segundo resultado importante de este Capítulo presentamos otra familia de perturbaciones integrables, que incluye una función homogénea de grado -2 en las coordenadas, junto con otra función radial arbitraria:

$$\begin{aligned} \mathcal{H}_r &= \frac{1}{2} B_+ + \delta_1 B_- + \mathcal{V}_{-2}(A_-, B_-) + \mathcal{J}(B_-) \\ &= \frac{1}{2} \sum_{i=1}^N p_i^2 + \delta_1 \sum_{i=1}^N q_i^2 + \mathcal{V}_{-2} \left(\sum_{i=1}^N \lambda_i q_i, \sum_{i=1}^N q_i^2 \right) + \mathcal{J} \left(\sum_{i=1}^N q_i^2 \right) \end{aligned} \quad (80)$$

donde $\delta_1, \lambda_i (i = 1, \dots, N)$ son parámetros libres.

Demostramos que la integral adicional de este sistema es

$$\mathcal{I}_r^{(N)} = \sum_{1 \leq i < j}^N (q_j p_i - q_i p_j)^2 + 2 \left(\sum_{i=1}^N q_i^2 \right) \mathcal{V}_{-2} \left(\sum_{i=1}^N \lambda_i q_i, \sum_{i=1}^N q_i^2 \right). \quad (81)$$

Una vez más, el aspecto más importante del sistema anterior recae en la libertad de elección de ambas funciones arbitrarias.

Un nuevo modelo ND integrable

Es muy interesante destacar que si tomamos en el sistema anterior todos los $\lambda_i = 1$ junto con

$$\mathcal{V}_{-2}(A_-, B_-) = \varepsilon \frac{A_-^2 - B_-}{B_- (MB_- - A_-^2)} \quad (82)$$

se llega al nuevo sistema integrable

$$\mathcal{H}_r^{(N)} = \frac{1}{2} \sum_{i=1}^N p_i^2 + \delta_1 \sum_{i=1}^N q_i^2 + \varepsilon \frac{2 \sum_{1 \leq i < j}^N q_i q_j}{\left(\sum_{i=1}^N q_i^2 \right) \sum_{1 \leq i < j}^N (q_i - q_j)^2} + \mathcal{J} \left(\sum_{i=1}^N q_i^2 \right) \quad (83)$$

que es la generalización clásica ND de un sistema cuántico exactamente soluble recientemente introducido como generalización 2D del hamiltoniano de Calogero-Moser.

Capítulo 6.

Extendiendo la integrabilidad de 2D a ND

En este último Capítulo se aborda la generalización ND con simetría axial de potenciales integrables 2D. Para ello, se mostrará en la Memoria el esquema algebraico utilizado de una manera pormenorizada. Es importante señalar, que no sólo se van a reobtener las expresiones conocidas de sistemas hamiltonianos integrables en un marco elegante como el de la simetría de coálgebra, sino que también este último hará posible la adición de términos centrífugos de la forma b_i/q_i^2 a los potenciales ND así obtenidos.

6.1 Desde 2D a ND a través de la simetría axial

En esta Sección se retoma la idea conocida de extensión de ciertos potenciales integrables 2D de la forma $\mathcal{V}(x^2, y)$, siempre que admitan un término centrífugo del tipo $1/x^2$, a sistemas ND integrables mediante una generalización axial del tipo

$$\mathcal{H}^{(N)} = \sum_{i=1}^{N-1} \frac{1}{2} p_{x_i}^2 + \frac{1}{2} p_y^2 + \mathcal{V} \left(\sqrt{\sum_{i=1}^{N-1} x_i^2}, y \right). \quad (84)$$

6.2 Sistemas hamiltonianos con simetría $sl(2, \mathbb{R}) \oplus h_6$

En esta Sección presentamos el fundamento algebraico de la construcción mencionada en 6.1, que permitirá la generalización ND. La idea básica es que las $(N - 1)$ primeras coordenadas canónicas de los hamiltonianos provienen de una coálgebra $sl(2, \mathbb{R})$, mientras que la coordenada N -ésima proviene de la realización a una partícula de h_6 .

$$J_+ = \sum_{i=1}^{N-1} p_i^2 \quad J_- = \sum_{i=1}^{N-1} q_i^2 \quad J_3 = \sum_{i=1}^{N-1} q_i p_i \quad (85)$$

$$\begin{aligned} A_+ &= p_N & A_- &= q_N & K &= q_N p_N - \frac{1}{2} \\ B_+ &= p_N^2 & B_- &= q_N^2 & M &= 1 \end{aligned} \quad (86)$$

En este esquema, es importante remarcar que $(N - 2)$ integrales van a provenir de $sl(2, \mathbb{R})$, mientras que la integral adicional se calcula de manera constructiva a partir de la integral del sistema 2D en su traducción adecuada al marco de $sl(2, \mathbb{R}) \oplus h_6$.

6.3 Integrabilidad completa

En esta Sección, se demuestra que cualquier hamiltoniano integrable 2D de la forma

$$\mathcal{H}^{(2)} = \frac{1}{2}(p_x^2 + p_y^2) + \mathcal{V}(x^2, y) \quad (87)$$

cuya integral a dos cuerpos se escribe como

$$\mathcal{I}^{(2)} = \mathcal{I}^{(2)}(p_x^2, x^2, x p_x, y p_y, p_y^2, y^2, p_y, y) \quad (88)$$

donde $x = q_1$ e $y = q_N = q_2$ es generalizable a ND, ya que puede escribirse como

$$\mathcal{H} = \mathcal{H}(J_+, J_-, J_3, K, B_+, B_-, A_+, A_-, M). \quad (89)$$

6.4 Sistemas ND de Hénon-Heiles

Como ejemplos importantes de esta construcción se generalizan a ND los tres potenciales conocidos integrables de tipo Hénon-Heiles en 2D: El sistema de Sawada-Kotera, el sistema KdV y el sistema de Kaup-Kuperschmidt.

Por ejemplo, el hamiltoniano de Sawada-Kotera en 2D tiene la siguiente expresión

$$\mathcal{H}^{(2)} = \frac{1}{2}(p_1^2 + p_2^2) + \delta(q_1^2 + q_2^2) + \alpha(q_1^2 q_2 + \frac{1}{3} q_2^3) \quad (90)$$

y en términos de $sl(2, \mathbb{R}) \oplus h_6$ se escribe como

$$\mathcal{H} = \frac{1}{2}(J_+ + B_+) + \delta(J_- + A_-^2) + \alpha \left(J_- A_- + \frac{1}{3} A_-^3 \right). \quad (91)$$

Utilizando el formalismo anteriormente descrito, la integral adicional del movimiento para (91) tiene la siguiente expresión

$$\begin{aligned} \mathcal{I} = & -2\alpha A_- J_3^2 - 3\delta A_+^2 A_-^2 - \alpha A_+^2 A_-^3 - 3\delta^2 A_-^4 - 2\alpha\delta A_-^5 - \frac{\alpha^2}{3} A_-^6 \\ & - 3\delta A_+^2 J_- - 3\alpha A_+^2 A_- J_- - 2\alpha\delta A_-^3 J_- - \frac{\alpha^2}{2} A_-^4 J_- \\ & - 3\delta^2 J_-^2 - 4\alpha\delta A_- J_-^2 - 2\alpha^2 A_-^2 J_-^2 + \frac{\alpha^2}{6} J_-^3 \\ & + 6\delta A_+ A_- J_3 + 3\alpha A_+ A_-^2 J_3 + \alpha A_+ J_- J_3 \\ & - J_+(A_-^2 + J_-)(3\delta + \alpha A_-) - \frac{3}{4} J_+^2 - \frac{3}{4} A_+^4. \end{aligned} \quad (92)$$

La generalización ND de este sistema junto con su integral adicional tienen la siguiente forma

$$\mathcal{H}^{(N)} = \sum_{i=1}^N \frac{1}{2} p_i^2 + \delta \sum_{i=1}^N q_i^2 + \alpha \left(q_N \sum_{i=1}^{N-1} q_i^2 + \frac{1}{3} q_N^3 \right) \quad (93)$$

$$\begin{aligned} \mathcal{I}^{(N)} = & -2\alpha q_N \left(\sum_{i=1}^{N-1} p_i q_i \right)^2 - p_N^2 q_N^2 (3\delta + \alpha q_N) - q_N^4 (3\delta^2 + 2\alpha\delta q_N) \\ & - \frac{\alpha^2}{3} q_N^6 - 3\delta p_N^2 \left(\sum_{i=1}^{N-1} q_i^2 \right) - 3\alpha p_N^2 q_N \left(\sum_{i=1}^{N-1} q_i^2 \right) \\ & - 2\alpha\delta q_N^3 \left(\sum_{i=1}^{N-1} q_i^2 \right) - \frac{\alpha^2}{2} q_N^4 \left(\sum_{i=1}^{N-1} q_i^2 \right) - 3\delta^2 \left(\sum_{i=1}^{N-1} q_i^2 \right)^2 \\ & - 4\alpha\delta q_N \left(\sum_{i=1}^{N-1} q_i^2 \right)^2 - 2\alpha^2 q_N^2 \left(\sum_{i=1}^{N-1} q_i^2 \right)^2 + \frac{\alpha^2}{6} \left(\sum_{i=1}^{N-1} q_i^2 \right)^3 \\ & - \left(\sum_{i=1}^{N-1} p_i^2 \right) (3\delta + \alpha q_N) \left[q_N^2 + \left(\sum_{i=1}^{N-1} q_i^2 \right) \right] - \frac{3}{4} \left(\sum_{i=1}^{N-1} p_i^2 \right)^2 - \frac{3}{4} p_N^4. \end{aligned} \quad (94)$$

6.5 Potenciales de Ramani ND

Otro caso muy interesante es el de los conocidos potenciales de Ramani en 2D, dados por

$$\mathcal{H}^{(2)} = \frac{1}{2} (p_1^2 + p_2^2) + \mathcal{V}_M(q_1, q_2) = \frac{1}{2} (p_1^2 + p_2^2) + \sum_{k=0}^{\lfloor \frac{M}{2} \rfloor} 2^{M-2k} \binom{M-k}{k} q_1^{2k} q_2^{M-2k}. \quad (95)$$

La integral del movimiento para el caso 2D tiene la siguiente expresión

$$\mathcal{I}^{(2)} = -q_2 p_1^2 + q_1 p_1 p_2 + q_1^2 \mathcal{V}_{M-1}. \quad (96)$$

Mediante la aplicación del formalismo anterior hemos conseguido la generalización ND de la superposición de toda la serie de potenciales de la serie dando lugar al siguiente sistema integrable:

$$\mathcal{H}_R^{(N)} = \frac{1}{2} \sum_{i=1}^N p_i^2 + \sum_{i=2}^R \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} \alpha_i 2^{i-2k} \binom{i-k}{k} \left(\sum_{j=1}^{N-1} q_j^2 \right)^k q_N^{i-2k} \quad (97)$$

cuya integral adicional es

$$\begin{aligned} \mathcal{I}_R^{(N)} = & -q_N \left(\sum_{i=1}^{N-1} p_i^2 \right) + p_N \left(\sum_{i=1}^{N-1} p_i q_i \right) \\ & + \left(\sum_{l=1}^{N-1} q_l^2 \right) \sum_{i=2}^R \sum_{k=0}^{\lfloor \frac{i-1}{2} \rfloor} \alpha_i 2^{i-1-2k} \binom{i-1-k}{k} \left(\sum_{j=1}^{N-1} q_j^2 \right)^k q_N^{i-1-2k}. \end{aligned} \quad (98)$$

6.6 Añadiendo términos centrífugos

Debido a la estructura algebraica subyacente a los modelos anteriores, todos ellos van a ser susceptibles de ser modificados mediante la inclusión de términos centrífugos. En este caso las $(N-2)$ integrales provenientes de la simetría de coálgebra van a escribirse en la forma:

$$\mathcal{C}^{(m)} = \sum_{i < j}^m (q_i p_j - q_j p_i)^2 + \sum_{i < j}^m \left(b_i \frac{q_j^2}{q_i} + b_j \frac{q_i^2}{q_j} \right) \quad m = 2, \dots, N-1. \quad (99)$$

Como ejemplo, vamos a mostrar algunos de estos casos, como por ejemplo el hamiltoniano generalizado de tipo Sawada-Kotera, que resulta ser

$$\mathcal{H}_b^{(N)} = \sum_{i=1}^{N-1} \frac{1}{2} \left(p_i^2 + \frac{b_i}{q_i^2} \right) + \frac{1}{2} p_N^2 + \delta \sum_{i=1}^N q_i^2 + \alpha \left(q_N \sum_{i=1}^{N-1} q_i^2 + \frac{1}{3} q_N^3 \right) \quad (100)$$

donde demostramos que la integral adicional tiene la siguiente forma:

$$\begin{aligned}
\mathcal{I}^{(N)} = & -2\alpha q_N \left(\sum_{i=1}^{N-1} p_i q_i \right)^2 - p_N^2 q_N^2 (3\delta + \alpha q_N) - q_N^4 (3\delta^2 + 2\alpha\delta q_N) \\
& - \frac{\alpha^2}{3} q_N^6 - 3\delta p_N^2 \left(\sum_{i=1}^{N-1} q_i^2 \right) - 3\alpha p_N^2 q_N \left(\sum_{i=1}^{N-1} q_i^2 \right) \\
& - 2\alpha\delta q_N^3 \left(\sum_{i=1}^{N-1} q_i^2 \right) - \frac{\alpha^2}{2} q_N^4 \left(\sum_{i=1}^{N-1} q_i^2 \right) - 3\delta^2 \left(\sum_{i=1}^{N-1} q_i^2 \right)^2 \\
& - 4\alpha\delta q_N \left(\sum_{i=1}^{N-1} q_i^2 \right)^2 - 2\alpha^2 q_N^2 \left(\sum_{i=1}^{N-1} q_i^2 \right)^2 + \frac{\alpha^2}{6} \left(\sum_{i=1}^{N-1} q_i^2 \right)^3 \\
& - \left[\sum_{i=1}^{N-1} \left(p_i^2 + \frac{b_i}{q_i^2} \right) (3\delta + \alpha q_N) \right] \left[q_N^2 + \left(\sum_{i=1}^{N-1} q_i^2 \right) \right] \\
& - \frac{3}{4} \left[\sum_{i=1}^{N-1} \left(p_i^2 + \frac{b_i}{q_i^2} \right) (3\delta + \alpha q_N) \right]^2 - \frac{3}{4} p_N^4.
\end{aligned}$$

Igualmente presentamos el ejemplo de sistema integrable de los potenciales de Ramani anterior, al que se le han añadido los correspondientes términos centrífugos:

$$\mathcal{H}_R^{(N)} = \sum_{i=1}^{N-1} \frac{1}{2} \left(p_i^2 + \frac{b_i}{q_i^2} \right) + \frac{1}{2} p_N^2 + \sum_{i=2}^R \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} \alpha_i 2^{i-2k} \binom{i-k}{k} \left(\sum_{j=1}^{N-1} q_j^2 \right)^k q_N^{i-2k} \quad (101)$$

y cuyas integrales del movimiento se proporcionan explícitamente.

6.7 La construcción $sl(2, \mathbb{R}) \oplus sl(2, \mathbb{R})$

Al igual que en la Sección anterior se procede a generalizar en ND los sistemas hamiltonianos 2D de la forma

$$\mathcal{H}^{(2)} = \frac{1}{2} (p_x^2 + p_y^2) + \mathcal{V}(x^2, y^2). \quad (102)$$

Para ello se utilizan dos copias del álgebra $sl(2, \mathbb{R})$ de manera que la primera copia se realiza en términos de los M primeros cuerpos y la segunda copia del álgebra $sl(2, \mathbb{R})$ proporciona desde $M + 1$ hasta N . Una vez más, lo más novedoso de los resultados obtenidos es la posibilidad de inclusión de términos centrífugos manteniendo la integrabilidad.

Como ejemplo de esta construcción, presentamos el siguiente hamiltoniano integrable que representa N osciladores no lineales con acoplamiento de orden cuatro y N términos centrífugos:

$$\begin{aligned}
\mathcal{H}_b^{(N)} = & \frac{1}{2} \sum_{i=1}^M \left(p_i^2 + \frac{b_i}{q_i^2} \right) + \frac{1}{2} \sum_{i=M+1}^N \left(p_i^2 + \frac{b_i}{q_i^2} \right) \\
& + \left(\sum_{i=1}^M q_i^2 \right)^2 + 6 \left(\sum_{i=1}^M q_i^2 \right) \left(\sum_{i=M+1}^N q_i^2 \right) + 8 \left(\sum_{i=M+1}^N q_i^2 \right)^2 \\
& + a \left(\sum_{i=1}^M q_i^2 + 4 \sum_{i=M+1}^N q_i^2 \right) + \frac{\lambda}{\left(\sum_{i=1}^M q_i^2 \right)} + \frac{\mu}{\left(\sum_{i=M+1}^N q_i^2 \right)}. \quad (103)
\end{aligned}$$

Capítulo 7. Suplementos

Al final de esta Memoria se incluyen una serie de apéndices a modo de Suplementos, en los que se presentan brevemente algunas de las nociones matemáticas necesarias para los desarrollos realizados, así como algunos resultados originales complementarios a los ya presentados en Capítulos anteriores.

El primer Suplemento se dedica a los conceptos fundamentales sobre álgebras de Lie, Poisson y funciones de Casimir.

El segundo Suplemento resume el test de integrabilidad de Yoshida-Ziglin para potenciales homogéneos ND. El test da una condición suficiente de la no integrabilidad de estos sistemas. Igualmente, se presentan dos ejemplos concretos de su aplicación.

El tercer Suplemento recoge otras realizaciones simplécticas para álgebras 5D con $R = 3$. Finalmente, el último de ellos se dedica al álgebra h_9 , extensión del álgebra h_6 generada por los Casimires de las subálgebras \mathcal{C}_{h_4} , $\mathcal{C}_{\bar{g}_+}$ y $\mathcal{C}_{\bar{g}_-}$.

Bibliografía

La Memoria finaliza con una amplia Bibliografía que incluye todas las referencias que están directamente relacionadas con el trabajo aquí presentado, así como las páginas del texto en inglés en las que aparecen cada una de ellas. Por otra parte, somos conscientes de la extensa bibliografía existente sobre integrabilidad de sistemas hamil-

tonianos, aunque nuestro intento ha sido ser exhaustivos con los trabajos existentes que hemos considerado conectados con nuestros resultados.

Conclusiones y problemas abiertos

Los resultados originales más importantes de la presente Tesis Doctoral pueden resumirse de la siguiente manera:

- Se ha introducido la condición necesaria para la realización simpléctica genérica de un álgebra de Poisson, de manera que se asegure la integrabilidad de los hamiltonianos ND que puedan ser obtenidos a partir de ella mediante el formalismo de coálgebra.
- Se han obtenido explícitamente las representaciones simplécticas genéricas para las diferentes coálgebras de Lie-Poisson de dimensiones 3, 4, 5 y 6, introduciendo además, una familia infinita de álgebras para la cual se mantiene el criterio de integrabilidad. En todos los casos estudiados se han presentado los correspondientes sistemas hamiltonianos ND junto con sus integrales de movimiento explícitas.
- Se ha probado cómo el hamiltoniano ND genérico definido a partir de la realización simpléctica de la coálgebra h_6 o ‘two-photon’, da lugar a sistemas cuasi-integrables, cuya integral adicional debe de encontrarse mediante métodos alternativos, para alcanzar la integrabilidad completa del sistema.
- Mediante la utilización de diferentes técnicas algebraicas, se han identificado algunas familias infinitas de hamiltonianos provenientes de la coálgebra ‘two-photon’, para las cuales dicha integral adicional existe. Además, esta integral ha sido encontrada de forma explícita. Entre estos sistemas se han propuesto nuevas familias de sistemas integrables naturales, de hamiltonianos electromagnéticos y de flujos geodésicos que definen espacios ND de curvatura no constante.
- Se han encontrado además dos nuevas familias de perturbaciones integrables no lineales del oscilador isótropo en ND. La primera depende de N parámetros y de dos funciones arbitrarias. Además esta familia incluye como casos particulares acoplamientos no lineales entre osciladores de grados cuatro y seis. La segunda familia de perturbaciones integrables, incluye funciones homogéneas de grado menos dos en las coordenadas y una función arbitraria radial. Como un ejemplo concreto de este tipo, se ha propuesto una generalización ND de un modelo 2D de tipo Calogero-Moser.
- Se ha demostrado cómo la generalización ND de muchos sistemas integrables 2D conocidos puede realizarse haciendo uso de la suma directa de algunas coálgebras. En particular, se ha probado que cualquier potencial integrable bidimensional del tipo $\mathcal{V}(x^2, y)$ puede ser generalizado a ND preservando la integrabilidad completa del mode-

lo. De forma similar, se ha obtenido un resultado análogo para potenciales de la forma $\mathcal{V}(x^2, y^2)$. Además, el formalismo permite la adición de términos centrífugos que preservan la integrabilidad. Como caso particular, se han encontrado en ND versiones generalizadas de los tres potenciales integrables de tipo Hénon-Heiles, de la serie de potenciales de Ramani, y se han obtenido nuevos sistemas ND de osciladores acoplados no lineales.

Problemas abiertos

Finalmente, se presentan varios problemas surgidos del presente trabajo, y que a nuestro juicio merecerán un estudio pormenorizado en el futuro.

- En general, las condiciones de integrabilidad presentadas en el Capítulo 2 pueden ser generalizadas en el caso en el que se utilicen realizaciones simplécticas de dimensiones diferentes en cada una de las copias del álgebra. Esta elección conduciría a sistemas hamiltonianos aparentemente distintos de los aquí presentados.
- Otra posibilidad abierta es la búsqueda de coálgebras de dimensión mayor de manera que, bajo la representación simpléctica adecuada, pudieran dar lugar a nuevos sistemas hamiltonianos integrables.
- Igualmente, sería interesante realizar un análisis pormenorizado de algunos de los nuevos hamiltonianos introducidos en esta Memoria, resolviendo de forma explícita sus ecuaciones de movimiento. Recordemos que este problema puede ser abordado mediante la técnica de las *cluster variables* [19]. Esta técnica hace uso de la simetría de coálgebra de cara a definir unas variables dinámicas colectivas adecuadas.
- En esta dirección, nos gustaría señalar que los sistemas hamiltonianos electromagnéticos y de flujos geodésicos presentados en esta Memoria merecen, en nuestra opinión, un estudio detallado en el futuro. Igualmente, la generalización del modelo de tipo Calogero-Moser introducido en el Capítulo 5 también deberá tratarse en mayor profundidad.
- Para la mayoría de los hamiltonianos presentados en esta Memoria, existen dos conjuntos diferentes de integrales del movimiento cuadráticas en los momentos. Este hecho pone de manifiesto la cuestión abierta concerniente a la separabilidad de todos estos modelos en al menos dos sistemas de coordenadas.
- Es claro que la construcción de coálgebra h_6 nos permite escribir muchos otros hamiltonianos a través de sus generadores. En cada caso, y por construcción, siempre se tendrían $(N - 2)$ ‘integrales universales’, de manera que la única tarea a realizar para alcanzar la integrabilidad completa de este tipo de sistemas es calcular la integral adicional, en caso de que ésta exista.

- Es igualmente conocido que la existencia de simetría de coálgebra permite la introducción natural de deformaciones a través de las q -deformaciones de la estructura de Poisson subyacente. En el caso de la coálgebra h_6 , su versión cuántica es conocida [10, 11], de manera que sus análogos en términos de Poisson pudieran ser utilizados para construir deformaciones integrables de algunos de los sistemas aquí presentados.
- Otra posibilidad interesante podría ser la construcción en espacios de curvatura constante de los sistemas ND aquí presentados sobre espacios Euclídeos. Esto ha sido posible mediante la utilización de $sl(2, \mathbb{R})$ [24, 26, 29], y lo sería para los sistemas aquí estudiados mediante la correspondiente construcción basada en h_6 .
- Finalmente, podría resultar ciertamente interesante el estudio de la versión cuántica de algunos de los modelos clásicos aquí presentados.

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Integrability of nonlinear Hamiltonian systems with N degrees of freedom

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Chapter 1

Introduction

*“The mathematician
plays a role the rules of which he creates himself,
as opposed to the physicist
who enacts a role the rules of which are determined by nature.
Having said that, it is undeniable that as time goes by
the rules the mathematician has primarily deemed as appropriate
will eventually end up matching those provided by nature.”*

P. A. M. Dirac

Certainly, the Hamiltonian formalism is the fundamental framework for our description of both Classical and Quantum dynamics. As a consequence, the most usual physical approach of a given system consists in providing either the Hamiltonian function (if the system is classical) or the corresponding Hamiltonian operator (if quantum) for it. On the other hand, from the mathematical viewpoint, the Hamiltonian formalism is based on Lie-algebraic structures which give the unified symmetry approach to Classical and Quantum Mechanics. In fact, although in this work we will concentrate on Classical Mechanical systems, the Lie approach that we adopt would indeed make possible the quantum generalization of all the results here presented.

Therefore, let us consider a system of classical particles with N degrees of freedom. Once the Hamiltonian $\mathcal{H}^{(N)}(q_1, q_2, \dots, q_N, p_1, p_2, \dots, p_N)$ for such system is given as a function depending on N pairs of canonical variables, Hamilton's equations

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} \quad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} \quad i = 1, \dots, N \quad (1.1)$$

will provide the dynamics of the system. In order to investigate it, the actual problem is to find the exact solutions of such differential equations that, in most of the physically interesting cases, are nonlinear and coupled.

This is the reason why, among all possible Hamiltonian systems, a distinguished subset called “exactly solvable systems” is frequently identified: they are those physical models for which Hamilton’s equations can be exactly solved. But the number of known exactly solvable Hamiltonians is very small, and usually the existence of exact solutions is deeply connected with the existence of underlying symmetries (in some sense) of the model under study [98, 135, 184].

This work deals with the construction of new exactly solvable Hamiltonians, and the cornerstone of our approach is the following fact: if for a given Hamiltonian we have enough number of independent constants of the motion, then Hamilton’s equations for such system can be integrated “by quadratures”. This means that (quoting [147]), “its solutions can be obtained by finitely many algebraic operations (including inverting functions) and “quadratures”, i.e., evaluating integrals of given functions”. In a more precise language, the following founding result holds (for a more technical discussion see [147], Chapter 1):

Let $\mathbb{R}^{2N} = \{q_1, \dots, q_N, p_1, \dots, p_N\}$ be the phase space of a Hamiltonian system $\mathcal{H}^{(N)}(q_1, q_2, \dots, q_N, p_1, p_2, \dots, p_N)$ endowed with the standard Poisson bracket

$$\{f, g\} = \sum_{i=1}^N \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i} \frac{\partial f}{\partial p_i} \right). \quad (1.2)$$

Then, if the system has $(N - 1)$ functionally independent integrals of the motion in involution $\{F_1, \dots, F_{N-1}\}$

$$\{\mathcal{H}, F_j\} = 0 \quad \{F_i, F_j\} = 0 \quad i, j = 1, \dots, (N - 1) \quad (1.3)$$

the solutions of Hamilton’s equations for $\mathcal{H}^{(N)}$ can be obtained by quadratures.

Therefore, a Hamiltonian system $\mathcal{H}^{(N)}$ with a (finite) number N of degrees of freedom is called *completely integrable* (in the abovementioned Liouville sense) if such set of $(N - 1)$ functionally independent (and globally defined) constants of the motion \mathcal{F}_k ($k = 1, \dots, N - 1$) does exist [162]. In this sense, complete integrability implies that we would be able to solve explicitly the dynamical behaviour of the system. As a consequence, integrable systems are the paradigm of regular motion, and can be interpreted as the opposite to chaotic systems. Anyhow, it is important to stress that integrability is not the rule but clearly the exception among dynamical systems.

Moreover, there exist some exceptional systems which have more than $(N - 1)$ functionally independent integrals in involution with $\mathcal{H}^{(N)}$: these are the so-called *superintegrable* ones (obviously, only $(N - 1)$ of these integrals can be mutually in involution)

[75, 80, 99, 123]. Moreover, a N -dimensional (hereafter ND) Hamiltonian $\mathcal{H}^{(N)}$ will be called *maximally superintegrable* (MS) if there exists a set of $(2N - 2)$ globally defined functionally independent constants of the motion Poisson-commuting with $\mathcal{H}^{(N)}$; among them, at least two different subsets of $(N - 1)$ constants in involution can be found. Finally, a Hamiltonian system will be called *quasi-maximally superintegrable* (QMS) if it has $(2N - 3)$ independent integrals with the abovementioned properties.

Only a small number of systems are MS in any dimension. The two outstanding examples of this class of systems are the harmonic oscillator and the Kepler potential. Superintegrability has deep geometrical consequences [64, 82], as for the abovementioned examples, in its classical version, all bounded trajectories are periodic. Thus, MS systems would be the paradigm of absolutely non-chaotic motion.

As a consequence of the abovementioned properties, the construction of finite-dimensional integrable Hamiltonian systems, has focused an intense research activity that makes use of many different approaches and techniques (see, for instance, [5, 6, 40, 52, 96, 106, 107, 117, 147, 162, 179, 180, 181]).

Nevertheless, a main problem in this field is the lack of general *sufficient* criterion and/or systematic methods in order to determine whether a given Hamiltonian system is integrable (or not). Several approaches to the characterization of necessary conditions for integrability have been developed (like the Painlevé test [166] or the Yoshida-Ziglin approach [119, 120, 125, 136, 151, 154, 182, 183, 184]) but the actual approach to integrable systems is mainly constructive, because finally one has to find the integrals of motion.

Moreover, even for 2D and 3D Hamiltonians the classification of integrable Hamiltonians is not trivial at all [51, 52, 73, 109, 135, 184], and in higher dimensions difficulties grow exponentially. Therefore, the number of known integrable systems (in the Liouville sense) that can be generalized for an arbitrary number N of degrees of freedom is very scarce. In most of the known cases, such a ND integrability is based on some underlying symmetry that allows for the appropriate propagation of the integrability properties to arbitrary dimension (see the systems described in [26, 37, 53, 54, 55, 81, 90, 111, 112, 113, 117, 127, 137, 157, 158, 160, 165, 169, 175, 180] and references therein).

The aim of this Ph.D. Thesis is to construct new ND integrable Hamiltonian systems with nonlinear interactions, obtaining explicitly their sets of functionally independent integrals of the motion. To face this problem we will use the coalgebra symmetry approach (see [9, 12, 13, 17, 18, 19, 21, 27, 31]) that has been recently introduced as a new symmetry framework that underlies the integrability properties of many systems, and that also allows for the construction of integrable deformations of many of such systems by making use of the theory of quantum groups and algebras [8, 36, 69, 168]. We

stress that in this scheme Casimir functions for the underlying coalgebra are essential in order to provide the integrals of the motion.

The coalgebra symmetry formalism is reviewed in Chapter 2. As a new contribution, the role of the dimensionality of the symplectic realization of a given Poisson coalgebra is analysed in order to propose a necessary condition for the complete integrability of the ND Hamiltonian systems that can be constructed from it. We also point out how the coalgebra symmetry is always providing a certain degree of superintegrability. A systematic study of the symplectic realizations of the 9 real 3D Lie coalgebras [150] is presented, and also some examples of q -Poisson coalgebras are recalled.

In Chapter 3 the full study of the symplectic realizations of all 4D and 5D real Lie coalgebras (see the algebra classification [59, 145]) is given. Afterwards, the completely integrable ND Hamiltonians that can be obtained from those symplectic realizations that fulfill the integrability criteria are presented. The analysis is completed by studying the same problem for all 6D nilpotent Lie coalgebras, and also for a infinite family of Lie coalgebras whose complete set of Casimir functions is found.

In the next Chapter the coalgebra formalism is used to construct new families of completely integrable ND systems which are defined on the 6D two-photon/Schrödinger coalgebra h_6 . In general, these Hamiltonians depend on several arbitrary functions and N free parameters. In particular, integrable natural systems, electromagnetic Hamiltonians and geodesic flows in arbitrary dimensions are presented. In order to prove the complete integrability of all these systems, we present different strategies which are mainly based on the rich subalgebra structure of h_6 .

In the fifth Chapter we further apply the h_6 coalgebra construction in order to introduce two different types of integrable perturbations/deformations of the ND isotropic harmonic oscillator Hamiltonian. The first one is a family of integrable perturbations depending on N parameters and two arbitrary functions. Afterwards, we discuss several examples of quartic and sextic coupled nonlinear oscillators that can be found as particular cases. The second family of ND perturbations includes homogeneous functions with degree -2 in the coordinates and a radial arbitrary function. Both families of systems can be considered as ND generalizations of certain subclasses of 3D integrable Hamiltonians that were proposed for the first time in [179].

The last chapter is devoted to show how the ND generalization of many known 2D integrable Hamiltonians can be achieved by making use the direct sum coalgebras $sl(2, \mathbb{R}) \oplus h_6$ and $sl(2, \mathbb{R}) \oplus sl(2, \mathbb{R})$. The former is used to show that any 2D integrable potential of the type $\mathcal{V}(x^2, y)$ can be generalized to ND by preserving its complete integrability, and by making use of the second coalgebra, a similar result is obtained for potentials of the type $\mathcal{V}(x^2, y^2)$. Moreover, the $sl(2, \mathbb{R})$ sector of the symmetry allows the systematic introduction of centrifugal terms, and the corresponding gener-

alized integrals of the motion are explicitly found. As outstanding examples of this construction, we present for the first time ND generalized versions of the three integrable Hénon-Heiles systems, as well as of the Ramani series of potentials and other ND systems of coupled nonlinear oscillators.

Finally, in a closing Chapter, the main conclusions obtained from our research program are summarized, together with a list of open problems for further investigation.

This work is completed with a Supplements Chapter in which several mathematical tools are reviewed. We also include there some original complementary results that we have decided to present separately from the main body of the text. Concerning the bibliography, we have tried to include all the references that are directly related to the results here presented, although we are aware of the extensive bibliography on integrable systems that can be found.

A significant part of the original results presented in this Ph.D. Thesis has been already included in the works [20, 23, 27, 28, 30, 31] and [32]. Nevertheless, this manuscript includes new results that are not contained in such papers, and also provides a unified approach to the application of coalgebra symmetries in order to construct new physically meaningful ND integrable systems.

Chapter 2

Integrable systems with coalgebra symmetry

“A system of differential equations is only more or less integrable.”

H. Poincaré

Coalgebras are either Poisson (commutative) or commutator algebras (non commutative) endowed with a compatible comultiplication structure, and have been recently shown to be the ‘hidden’ symmetries underlying the integrability properties of a wide class of ND (super)integrable classical and quantum Hamiltonian systems (see [12, 13, 14, 16, 19, 21, 22, 24, 25] and references therein). In this construction, once a symplectic (resp. operatorial) realization of the coalgebra is given, their generators play the role of dynamical symmetries of the Hamiltonian, while the coproduct map of the coalgebra is used to ‘propagate’ the integrability to arbitrary dimension, and to give the explicit form of the integrals of motion as the coproducts of the Casimirs of the initial algebra.

From this coalgebra approach, several well-known (super)integrable systems have been recovered. Also some integrable deformations for them, as well as new integrable systems, have been obtained. In particular, this integrability-preserving coalgebra deformation procedure has been used to introduce integrable free motions on spaces with either constant or variable curvature, and (super)integrable potentials can also be defined on such spaces through the coalgebra approach [22, 24, 25, 26, 29].

Although we shall concentrate on classical mechanical systems obtained from (commutative) Poisson coalgebras, we stress that all the (super)integrability properties of the coalgebra symmetric systems that we are going to describe are preserved at the

quantum mechanical level by using the corresponding (noncommutative) operator coalgebras. In this way, the coalgebra approach has also been used, for instance, to solve in [132, 133] the quantum Calogero–Gaudin system [56], as well as supersymmetric generalizations of this model and of its q -deformations, which have been constructed by starting from the underlying supersymmetric coalgebra structures [45, 134]. Finally we recall that a comodule algebra generalization of the coalgebra symmetry has been introduced in [18].

In this Chapter we review from a more systematic viewpoint this algebraic approach to Hamiltonian integrability, and the role of symplectic realizations of Poisson coalgebras in the formalism will be emphasized. In this way we present a more global perspective on both the possibilities and limitations of the coalgebra symmetry approach. In particular, we perform a detailed analysis of 3-dimensional Lie coalgebras that admit symplectic realizations that lead to new infinite families of completely integrable (in fact, superintegrable) Hamiltonian systems with N degrees of freedom. Such systems are fully constructed, together with the explicit form of their integrals of the motion. In the final Section, we present some examples of q -Poisson coalgebras in order to show how they can be used, in order to construct integrable deformations of ND Hamiltonian systems.

2.1 Hamiltonian systems on Poisson coalgebras

We recall that a coalgebra (A, Δ) is a (unital, associative) algebra A endowed with a coproduct map [79]:

$$\Delta : A \rightarrow A \otimes A, \quad (2.1)$$

which is coassociative

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta \quad (2.2)$$

that is, the following diagram is commutative:

$$\begin{array}{ccccc}
 & & A \otimes A & & \\
 & \nearrow \Delta & & \searrow \Delta \otimes id & \\
 A & & & & A \otimes A \otimes A \\
 & \searrow \Delta & & \nearrow id \otimes \Delta & \\
 & & A \otimes A & &
 \end{array}$$

Due to the coassociativity property, the comultiplication Δ provides a ‘two-fold way’ for the definition of the objects on $A \otimes A \otimes A$ that, as we shall explain in Section (2.2.3), will be deeply connected with *superintegrability* properties.

In addition, Δ has to be an algebra homomorphism from A to $A \otimes A$:

$$\Delta(ab) = \Delta(a)\Delta(b) \quad \forall a, b \in A. \quad (2.3)$$

Moreover, if A is a Poisson algebra and

$$\Delta(\{a, b\}_A) = \{\Delta(a), \Delta(b)\}_{A \otimes A} \quad \forall a, b \in A, \quad (2.4)$$

we shall say that (A, Δ) is a Poisson coalgebra.

Let us summarize the general construction of ref. [12]. Let (A, Δ) be a Poisson coalgebra with l generators X_i ($i = 1, \dots, l$), and r is the the number of functionally independent Casimir functions $\mathcal{C}_j(X_1, \dots, X_l)$ (with $j = 1, \dots, r$). Among them, a certain subset of R Casimir funtions will be non-linear functions of the generators of the coalgebra, and will be the relevant ones as far as integrability is concerned.

The coassociative coproduct $\Delta \equiv \Delta^{(2)}$ has to be a Poisson map with respect to the usual Poisson bracket on $A \otimes A$:

$$\{X_i \otimes X_j, X_r \otimes X_s\}_{A \otimes A} = \{X_i, X_r\}_A \otimes X_j X_s + X_i X_r \otimes \{X_j, X_s\}_A. \quad (2.5)$$

Then, the m -th coproduct map $\Delta^{(m)}(X_i)$

$$\Delta^{(m)} : A \rightarrow A \otimes A \otimes \dots \otimes A \quad (2.6)$$

can be defined by applying recursively the coproduct $\Delta^{(2)}$ in the form

$$\Delta^{(m)} := (id \otimes id \otimes \dots \otimes id \otimes \Delta^{(2)}) \circ \Delta^{(m-1)}. \quad (2.7)$$

Such an induction ensures that $\Delta^{(m)}$ is also a Poisson map.

Table 1. Functions obtained by applying the coproduct map.

X_1	X_2	...	X_l	\mathcal{C}_1	\mathcal{C}_2	...	\mathcal{C}_r
$\Delta^{(2)}(X_1)$	$\Delta^{(2)}(X_2)$...	$\Delta^{(2)}(X_l)$	$\Delta^{(2)}(\mathcal{C}_1)$	$\Delta^{(2)}(\mathcal{C}_2)$...	$\Delta^{(2)}(\mathcal{C}_r)$
$\Delta^{(3)}(X_1)$	$\Delta^{(3)}(X_2)$...	$\Delta^{(3)}(X_l)$	$\Delta^{(3)}(\mathcal{C}_1)$	$\Delta^{(3)}(\mathcal{C}_2)$...	$\Delta^{(3)}(\mathcal{C}_r)$
\vdots	\vdots		\vdots	\vdots	\vdots	\vdots	\vdots
$\Delta^{(N)}(X_1)$	$\Delta^{(N)}(X_2)$...	$\Delta^{(N)}(X_l)$	$\Delta^{(N)}(\mathcal{C}_1)$	$\Delta^{(N)}(\mathcal{C}_2)$...	$\Delta^{(N)}(\mathcal{C}_r)$

In this way, we can construct the set of functions shown in Table 1. From them, given a smooth function $\mathcal{H}(X_1, \dots, X_l)$, the N -sites Hamiltonian is defined as the N -th coproduct of \mathcal{H} :

$$\mathcal{H}^{(N)} := \Delta^{(N)}(\mathcal{H}(X_1, \dots, X_l)) = \mathcal{H}(\Delta^{(N)}(X_1), \dots, \Delta^{(N)}(X_l)). \quad (2.8)$$

From [12] it can be proven that the set of $r \cdot N$ functions ($m = 1, \dots, N; j = 1, \dots, r$)

$$\mathcal{C}_j^{(m)} := \Delta^{(m)}(\mathcal{C}_j(X_1, \dots, X_l)) = \mathcal{C}_j(\Delta^{(m)}(X_1), \dots, \Delta^{(m)}(X_l)) \quad (2.9)$$

Poisson-commute with the Hamiltonian

$$\left\{ \mathcal{C}_j^{(m)}, \mathcal{H}^{(N)} \right\}_{A \otimes A \otimes \dots \otimes A} = 0 \quad (2.10)$$

and, is in involution:

$$\left\{ \mathcal{C}_i^{(m)}, \mathcal{C}_j^{(n)} \right\}_{A \otimes A \otimes \dots \otimes A} = 0 \quad m, n = 1, \dots, N \quad i, j = 1, \dots, r. \quad (2.11)$$

This construction is completely general, and holds when the Poisson bracket is replaced by the commutator, thus providing an immediate generalization of this approach to the case of quantum mechanical systems, for which the usual ordering prescriptions are taken into account.

There are two main classes of coalgebras to be used to generate ND integrable systems through this procedure. The first one are Lie–Poisson algebras g^* with generators X_i ($i = 1, \dots, l$) and Casimir functions $\mathcal{C}_j(X_1, \dots, X_l)$ ($j = 1, \dots, r$), which are always endowed with the (primitive) coalgebra structure

$$\Delta(X_i) = X_i \otimes 1 + 1 \otimes X_i. \quad (2.12)$$

Then, a very natural choice is to consider Hamiltonians $\mathcal{H}(X_1, \dots, X_l)$ that, under the iterated application of the coproduct map, will generate dynamical systems on $g^* \otimes g^* \otimes \dots \otimes g^*$ with constants of the motion in involution coming from the non-linear Casimir functions. These will be the kind of systems that will be systematically explored in the two following Chapters.

The second one are the Poisson analogues of quantum algebras and groups [69, 79, 168], (the so-called q -Poisson algebras) which are also (deformed) coalgebras (A_z, Δ_z) (where z is the deformation parameter). Consequently, any function of the generators of a given ‘quantum’ Poisson algebra (with deformed Casimir elements $\mathcal{C}_{z,j}$) will provide a deformation of the Hamiltonian constructed in terms of the undeformed Lie–Poisson coalgebra. Some of this systems have been already explored in the previous literature

(for instance, in [12, 13, 15, 22]), and we stress that all the systems here presented can be deformed without altering their integrability properties provided that appropriate coalgebra deformations are constructed. In this respect we stress that the problem of the classification of quantum deformations is only fully solved for all 3D Lie algebras and for some isolated cases in slightly higher dimensions (see [14] and references therein). For a deeper insight into coalgebras, Hopf algebras and quantum group theory, we refer to [8, 36, 69, 79, 168]. In the final Section of this Chapter, we will explicitly present an example of the coalgebra construction for a q -Poisson coalgebra.

Some remarks are in order.

- In general, we cannot say *a priori* that $\mathcal{H}^{(N)}$ is a completely integrable Hamiltonian system because [27]: (i) We have to determine the number of degrees of freedom of $\mathcal{H}^{(N)}$ by choosing an explicit symplectic realization of the coalgebra (A, Δ) . (ii) Once this number is fixed, we have to check whether the number of independent invariants extracted from $\mathcal{C}_j^{(m)}$ (under such a specific symplectic realization) is enough to guarantee complete integrability.
- However, if the coalgebra symmetry provides the complete integrability for $\mathcal{H}^{(N)}$, the integrals of the motion can always be obtained in arbitrary dimension N in an explicit form. Hence the coalgebra symmetry arises as a unified approach to integrability since we obtain families of ND systems that share a very large common set of integrals of the motion. Furthermore, in this case $\mathcal{H}^{(N)}$ will probably be superintegrable (to some extent) [21].
- In the case of q -deformations of Poisson coalgebras, the Hamiltonians obtained through the coalgebra approach are integrable deformations of the $z = 0$ ($q = 1$) cases in such a manner that different quantum algebras give rise to different integrable deformations. Moreover, this fact can provide some relevant information concerning the geometric/physical interpretation of the deformation parameter.
- This construction holds for noncommutative coalgebras as well. Thus, quantum mechanical systems can also be constructed (although ordering problems have to be fixed).
- From the point of view of the Poisson algebra of the constants of the motion (see [75]), the coalgebra approach is also relevant: once the coalgebra A is fixed, the $\{\mathcal{C}_i^{(m)}, \mathcal{C}_j^{(n)}\}$ brackets are fixed for any \mathcal{H} defined on A . The only brackets depending on the specific choice of the Hamiltonian are $\{\mathcal{H}, \mathcal{C}_j^{(m)}\} \forall j$ and $\forall m$. These are the only brackets that would distinguish two different systems \mathcal{H}_1 and \mathcal{H}_2 with the same coalgebra symmetry.

Before entering into explicit examples of the coalgebra construction, two important and general aspects have to be more deeply analysed in order to provide a global overview on the subject: they are, the role of the symplectic realizations [65] (which has been thoroughly studied in [27]) and the superintegrability features of the coalgebra symmetry (which were presented for the first time in [21]).

2.2 Symplectic realizations and complete integrability

For any l -dimensional Poisson coalgebra (A, Δ) and for any smooth Hamiltonian function \mathcal{H} depending on l variables, we can construct a Hamiltonian system on the Poisson manifold $A^{\otimes N}$ constructed as N -tensor copies of A . This is a ‘cluster-type’ dynamical system [19] with $l \cdot N$ dynamical variables whose evolution equations are

$$\dot{X}_{(i,m)} = \left\{ X_{(i,m)}, \mathcal{H}^{(N)} \right\} \quad i = 1, \dots, l \quad m = 1, \dots, N \quad (2.13)$$

where $X_{(i,m)}$ denotes the generator X_i living on the m -th copy of A . The r Casimir functions of A generate a maximum number of $r \cdot N$ integrals of the motion for $\mathcal{H}^{(N)}$ (see Table 1) but since, in general, $l - r \geq 2$ we have always less than $l \cdot N - 1$ integrals and, therefore, complete integrability for \mathcal{H} cannot be reached in terms of the ‘algebraic’ dynamical variables $X_{(i,m)}$.

However, we can get a specialization of the coalgebra formalism by working on the symplectic leaves of the initial Poisson coalgebra A , that can be parametrized through suitable symplectic realizations. If A has r independent Casimir functions, a symplectic leaf of A (always even dimensional) will be denoted by $A_{(k_1, k_2, \dots, k_r)}$, where the leaf is characterized by a given set of constant values (k_1, k_2, \dots, k_r) for the Casimir functions.

An s -dimensional *symplectic realization* D for $A_{(k_1, k_2, \dots, k_r)}$ is given (locally) in terms of s pairs (q_i, p_i) of canonical Darboux variables

$$D : x \rightarrow x(q_1, p_1, q_2, p_2, \dots, q_s, p_s) \quad (2.14)$$

where x is any point on $A_{(k_1, k_2, \dots, k_r)}$. Different symplectic leaves $A_{(k_1, i, k_2, i, \dots, k_r, i)}$ can be chosen for the different copies of A within $A^{\otimes N}$. In this way $\mathcal{H}^{(N)}$ is defined on the N -th tensor product of the symplectic leaves

$$A_{(k_{1,1}, k_{2,1}, \dots, k_{r,1})} \otimes A_{(k_{1,2}, k_{2,2}, \dots, k_{r,2})} \otimes \dots \otimes A_{(k_{1,N}, k_{2,N}, \dots, k_{r,N})} \quad (2.15)$$

where $k_{i,m}$ is the value of the i -th Casimir for the m -th symplectic leaf.

If we consider symplectic realizations with the same dimension s for all the sites in the tensor product chain, $\mathcal{H}^{(N)}$ given by (2.8) turns out to be a function of $N \cdot s$ pairs of

canonical variables, i.e., it defines a Hamiltonian system with $N \cdot s$ degrees of freedom. Therefore, we need a number of $(N \cdot s - 1)$ independent and globally defined constants of the motion in involution to state that such Hamiltonian defines a completely integrable system.

2.2.1 Integrability conditions

At this point, we have to characterize the number of integrals that the coalgebra formalism provides. First of all, note that under the symplectic realizations that we are going to consider, the coproducts of linear Casimirs give just a sum of numerical constants with no dynamical information. Thus, as far as integrability properties are concerned, we have to consider only non-linear Casimirs, and for each of them we get $(N - 1)$ integrals coming from the $2, \dots, N$ -th coproducts (the one-site evaluation of such Casimirs gives, by construction, constants). Since we have R non-linear Casimirs (hereafter we will assume that $R \geq 1$, since coalgebras without any non-linear Casimir will be excluded) we get a maximum possible number of integrals in involution given by

$$(N - 1) \cdot R. \quad (2.16)$$

In order to get complete integrability we should have enough number of integrals, which means that

$$N \cdot s - 1 \leq (N - 1) \cdot R \quad (2.17)$$

thus, we need that the chosen symplectic realization D of A fulfils

$$s \leq R - \frac{R - 1}{N}. \quad (2.18)$$

Therefore the necessary condition for complete integrability is:

- The symplectic realization has to be of the type $s = 1$ for coalgebras with $R = 1$.
- The symplectic realization has to be of the type $s < R$ for coalgebras with $R > 1$.

Obviously, this condition is not sufficient since the functional independence of a $(N \cdot s - 1)$ dimensional subset of integrals has to be explicitly checked in each case. Note that this condition *does not depend explicitly* of the dimension l of the coalgebra and that low values of s are certainly preferred from the integrability viewpoint.

2.2.2 Generic symplectic realizations

Now let us consider a particular type of symplectic realizations whose s is always fixed by the dimension l and the number r of independent Casimirs of the coalgebra. We shall call “generic” to the symplectic realization with dimension $s = s_m$ given by

$$s_m = \frac{l - r}{2}. \quad (2.19)$$

This symplectic realization is “generic” in the sense that will incorporate as many parameters (k_1, \dots, k_r) as the number of Casimir functions of the coalgebra. Certainly, symplectic realizations with $s < s_m$ could exist, but they are not “generic” in the sense that can give rise to degeneracies in the Casimir functions (for an explicit example, see [16]).

The sufficient condition for integrability on the generic symplectic realization leads to

$$s_m = \frac{l - r}{2} \leq R - \frac{R - 1}{N} \quad (2.20)$$

which leads to the final expression

$$l \leq (2R + r) - \frac{2}{N}(R - 1). \quad (2.21)$$

The former expression (2.21), gives us the integrability condition in terms of the dimension l and the number R . Thus, we get that the complete integrability for the generic symplectic realization is achieved provided that:

- The dimension $l \leq 2R + r = 2 + r$ in the case of $R = 1$ coalgebras.
- The dimension $l < 2R + r$ in the case of $R > 1$ coalgebras.

In this work we will systematically consider all the Lie coalgebras with $l \leq 5$ for which their generic symplectic realizations fulfill the integrability condition, and we will construct all the completely integrable systems associated with them. Through this work, we shall follow the Lie algebra classifications and notation summarized in [145].

Two remarks are in order:

- For *the classical simple Lie algebras*, $R = r$ so we have that for $R = 1$ coalgebras the condition is $l \leq 3$ and for $R > 1$ coalgebras we should have that $l < 3R$. Therefore, apart from the $r = 1$ cases, this result excludes all simple Lie algebras to provide complete integrable systems in the generic symplectic realization.

• Let us consider *any Lie coalgebra in which the generic symplectic realization has $s_m = 1$* . In that case, provided that $R \geq 1$, the coalgebra fulfills always the integrability condition under that symplectic realization and integrable systems can be always constructed.

2.2.3 Coalgebra superintegrability

Instead of (2.7), another recursion relation for the m -th coproduct map can be defined as [21]:

$$\Delta_R^{(m)} := (\Delta^{(2)} \otimes \text{id} \otimes \dots \otimes \text{id} \otimes \text{id}) \circ \Delta_R^{(m-1)}. \quad (2.22)$$

Due to the coassociativity property of the coproduct, this new expression will provide exactly the same expressions for the N -th coproduct of any generator of A :

$$\Delta^{(N)}(X_i) \equiv \Delta_R^{(N)}(X_i). \quad (2.23)$$

However, if we label from 1 to N the sites of the chain $A \otimes A \otimes \dots \otimes A$, lower dimensional m th-coproducts (with $2 < m < N$) will be ‘different’ in the sense that the ‘left’ coproducts $\Delta^{(m)}$ (2.7) will contain objects living on the tensor product space $1 \otimes 2 \otimes \dots \otimes m$, whilst the ‘right’ coproducts $\Delta_R^{(m)}$ will be defined on the sites $(N - m + 1) \otimes (N - m + 2) \otimes \dots \otimes N$.

Therefore, the coalgebra symmetry of a given Hamiltonian gives rise to two ‘pyramidal’ sets of $r \cdot N$ integrals of the motion in involution that Poisson-commute with $\mathcal{H}^{(N)}$ [21], under the corresponding symplectic realization D (2.14). Namely, both sets are just the ‘left’ integrals above considered $\mathcal{C}_j^{(m)}$ (2.9) together with the ‘right’ ones given by

$$\mathcal{C}_{j,(m)} := \Delta_R^{(m)}(\mathcal{C}_j(X_1, \dots, X_l)) = \mathcal{C}_j(\Delta_R^{(m)}(X_1), \dots, \Delta_R^{(m)}(X_l)) \quad (2.24)$$

such that $\mathcal{C}_j^{(N)} \equiv \mathcal{C}_{j,(N)}$, while both $\mathcal{C}_j^{(1)}$ and $\mathcal{C}_{j,(1)}$ are constants.

The very same arguments discussed in Section 2.2 on the connection between integrability and the dimensionality of the chosen symplectic realizations can be applied to the *additional* set of ‘right’ integrals $\mathcal{C}_{j,(m)}$. In this way, a completely integrable Hamiltonian $\mathcal{H}^{(N)}$ with coalgebra symmetry will be, in principle, *superintegrable*.

For the sake of simplicity let us assume that we are dealing with a Poisson coalgebra with $R = 1$, that is, with N degrees of freedom and with a single nonlinear Casimir. Therefore if we take a symplectic realization with $s = 1$ the necessary condition for integrability (2.18) is fulfilled. In this case, the coalgebra approach gives rise to $(2N - 3)$ functionally independent integrals, which are displayed in Table 2, and

Table 2. Coalgebra symmetry and quasi-maximal syperintegrability

‘Left’ set of $(N - 1)$ integrals $\mathcal{C}^{(m)}$ in involution	Tensor product space
$\mathcal{C}^{(2)} \equiv \Delta^{(2)}(\mathcal{C})$	$1 \otimes 2$
$\mathcal{C}^{(3)} \equiv \Delta^{(3)}(\mathcal{C})$	$1 \otimes 2 \otimes 3$
\vdots	\vdots
$\mathcal{C}^{(m)} \equiv \Delta^{(m)}(\mathcal{C})$	$1 \otimes 2 \otimes \cdots \otimes m$
\vdots	\vdots
$\mathcal{C}^{(N)} \equiv \Delta^{(N)}(\mathcal{C})$	$1 \otimes 2 \otimes \cdots \otimes (N - 1) \otimes N$
‘Right’ set of $(N - 1)$ integrals $\mathcal{C}^{(m)}$ in involution	Tensor product space
$\mathcal{C}_{(2)} \equiv \Delta_R^{(2)}(\mathcal{C})$	$(N - 1) \otimes N$
$\mathcal{C}_{(3)} \equiv \Delta_R^{(3)}(\mathcal{C})$	$(N - 2) \otimes (N - 1) \otimes N$
\vdots	\vdots
$\mathcal{C}_{(m)} \equiv \Delta_R^{(m)}(\mathcal{C})$	$(N - m + 1) \otimes (N - m + 2) \otimes \cdots \otimes N$
\vdots	\vdots
$\mathcal{C}_{(N)} = \mathcal{C}^{(N)} \equiv \Delta_R^{(N)}(\mathcal{C})$	$1 \otimes 2 \otimes \cdots \otimes (N - 1) \otimes N$

with $\Delta_L^{(N)}(\mathcal{C}) \equiv \Delta_R^{(N)}(\mathcal{C})$ being a common ‘left-right’ integral. Then each set of ‘left’ and ‘right’ integrals is then formed by $(N - 1)$ functions in involution. Therefore there is *only one* missing integral in order to ensure the *maximal superintegrability* of the system, which means that $\mathcal{H}^{(N)}$ is always, at least, QMS. Such a ‘remaining’ integral, in case it does exist, will not be provided by the coalgebra symmetry and it will have to be found by alternative methods. Hence, we can conclude that the coalgebra symmetry for this class of Hamiltonian systems with N degrees of freedom implies the following integrability hierarchy:

- $N = 2$. The Hamiltonian is only *integrable* with a single constant of motion $\mathcal{C}^{(2)} = \mathcal{C}_{(2)}$.
- $N = 3$. The Hamiltonian is *minimally* (or weakly) superintegrable with three integrals given by $\{\mathcal{C}^{(2)}, \mathcal{C}^{(3)} = \mathcal{C}_{(3)}, \mathcal{C}_{(2)}\}$.
- $N > 3$. The Hamiltonian is *QMS* with $(2N - 3)$ integrals $\{\mathcal{C}^{(m)}, \mathcal{C}^{(N)} = \mathcal{C}_{(N)}, \mathcal{C}_{(m)}\}$ for $m = 2, \dots, N - 1$.

2.3 The $sl(2, \mathbb{R})$ Lie-Poisson coalgebra

We stress that the coalgebra approach to complete integrability is completely general and constructive for any Poisson coalgebra endowed with a *suitable* symplectic realization. In order to illustrate the above ideas, we will firstly consider the coalgebra construction for the $sl(2, \mathbb{R})$ [178] Lie-Poisson algebra.

This coalgebra is defined by the following Lie-Poisson brackets and comultiplication map:

$$\{J_3, J_+\} = 2J_+ \quad \{J_3, J_-\} = -2J_- \quad \{J_-, J_+\} = 4J_3 \quad (2.25)$$

$$\Delta(1) = 1 \otimes 1 \quad \Delta(J_i) = J_i \otimes 1 + 1 \otimes J_i \quad i = +, -, 3. \quad (2.26)$$

The Casimir function for $sl(2, \mathbb{R})$ reads

$$\mathcal{C} = J_- J_+ - J_3^2. \quad (2.27)$$

A one-particle symplectic realization of this coalgebra is given by

$$D(J_-) = q_1^2 \quad D(J_+) = p_1^2 + \frac{b_1}{q_1^2} \quad D(J_3) = q_1 p_1 \quad (2.28)$$

where $\{q_1, p_1\} = 1$, with the standard Poisson-bracket. Note that, under this realization, $\mathcal{C}^{(1)} = D(\mathcal{C}) = b_1$. Hence, according to the notation and results presented in Section 2.1 we are dealing with a Poisson coalgebra with dimension $l = 3$ and with a single nonlinear Casimir. Thus $r = R = 1$ and this implies that (2.28) is the generic symplectic realization with $s \equiv s_m = (l - r)/2 = 1$. The corresponding N -particle symplectic realization of $sl(2, \mathbb{R})$, living on $sl(2, \mathbb{R}) \otimes \dots \otimes^N sl(2, \mathbb{R})$, is obtained by applying

$$J_i^{(N)} = (D \otimes D \otimes \dots \otimes^N D)(\Delta^{(N)}(J_i))$$

which gives [13, 24]

$$\begin{aligned} J_-^{(N)} &= \sum_{i=1}^N q_i^2 \equiv \mathbf{q}^2 & J_3^{(N)} &= \sum_{i=1}^N q_i p_i \equiv \mathbf{q} \cdot \mathbf{p} \\ J_+^{(N)} &= \sum_{i=1}^N \left(p_i^2 + \frac{b_i}{q_i^2} \right) \equiv \mathbf{p}^2 + \sum_{i=1}^N \frac{b_i}{q_i^2} \end{aligned} \quad (2.29)$$

where b_i , are N arbitrary real parameters. This means that the N -particle generators (2.29) fulfil the commutation rules (2.25) with respect to the ND canonical Poisson bracket.

Next, since $s_m = 1$ and $R = 1$ we are in the case of a Poisson coalgebra endowed with the $(2N - 3)$ integrals displayed in Table 2; these turn out to be [21, 24]:

$$\mathcal{C}^{(m)} = \sum_{1 \leq i < j}^m I_{ij} + \sum_{i=1}^m b_i \quad \mathcal{C}_{(m)} = \sum_{N-m+1 \leq i < j}^N I_{ij} + \sum_{i=N-m+1}^N b_i \quad (2.30)$$

where $m = 2, \dots, N$ and

$$\mathcal{I}_{ij} = (q_i p_j - q_j p_i)^2 + \left(b_i \frac{q_j^2}{q_i^2} + b_j \frac{q_i^2}{q_j^2} \right) \quad (2.31)$$

are the b_i -generalization of the square of the ‘angular momentum’ generators $J_{ij} = (q_i p_j - q_j p_i)$ which span an $so(N)$ Lie–Poisson algebra. As a consequence of the coalgebra symmetry, the generators (2.29) Poisson commute with these $(2N - 3)$ functions. Therefore, any arbitrary function \mathcal{H} defined as

$$\mathcal{H}^{(N)} = \mathcal{H} \left(J_-^{(N)}, J_+^{(N)}, J_3^{(N)} \right) = \mathcal{H} \left(\mathbf{q}^2, \mathbf{p}^2 + \sum_{i=1}^N \frac{b_i}{q_i^2}, \mathbf{q} \cdot \mathbf{p} \right) \quad (2.32)$$

gives rise to an ND QMS Hamiltonian system which is always endowed, at least, with the $(2N - 3)$ integrals (2.30).

Therefore, the following Hamiltonian

$$\mathcal{H}^{(N)} = \frac{1}{2} \sum_{i=1}^N \left(p_i^2 + \frac{b_i}{q_i^2} \right) + \mathcal{F} \left(\sum_{i=1}^N q_i^2 \right) = \frac{1}{2} J_+ + \mathcal{F}(J_-), \quad (2.33)$$

represents the motion of a particle on the ND Euclidean space under the action of N ‘centrifugal barriers’ determined by the b_i -terms and an arbitrary central potential given by the function \mathcal{F} . As particular cases, this Hamiltonian reproduces the Smorodinsky–Winternitz system [89] for $\mathcal{F} = \omega J_-$ and provides, a ND generalization of the Kepler potential when $\mathcal{F} = \frac{\gamma}{\sqrt{J_-}}$. In fact, this coalgebra has been recently shown to underlie the integrability of the oscillator and Kepler potentials on the ND spaces with constant curvature [24] and also of the ND spherically symmetric generalization of certain spaces with non-constant curvature [25, 26, 29].

2.4 ND integrable systems from 3D Lie coalgebras

After the previous generic presentation of the coalgebra formalism, we are going to analyse in detail, the generic symplectic realizations for 3D real Lie algebras (in this

Chapter) and for some 4D, 5D and 6D real Lie algebras (in the following chapter) fulfilling the integrability condition. In this way, we shall illustrate the type of ND integrable systems that can be associated to these Lie-coalgebras symmetries.

By following the known classifications summarized in [145] we firstly consider the set of 9 non-isomorphic 3D ($l = 3$) real Lie algebras, all of them with $r = 1$ (note that the generators e_i in [145] are now written as J_i). Their generic symplectic realizations have been computed and are given in Table 3.

Only one of these algebras (namely, the Heisenberg algebra $A_{3,1}$) has $R = 0$ (only one linear Casimir). This will be the only case in which the construction does not provide any dynamically relevant constant of the motion, since the Casimir coincides with the central generator J_1 and its m -th coproducts are just numerical constants.

Table 3. Generic symplectic realizations and Casimirs for 3D algebras.

	J_1	J_2	J_3	\mathcal{C}	
$A_{3,1}$	k	p	$-kq$	J_1	$k \neq 0$
$A_{3,2}$	$ke^{\frac{p}{k}}$	$pe^{\frac{p}{k}}$	$-kq$	$J_1 e^{-\frac{J_2}{J_1}}$	$k \neq 0$
$A_{3,3}$	$\frac{p^2}{2k}$	$\frac{p^2}{2}$	$-\frac{pq}{2}$	$\frac{J_2}{J_1}$	$k \neq 0$
$A_{3,4}$	ke^p	e^{-p}	$-q$	$J_1 J_2$	$k \neq 0$
$A_{3,5}^\alpha$	$e^{\frac{p}{\alpha}}$	ke^p	$-\alpha q$	$J_2 J_1^{-\alpha}$	$k \neq 0$
$A_{3,6}$	$\sqrt{k} \cos p$	$\sqrt{k} \sin p$	$-q$	$J_1^2 + J_2^2$	$k > 0$
$A_{3,7}^\alpha$	$\sqrt{k} e^{\alpha p} \cos p$	$\sqrt{k} e^{\alpha p} \sin p$	$-q$	$(J_1^2 + J_2^2) e^{-2\alpha \arctan(\frac{J_2}{J_1})}$	$k > 0$
$A_{3,8}$	$\frac{e^q}{2}(k - 2p^2)$	p	e^{-q}	$2J_2^2 + J_1 J_3 + J_3 J_1$	$\forall k$
(***)	$\frac{q^2}{2}$	$\frac{pq}{2}$	$-\frac{p^2}{2} + \frac{k}{q^2}$		$\forall k, q \neq 0$
(**)	$-pq^2 + \sqrt{2k}q$	$pq - \sqrt{\frac{k}{2}}$	p		$k \geq 0$
(*)	$p \sin q + p$	$p \cos q$	$p \sin q - p$		$k = 0$
$A_{3,9}$	p	$\sqrt{k - p^2} \cos q$	$\sqrt{k - p^2} \sin q$	$J_1^2 + J_2^2 + J_3^2$	$k > 0$

The rest of the 3D Lie coalgebras have $R = 1$ and, since $s_m = (3 - 1)/2 = 1$, we do have complete integrability for all the latter cases. Therefore, the ‘one-particle’ symplectic realizations given in Table 3 provide *automatically* infinite families of ND completely integrable (and in this case, quasi-maximally superintegrable) systems. In Table 3, the constant k is just the value that the Casimir \mathcal{C} takes under the given symplectic realization. Note that, in many cases, if $k = 0$ we would get a lower dimensional algebra, a case that we do not consider. We also point out that two symplectic realizations with the same value for k can always be related through a canonical transformation. This is the case of the $A_{3,8}^\alpha$ algebra, for which four different

symplectic realizations are explicitly provided in order to illustrate the multiplicity of apparently different systems that share the same underlying coalgebra symmetry. In order to get classical Hamiltonian systems defined through real symplectic realizations, we will also restrict the values of the constant k , if needed.

To shorten the presentation of the results, for each Lie coalgebra we will give:

- a) The explicit non-vanishing commutation rules (that have to be understood as Poisson brackets) that are fulfilled by the given symplectic realization.
- b) The most general integrable Hamiltonian $\mathcal{H}^{(N)}$ that can be constructed as an arbitrary function of the N -th coproduct of the three generators.
- c) The explicit form of the constants of the motion $\mathcal{C}^{(m)}$ ($m = 2, \dots, N$) coming from the Casimir function.

The ‘right’ set of constants of the motion [21] that gives rise to the quasi-maximal superintegrability of all these systems can be obtained from a given set $\mathcal{C}^{(m)}$ by performing the appropriate permutation of indices, as explained in [21], and we will not write them explicitly. Note that for each Lie coalgebra we obtain an *infinite* family of superintegrable ND Hamiltonians (the function \mathcal{H} is arbitrary) depending on N *arbitrary constants* k_i that label the specific symplectic realization that we have chosen on each of the N copies of the Lie algebra.

• **$A_{3,2}$ integrable systems**

$$\{J_1, J_3\} = J_1 \quad \{J_2, J_3\} = J_1 + J_2 \quad (2.34)$$

$$\mathcal{H}^{(N)} = \mathcal{H} \left(\sum_{i=1}^N k_i e^{\frac{p_i}{k_i}}, \sum_{i=1}^N p_i e^{\frac{p_i}{k_i}}, - \sum_{i=1}^N k_i q_i \right) \quad (2.35)$$

$$\mathcal{C}^{(m)} = \left(\sum_{i=1}^m k_i e^{\frac{p_i}{k_i}} \right) e^{- \left(\frac{\sum_{i=1}^m p_i e^{\frac{p_i}{k_i}}}{\sum_{i=1}^m k_i e^{\frac{p_i}{k_i}}} \right)}. \quad (2.36)$$

Note that in this case (and in some of the remaining examples) the integrals do not depend on the canonical coordinates q_i . Therefore, by taking into account the $\mathcal{C}^{(m)}$ and the $\mathcal{C}_{(m)}$ constants, we could obtain at most N functionally independents integrals of motion in involution with $\mathcal{H}^{(N)}$.

• **$A_{3,3}$ integrable systems**

$$\{J_1, J_3\} = J_1 \quad \{J_2, J_3\} = J_2 \quad (2.37)$$

$$\mathcal{H}^{(N)} = \mathcal{H} \left(\sum_{i=1}^N \frac{p_i^2}{2k_i}, \sum_{i=1}^N \frac{p_i^2}{2}, - \sum_{i=1}^N \frac{p_i q_i}{2} \right) \quad (2.38)$$

$$\mathcal{C}^{(m)} = \frac{\left(\sum_{i=1}^m p_i^2 \right)}{\left(\sum_{i=1}^m \frac{p_i^2}{k_i} \right)}. \quad (2.39)$$

Surprisingly enough, in this case we lose the integrability if we take the same symplectic realization on all the copies of the A algebra (i.e., if $k_1 = k_2 = \dots = k_N$), since all the integrals are transformed into constants. This fact, emphasizes the role of symplectic representations throughout the formalism.

- $A_{3,4}$ integrable systems

$$\{J_1, J_3\} = J_1 \quad \{J_2, J_3\} = -J_2 \quad (2.40)$$

$$\mathcal{H}^{(N)} = \mathcal{H} \left(\sum_{i=1}^N k_i e^{p_i}, \sum_{i=1}^N e^{-p_i}, - \sum_{i=1}^N q_i \right) \quad (2.41)$$

$$\mathcal{C}^{(m)} = \sum_{i=1}^m k_i + \sum_{\substack{i,j=1 \\ i \neq j}}^m k_i e^{p_i - p_j}. \quad (2.42)$$

This algebra is just the $(1 + 1)$ Poincaré algebra. An analogue of the Ruijsenaars-Schneider Hamiltonian [160] was obtained in [12] by using a quantum deformation of this realization, and will be presented in the next Section.

- $A_{3,5}^\alpha$ integrable systems $(0 < |\alpha| < 1)$

$$\{J_1, J_3\} = J_1 \quad \{J_2, J_3\} = \alpha J_2 \quad (2.43)$$

$$\mathcal{H}^{(N)} = \mathcal{H} \left(\sum_{i=1}^N e^{\frac{p_i}{\alpha}}, \sum_{i=1}^N k_i e^{p_i}, - \sum_{i=1}^N \alpha q_i \right) \quad (2.44)$$

$$\mathcal{C}^{(m)} = \frac{\sum_{i=1}^m k_i e^{p_i}}{\left(\sum_{i=1}^m e^{\frac{p_i}{\alpha}} \right)^\alpha}. \quad (2.45)$$

- $A_{3,6}$ integrable systems

$$\{J_1, J_3\} = -J_2 \quad \{J_2, J_3\} = J_1, \quad (2.46)$$

$$\mathcal{H}^{(N)} = \mathcal{H} \left(\sum_{i=1}^N \sqrt{k_i} \cos p_i, \sum_{i=1}^N \sqrt{k_i} \sin p_i, -\sum_{i=1}^N q_i \right) \quad (2.47)$$

$$\mathcal{C}^{(m)} = \sum_{i=1}^m k_i + \sum_{\substack{i,j=1 \\ i \neq j}}^m \sqrt{k_i k_j} \cos(p_i - p_j). \quad (2.48)$$

Note that, this algebra is the two dimensional Euclidean algebra.

• $A_{3,7}^\alpha$ integrable systems ($\alpha > 0$)

$$\{J_1, J_3\} = \alpha J_1 - J_2 \quad \{J_2, J_3\} = J_1 + \alpha J_2 \quad (2.49)$$

$$\mathcal{H}^{(N)} = \mathcal{H} \left(\sum_{i=1}^N \sqrt{k_i} e^{\alpha p_i} \cos p_i, \sum_{i=1}^N \sqrt{k_i} e^{\alpha p_i} \sin p_i, -\sum_{i=1}^N q_i \right) \quad (2.50)$$

$$\mathcal{C}^{(m)} = \left(\sum_{i=1}^m k_i e^{2\alpha p_i} + \sum_{\substack{i,j=1 \\ i \neq j}}^m \sqrt{k_i k_j} e^{\alpha(p_i + p_j)} \cos(p_i - p_j) \right) e^{-2\alpha \arctan \left(\frac{\sum_{i=1}^m \sqrt{k_i} e^{\alpha p_i} \sin p_i}{\sum_{i=1}^m \sqrt{k_i} e^{\alpha p_i} \cos p_i} \right)}. \quad (2.51)$$

• $A_{3,8}$ integrable systems

$$\{J_1, J_2\} = J_1 \quad \{J_1, J_3\} = -2J_2 \quad \{J_2, J_3\} = J_3 \quad (2.52)$$

$$\mathcal{H}^{(N)} = \mathcal{H} \left(\sum_{i=1}^N \frac{e^{q_i}}{2} (k_i - 2p_i^2), \sum_{i=1}^N p_i, \sum_{i=1}^N e^{-q_i} \right) \quad (2.53)$$

$$\mathcal{C}^{(m)} = \sum_{i=1}^m k_i + \sum_{\substack{i,j=1 \\ i \neq j}}^m e^{q_j - q_i} (k_j - 2p_j^2) + 2 \sum_{\substack{i,j=1 \\ i \neq j}}^m p_i p_j. \quad (2.54)$$

This is the $sl(2, \mathbb{R}) \simeq so(2, 1)$ Lie coalgebra, and many different and important integrable systems can be obtained as $\mathcal{H}^{(N)}$ by making use of different symplectic realizations. For instance, the Calogero–Gaudin Hamiltonian [55, 91, 112, 113]

$$\mathcal{H}^{(N)} = \sum_{i < j}^N 2p_i p_j (1 - \cos(q_i - q_j)), \quad (2.55)$$

comes from the $k = 0$ symplectic realization (*) of $A_{3,8}$ (the $sl(2)$ algebra) by taking as the Hamiltonian the Casimir operator \mathcal{C} (see [12, 113] for a detailed discussion of this

system and its integrable deformations, and the papers [132, 133] for its q -deformed quantum mechanical version). In general, note that the choice of the symplectic realization drastically changes the ‘shape’ of the Hamiltonian. For example, by using the Gel’fand–Dyson symplectic map (***) with $k = 0$ the very same Calogero–Gaudin system reads

$$\mathcal{H}^{(N)} = - \sum_{i < j}^N p_i p_j (q_i - q_j)^2. \quad (2.56)$$

Finally, the symplectic realization (***) has been fully analysed in the previous Section.

• $A_{3,9}$ integrable systems

$$\{J_1, J_2\} = J_3 \quad \{J_1, J_3\} = -J_2 \quad [J_2, J_3] = J_1 \quad (2.57)$$

$$\mathcal{H}^{(N)} = \mathcal{H} \left(\sum_{i=1}^N p_i, \sum_{i=1}^N \sqrt{k_i - p_i^2} \cos q_i, \sum_{i=1}^N \sqrt{k_i - p_i^2} \sin q_i \right) \quad (2.58)$$

$$\mathcal{C}^{(m)} = - \sum_{i=1}^m k_i + 2 \sum_{i=1}^m p_i^2 + \sum_{\substack{i,j \\ i \neq j}}^m p_i p_j + \sum_{\substack{i,j \\ i \neq j}}^m \cos(q_i - q_j) \sqrt{p_i^2 - k_i} \sqrt{p_j^2 - k_j}. \quad (2.59)$$

These would be the classical integrable systems provided by N copies of the $so(3)$ algebra.

2.5 Integrable deformations from q -Poisson coalgebras

The Poisson analogues of quantum algebras and groups [66, 67, 69, 79] are also (deformed) coalgebras (A_z, Δ_z) ($q = e^z$), which means that any function of the generators of a given ‘quantum’ Poisson algebra (with deformed Casimir elements $\mathcal{C}_{z,j}$) will provide a deformation of the Hamiltonian generated by the undeformed structure. Such a q -coalgebraic deformation will preserve, by construction, the (super)integrability properties of the system defined on the undeformed Lie–Poisson coalgebra.

Therefore, q -deformations can be understood in this context as the algebraic machinery suitable for generating integrable deformations of Hamiltonian systems. This was explicitly shown for the first time by constructing the following deformed CG system [9]:

$$\mathcal{H}_z^{(N)} = \sum_{1 \leq i < j}^N 2 \pi_i \pi_j (1 - \cos(q_i - q_j)) \quad (2.60)$$

where the non-local deformations of the momenta are

$$\pi_k = 2 \frac{\sinh(\frac{z}{2} p_k)}{z} \prod_{i=1}^{k-1} e^{-\frac{z}{2} p_i} \prod_{j=k+1}^N e^{\frac{z}{2} p_j}.$$

The corresponding constants of the motion come from the (deformed) coproduct of the (deformed) Casimir of $so_z(2, 1)$ (the standard deformation of $so(2, 1)$) and, as expected, in the limit $z \rightarrow 0$ we recover the ‘classical’ CG system given in (2.55). We remark that the quantum mechanical version of this deformed CG system has been explicitly solved [132, 133].

Another interesting example of coalgebra-invariant system is the following analogue [12] of the Ruijsenaars–Schneider model [160]:

$$\mathcal{H}_z^{(N)} = \sum_{i=1}^N \cosh \theta_i \exp \left(-\frac{z}{2} \left(\sum_{j=1}^{i-1} q_j \right) + \frac{z}{2} \left(\sum_{k=i+1}^N q_k \right) \right) \quad (2.61)$$

where (q_i, θ_i) are canonically conjugate variables such that $\{q_i, \theta_j\} = \delta_{ij}$. This completely integrable Hamiltonian was obtained by using the Poisson analogue of the quantum deformation of the (1+1)D Poincaré algebra introduced in [173].

2.5.1 The q -Poisson coalgebra $sl_z(2, \mathbb{R})$

In order to illustrate explicitly this construction, let us focus on the non-standard $sl_z(2, \mathbb{R})$ Poisson coalgebra, which is defined by the following (deformed) Poisson brackets and coproduct map (see [13, 142]):

$$\{J_3, J_+\} = 2J_+ \cosh zJ_- \quad \{J_3, J_-\} = -2 \frac{\sinh zJ_-}{z} \quad \{J_-, J_+\} = 4J_3 \quad (2.62)$$

$$\begin{aligned} \Delta_z(1) &= 1 \otimes 1 & \Delta_z(J_-) &= J_- \otimes 1 + 1 \otimes J_- \\ \Delta_z(J_i) &= J_i \otimes e^{zJ_-} + e^{-zJ_-} \otimes J_i & (i = +, 3). \end{aligned} \quad (2.63)$$

The Casimir function for $sl_z(2, \mathbb{R})$ reads

$$\mathcal{C}_z = \frac{\sinh zJ_-}{z} J_+ - J_3^2. \quad (2.64)$$

A one-particle (deformed) symplectic realization of $sl_z(2, \mathbb{R})$ is:

$$D_z(J_-) = q_1^2 \quad D_z(J_+) = \frac{\sinh zq_1^2}{zq_1^2} p_1^2 + \frac{zb_1}{\sinh zq_1^2} \quad D_z(J_3) = \frac{\sinh zq_1^2}{zq_1^2} q_1 p_1 \quad (2.65)$$

such that $\mathcal{C}_z^{(1)} = D_z(\mathcal{C}_z) = b_1$. Hence we are dealing again with a coalgebra with $l = 3$, $r = R = 1$ and $s_m = 1$. The N th-coproduct of (2.63) through the one-particle representation (2.65) gives rise to an N -particle symplectic realization on $sl_z(2, \mathbb{R}) \otimes \dots^N \otimes sl_z(2, \mathbb{R})$ through

$$J_i^{(N)} = (D_z \otimes D_z \otimes \dots^N \otimes D_z)(\Delta_z^{(N)}(J_i)),$$

namely

$$\begin{aligned} J_-^{(N)} &= \sum_{i=1}^N q_i^2 \equiv \mathbf{q}^2 & J_3^{(N)} &= \sum_{i=1}^N \frac{\sinh zq_i^2}{zq_i^2} q_i p_i \exp \left\{ -z \sum_{k=1}^{i-1} q_k^2 + z \sum_{l=i+1}^N q_l^2 \right\} \equiv (\mathbf{q} \cdot \mathbf{p})_z \\ J_+^{(N)} &= \sum_{i=1}^N \left(\frac{\sinh zq_i^2}{zq_i^2} p_i^2 + \frac{zb_i}{\sinh zq_i^2} \right) \exp \left\{ -z \sum_{k=1}^{i-1} q_k^2 + z \sum_{l=i+1}^N q_l^2 \right\} \equiv \tilde{\mathbf{p}}_z^2 \end{aligned} \quad (2.66)$$

where the b_i 's are again N arbitrary real parameters that label the representation on each 'lattice' site.

In this case the $(2N-3)$ functions written in Table 2 are explicitly given by [13, 21]:

$$\begin{aligned} \mathcal{C}_z^{(m)} &= \sum_{1 \leq i < j}^m I_{ij}^z \exp \left\{ -2z \sum_{k=1}^{i-1} q_k^2 - zq_i^2 + zq_j^2 + 2z \sum_{l=j+1}^m q_l^2 \right\} \\ &\quad + \sum_{i=1}^m b_i \exp \left\{ -2z \sum_{k=1}^{i-1} q_k^2 + 2z \sum_{l=i+1}^m q_l^2 \right\} \\ \mathcal{C}_{z,(m)} &= \sum_{N-m+1 \leq i < j}^N I_{ij}^z \exp \left\{ -2z \sum_{k=N-m+1}^{i-1} q_k^2 - zq_i^2 + zq_j^2 + 2z \sum_{l=j+1}^N q_l^2 \right\} \\ &\quad + \sum_{i=N-m+1}^N b_i \exp \left\{ -2z \sum_{k=N-m+1}^{i-1} q_k^2 + 2z \sum_{l=i+1}^N q_l^2 \right\} \end{aligned} \quad (2.67)$$

where $m = 2, \dots, N$ and

$$\mathcal{I}_{ij}^z = \frac{\sinh zq_i^2}{zq_i^2} \frac{\sinh zq_j^2}{zq_j^2} (q_i p_j - q_j p_i)^2 + \left(b_i \frac{\sinh zq_j^2}{\sinh zq_i^2} + b_j \frac{\sinh zq_i^2}{\sinh zq_j^2} \right). \quad (2.68)$$

Consequently, any smooth function \mathcal{H}_z defined on the N -particle symplectic realization (2.66) of the generators of $sl_z(2, \mathbb{R})$ in the form

$$\mathcal{H}_z^{(N)} = \mathcal{H}_z \left(J_-^{(N)}, J_+^{(N)}, J_3^{(N)} \right) = \mathcal{H}_z \left(\mathbf{q}^2, \tilde{\mathbf{p}}_z^2, (\mathbf{q} \cdot \mathbf{p})_z \right) \quad (2.69)$$

defines a QMS Hamiltonian system. We stress that all the choices for \mathcal{H}_z share the ‘universal’ set of $(2N - 3)$ constants of motion given by (2.67).

Clearly, the non-deformed limit $z \rightarrow 0$ of all the above expressions gives rise to the ones corresponding to the $sl(2, \mathbb{R})$ -coalgebra presented in Section 2.3, which shows that quantum algebras may provide (super)integrable generalizations of non-deformed Hamiltonian systems.

We also remark that this quantum deformation has been interpreted in [13] as an algebraic way to introduce long-range interactions in the underlying undeformed systems due to the exponentials of the type $\exp(z \sum_j q_j^2)$ coming from the deformed symplectic realization (2.66) (compare with (2.29) and (2.32)). Another physically meaningful application has been presented in [22], where the geodesic flow hamiltonians defined as certain functions of the generators of $sl_z(2, \mathbb{R})$ have been shown to generate different curved spaces. In all of them, the deformation parameter can be interpreted as a curvature constant.

Chapter 3

Higher dimensional coalgebras

*“Don’t worry about your difficulties in mathematics,
I can assure you that mine are still greater.”*

A. Einstein

In this Chapter, we continue with the systematic construction of generic symplectic realizations for low-dimensional real Lie-Poisson coalgebras. The four and five dimensional cases will be fully given, but we recall that in dimensions higher than five, classifications of real Lie algebras and their Casimir invariants are partial and restricted to certain simple, solvable or nilpotent subclasses, although a significant number of the latter are known (see, for instance, [44, 57, 61, 63, 148, 164, 171]). Thus, can be generated many new families of integrable systems provided that, for a given Lie coalgebra with known Casimir invariants, the integrability criterion is checked in order to determine *a priori* which are the symplectic realizations that can lead to integrable systems. Nevertheless, this analysis shows that, most of these systems have integrals of motion depending only on the momenta, thus restricting its potential dynamical interest.

3.1 Integrable systems from 4D Lie coalgebras

In this case the classification [145] provide a set of 12 non-isomorphic 4D ($l = 4$) real Lie algebras. Among them, 4 algebras have $R = 0$, and will not give rise to integrable systems under our approach.

3.1.1 Algebras with $R = 1$

We have four Lie algebras with $r = 2$ and $R = 1$ (two Casimirs, one of them linear). In all these cases $s_m = (4 - 2)/2 = 1$ and the integrability condition is fulfilled. The explicit generic symr of these four algebras, together with the explicit form of the non-linear Casimir, are given in Table 4.

Table 4. Symplectic realizations for 4D Lie-Poisson algebras with ($R = 1$) .

	$A_{4,1}$	$A_{4,3}$	$A_{4,8}$	$A_{4,10}$
J_1	k_1	e^p	k_1	k_1
J_2	p	k_1	$\sqrt{p}e^q$	k_1q
J_3	$\frac{p^2 - k_2}{2k_1}$	$k_1(p - \log k_2)$	$k_1\sqrt{p}e^{-q}$	p
J_4	$-k_1q$	$-q$	$-\frac{k_2}{2k_1} + p$	$\frac{-p^2 - k_1^2q_1^2 + k_2}{2k_1}$
\mathcal{C}_2	$J_2^2 - 2J_1J_3$	$J_1e^{-\frac{J_3}{J_2}}$	$J_2J_3 + J_3J_2 - 2J_1J_4$	$2J_1J_4 + J_2^2 + J_3^2$
	$k_1 \neq 0$	$k_1 \neq 0, k_2 > 0$	$k_1 \neq 0$	$k_1 \neq 0$

• $A_{4,1}$ integrable systems

$$\{J_2, J_4\} = J_1 \quad \{J_3, J_4\} = J_2 \quad (3.1)$$

$$\mathcal{H}^{(N)} = \mathcal{H} \left(\sum_{i=1}^N k_{1,i}, \sum_{i=1}^N p_i, \sum_{i=1}^N \frac{p_i^2 - k_{2,i}}{2k_{1,i}}, - \sum_{i=1}^N k_{1,i} q_i \right) \quad (3.2)$$

$$\mathcal{C}^{(m)} = \sum_{i=1}^m k_{2,i} + \sum_{\substack{i,j=1 \\ i \neq j}}^m p_i p_j + \sum_{\substack{i,j=1 \\ i \neq j}}^m k_{1,i} \frac{k_{2,j}}{k_{1,j}} - \sum_{\substack{i,j=1 \\ i \neq j}}^m \frac{p_i^2}{k_{1,i}} k_{1,j}. \quad (3.3)$$

The algebra $A_{4,1}$ is the (1+1) extended Galilei Lie algebra, and their associated integrable systems have been constructed in [14], as well as their integrable deformations.

• $A_{4,3}$ integrable systems

$$\{J_1, J_4\} = J_1 \quad \{J_3, J_4\} = J_2 \quad (3.4)$$

$$\mathcal{H}^{(N)} = \mathcal{H} \left(\sum_{i=1}^N e^{p_i}, \sum_{i=1}^N k_{1,i}, \sum_{i=1}^N k_{1,i} (p_i - \log k_{2,i}), - \sum_{i=1}^N q_i \right) \quad (3.5)$$

$$\mathcal{C}^{(m)} = \left(\sum_{i=1}^m e^{p_i} \right) e^{-\frac{\sum_{i=1}^m k_{1,i} (p_i - \log k_{2,i})}{\sum_{i=1}^m k_{1,i}}}. \quad (3.6)$$

• $A_{4,8}$ integrable systems

$$\{J_2, J_3\} = J_1 \quad \{J_2, J_4\} = J_2 \quad \{J_3, J_4\} = -J_3 \quad (3.7)$$

$$\mathcal{H}^{(N)} = \mathcal{H} \left(\sum_{i=1}^N k_{1,i}, \sum_{i=1}^N \sqrt{p_i} e^{q_i}, \sum_{i=1}^N k_{1,i} \sqrt{p_i} e^{-q_i}, \sum_{i=1}^N \left(-\frac{k_{2,i}}{2k_{1,i}} + p_i \right) \right) \quad (3.8)$$

$$\mathcal{C}^{(m)} = \sum_{i=1}^m k_{2,i} + \sum_{\substack{i,j=1 \\ i \neq j}}^m \frac{k_{2,i}}{k_{1,i}} k_{1,j} - 2 \sum_{\substack{i,j=1 \\ i \neq j}}^m p_i k_{1,j} + \sum_{\substack{i,j=1 \\ i \neq j}}^m e^{-q_i + q_j} k_{1,i} \sqrt{p_i p_j}. \quad (3.9)$$

The algebra $A_{4,8}$ is also known as the oscillator algebra h_4 . If we take the \mathcal{H} function

$$\mathcal{H} = \lambda J_4 + \mu J_2 J_3, \quad (3.10)$$

under the realization with $(k_{1,i} = 1)$ and $(k_{2,i} = 0)$ we get the following integrable Hamiltonian

$$\mathcal{H}^{(N)} = (\lambda + \mu) \sum_{i=1}^N p_i + 2\mu \sum_{i < j}^N \sqrt{p_i p_j} \cosh(q_i - q_j), \quad (3.11)$$

whose quantum mechanical version was introduced in [56]. The integrals of the motion in involution in the chosen realization read:

$$\mathcal{C}^{(m)} = -2 \sum_{i=1}^m p_i + 2 \sum_{\substack{i,j=1 \\ i \neq j}}^m \sqrt{p_i p_j} \cosh(q_i - q_j) = 2 \sum_{\substack{i,j=1 \\ i \neq j}}^m \sqrt{p_i p_j} \cosh(q_i - q_j). \quad (3.12)$$

• $A_{4,10}$ integrable systems

$$\{J_2, J_3\} = J_1 \quad \{J_2, J_4\} = -J_3 \quad \{J_3, J_4\} = J_2 \quad (3.13)$$

$$\mathcal{H}^{(N)} = \mathcal{H} \left(\sum_{i=1}^N k_{1,i}, \sum_{i=1}^N k_{1,i} q_i, \sum_{i=1}^N p_i, -\sum_{i=1}^N \frac{p_i^2 + k_{1,i}^2 q_i^2 - k_{2,i}}{2k_{1,i}} \right) \quad (3.14)$$

$$\mathcal{C}^{(m)} = \sum_{i,j=1}^m \frac{k_{2,i}k_{1,j}}{k_{1,i}} - \sum_{\substack{i,j=1 \\ i \neq j}}^m \frac{p_i^2 k_{1,j}}{k_{1,i}} + \sum_{\substack{i,j=1 \\ i \neq j}}^m k_{1,i}k_{1,j}q_iq_j + \sum_{\substack{i,j=1 \\ i \neq j}}^m p_i p_j - \sum_{\substack{i,j=1 \\ i \neq j}}^m q_i^2 k_{1,i}k_{1,j}. \quad (3.15)$$

In particular, the N -particle Hamiltonian given by $\mathcal{H} = -J_4 + \mathcal{F}(J_2)$ gives

$$\mathcal{H}^{(N)} = \sum_{i=1}^N \frac{p_i^2 + k_{1,i}^2 q_i^2 - k_{2,i}}{2k_{1,i}} + \mathcal{F}\left(\sum_{i=1}^N k_{1,i} q_i\right) \quad (3.16)$$

which is a completely integrable Hamiltonian for any choice of the function \mathcal{F} , with integrals of the motion independent of \mathcal{F} and given by (3.15). This system, is just an anisotropic ND oscillator plus an arbitrary function $\mathcal{F}(J_2)$.

3.1.2 Algebras with $R = 2$

We have four more 4D algebras with $R = 2$. Again, $s_m = (4 - 2)/2 = 1$ and the integrability condition is fulfilled. The generic symplectic realizations of these four algebras are given in Table 5.

It is important to stress that, since we have two non-linear Casimirs, we obtain two sets $\{\mathcal{C}_1^{(m)}\}$ and $\{\mathcal{C}_2^{(m)}\}$ of $(N - 1)$ constants of the motion in involution with the Hamiltonian, besides the two additional sets given by the ‘right’ coproducts. However, in all these cases the Casimir function $\{\mathcal{C}_j^{(m)}\}$ depends only on the momenta (p_1, p_2, \dots, p_m) . Therefore, we can find at most N functionally independent objects among the full set of $\{\mathcal{C}_j^{(m)}\}$ functions.

- $A_{4,2}^\alpha$ integrable systems ($\alpha \neq 0$)

$$\{J_1, J_4\} = \alpha J_1 \quad \{J_2, J_4\} = J_2 \quad \{J_3, J_4\} = J_2 + J_3 \quad (3.17)$$

$$\mathcal{H}^{(N)} = \mathcal{H}\left(\sum_{i=1}^N \frac{k_{1,i}^\alpha}{k_{2,i}} e^{p_i}, \sum_{i=1}^N k_{1,i} e^{p_i}, \sum_{i=1}^N k_{1,i} p_i e^{p_i}, -\sum_{i=1}^N q_i\right) \quad (3.18)$$

$$\mathcal{C}_1^{(m)} = \left(\sum_{i=1}^m k_{1,i} e^{p_i}\right) e^{-\left(\frac{\sum_{i=1}^m k_{1,i} p_i e^{p_i}}{\sum_{i=1}^m k_{1,i} e^{p_i}}\right)} \quad \mathcal{C}_2^{(m)} = \frac{\left(\sum_{i=1}^m k_{1,i} e^{p_i}\right)^\alpha}{\left(\sum_{i=1}^m \frac{k_{1,i}^\alpha}{k_{2,i}} e^{p_i}\right)}. \quad (3.19)$$

Table 5. Symplectic realizations for 4D Lie-Poisson algebras with $R = 2$.

	$A_{4,2}^\alpha$	$A_{4,4}$	$A_{4,5}^{a,b}$	$A_{4,6}^{a,b}$
J_1	$\frac{k_1^\alpha e^p}{k_2}$	$k_1 e^p$	$(k_1 a)^{\frac{1}{a}} e^p$	$\left(k_1 k_2 e^{\frac{b\pi}{2}}\right)^{\frac{a}{2b}} e^{ap}$
J_2	$k_1 e^p$	$k_1 p e^p$	$a e^{ap}$	$\sqrt{\frac{e^{\frac{b\pi}{2}} k_2}{2}} e^{bp} f(p)$
J_3	$k_1 p e^p$	$\frac{k_1 e^p}{2} (p^2 + k_2)$	$\frac{(k_1 a)^{\frac{b}{a}}}{k_2} e^{bp}$	$\sqrt{\frac{e^{\frac{b\pi}{2}} k_2}{2}} e^{bp} g(p)$
J_4	$-q$	$-q$	$-q$	$-q$
\mathcal{C}_1	$J_2 e^{-\frac{J_3}{J_2}}$	$J_1 e^{-\frac{J_2}{J_1}}$	$\frac{J_1^\alpha}{J_2}$	$\frac{J_1^{\frac{a}{2b}}}{J_2^2 + J_3^2}$
\mathcal{C}_2	$\frac{J_2^\alpha}{J_1}$	$\frac{2J_1 J_3 - J_2^2}{J_1^2}$	$\frac{J_1^b}{J_3}$	$(J_2^2 + J_3^2) e^{-2b \arctan\left(\frac{J_3}{J_2}\right)}$
	$k_1 \neq 0$	$k_1 \neq 0$	$k_1 \neq 0$	$k_1 \neq 0, k_2 > 0$

where

$$f(p) = (\cos p - \sin p), \quad g(p) = (\cos p + \sin p) \quad (3.20)$$

are the functions involved in $A_{4,6}^{a,b}$.

• $A_{4,4}$ integrable systems

$$\{J_1, J_4\} = J_1 \quad \{J_2, J_4\} = J_1 + J_2 \quad \{J_3, J_4\} = J_2 + J_3 \quad (3.21)$$

$$\mathcal{H}^{(N)} = \mathcal{H} \left(\sum_{i=1}^N k_{1,i} e^{p_i}, \sum_{i=1}^N k_{1,i} p_i e^{p_i}, \sum_{i=1}^N \frac{k_{1,i} e^{p_i}}{2} (p_i^2 + k_{2,i}), -\sum_{i=1}^N q_i \right) \quad (3.22)$$

$$\begin{aligned} \mathcal{C}_1^{(m)} &= \left(\sum_{i=1}^m k_{1,i} e^{p_i} \right) e^{-\left(\frac{\sum_{i=1}^m k_{1,i} p_i e^{p_i}}{\sum_{i=1}^m k_{1,i} e^{p_i}} \right)} \\ \mathcal{C}_2^{(m)} &= \frac{2 \left(\sum_{i=1}^m k_{1,i} e^{p_i} \right) \left(\sum_{i=1}^m \frac{k_{1,i} e^{p_i}}{2} (p_i^2 + k_{2,i}) \right) - \left(\sum_{i=1}^m k_{1,i} p_i e^{p_i} \right)^2}{\left(\sum_{i=1}^m k_{1,i} e^{p_i} \right)^2}. \end{aligned} \quad (3.23)$$

- $A_{4,5}^{a,b}$ integrable systems $(ab \neq 0) \quad (-1 \leq a \leq b \leq 1)$

$$\{J_1, J_4\} = J_1 \quad \{J_2, J_4\} = a J_2 \quad \{J_3, J_4\} = b J_3 \quad (3.24)$$

$$\mathcal{H}^{(N)} = \mathcal{H} \left(\sum_{i=1}^N (k_{1,i} a)^{\frac{1}{a}} e^{p_i}, \sum_{i=1}^N a e^{ap_i}, \sum_{i=1}^N \frac{(k_{1,i} a)^{\frac{b}{a}}}{k_{2,i}} e^{bp_i}, -\sum_{i=1}^N q_i \right) \quad (3.25)$$

$$\mathcal{C}_1^{(m)} = \frac{\left(\sum_{i=1}^m (k_{1,i} a)^{\frac{1}{a}} e^{p_i} \right)^a}{\left(a \sum_{i=1}^m e^{ap_i} \right)} \quad \mathcal{C}_2^{(m)} = \frac{\left(\sum_{i=1}^m (k_{1,i} a)^{\frac{1}{a}} e^{p_i} \right)^b}{\left(\sum_{i=1}^m \frac{(k_{1,i} a)^{\frac{b}{a}}}{k_{2,i}} e^{bp_i} \right)}. \quad (3.26)$$

- $A_{4,6}^{a,b}$ integrable systems $(a \neq 0, \quad b \geq 0)$

$$\{J_1, J_4\} = a J_1 \quad \{J_2, J_4\} = b J_2 - J_3 \quad \{J_3, J_4\} = J_2 + b J_3 \quad (3.27)$$

$$\begin{aligned} J_1^{(N)} &= \sum_{i=1}^N \left(k_{1,i} k_{2,i} e^{\frac{b\pi}{2}} \right)^{\frac{a}{2b}} e^{ap_i} & J_2^{(N)} &= \sum_{i=1}^N \sqrt{\frac{e^{\frac{b\pi}{2}} k_{2,i}}{2}} e^{bp_i} (\cos p_i - \sin p_i) \\ J_3^{(N)} &= \sum_{i=1}^N \sqrt{\frac{e^{\frac{b\pi}{2}} k_{2,i}}{2}} e^{bp_i} (\cos p_i + \sin p_i) & J_4^{(N)} &= -\sum_{i=1}^N q_i \end{aligned} \quad (3.28)$$

$$\mathcal{H}^{(N)} = \mathcal{H} \left(J_1^{(N)}, J_2^{(N)}, J_3^{(N)}, J_4^{(N)} \right) \quad (3.29)$$

$$\begin{aligned} \mathcal{C}_1^{(m)} &= \frac{\sum_{i=1}^m (k_{1,i} k_{2,i}) e^{2bp_i}}{\left(\sum_{i=1}^m k_{2,i} e^{2bp_i} + \sum_{\substack{i,j=1, \\ i \neq j}}^m \sqrt{k_{2,i} k_{2,j}} e^{b(p_i+p_j)} \cos(p_i - p_j) \right)} \\ \mathcal{C}_2^{(m)} &= e^{\frac{b\pi}{2}} \left(\sum_{i=1}^m k_{2,i} e^{2bp_i} + \sum_{\substack{i,j=1, \\ i \neq j}}^m \sqrt{k_{2,i} k_{2,j}} e^{b(p_i+p_j)} \cos(p_i - p_j) \right) e^{-2b \arctan(\mathcal{F}(p_1, p_2, \dots, p_m))} \end{aligned} \quad (3.30)$$

being

$$\mathcal{F}(p_1, p_2, \dots, p_m) = \frac{\left(\sum_{i=1}^m \sqrt{\frac{e^{\frac{b\pi}{2}} k_{2,i}}{2}} e^{bp_i} (\cos p_i + \sin p_i) \right)}{\left(\sum_{i=1}^m \sqrt{\frac{e^{\frac{b\pi}{2}} k_{2,i}}{2}} e^{bp_i} (\cos p_i - \sin p_i) \right)}. \quad (3.31)$$

As it can be easily appreciated, the complexity of the symplectic realizations increases with the number of non-linear Casimirs that constrain the realization.

3.2 Integrable systems from 5D Lie coalgebras

Non-isomorphic real Lie algebras of dimension 5 are also fully classified (we use the notation given in [145] for Mubarakzyanov [128, 129, 130, 131] results). In fact, there are 40 different 5D Lie algebras with $r = 1, 3$.

It is easy to check that all the $r = 1$ cases do not fulfill the integrability condition. This is obvious for the 7 cases in which $R = 0$. There are also 18 cases with $R = r = 1$, but for them $s_m = (5 - 1)/2 = 2 > R$.

So, we are left with the 15 cases with $r = 3$. For all of them the integrability condition holds, since $R = 1, 2, 3$ and $s_m = (5 - 3)/2 = 1$. Hereafter, as far as constants are concerned and so as to shorten the presentation in the tables, expressions like $(k_1, k_2 \neq 0)$, must be understood as $(k_1 \neq 0, k_2 \neq 0)$. For the sake of making the presentation more readable, we postpone some of the cases to the Supplement 8.3.

3.2.1 Algebras with $R = 1$

Table 6. Symplectic realizations for 5D Lie-Poisson algebras with $R = 1$.

	J_1	J_2	J_3	J_4	J_5	\mathcal{C}_3	
$A_{5,1}$	k_1	k_2	$k_1 p + \frac{k_3}{k_2}$	$k_2 p$	$-q$	$J_2 J_3 - J_1 J_4$	$k_1, k_2 \neq 0$
$A_{5,3}$	k_1	k_2	p	$-k_2 q - \frac{k_3}{2k_1}$	$-k_1 q - \frac{p^2}{2k_2}$	$J_3^2 + 2J_2 J_5 - 2J_1 J_4$	$k_1, k_2 \neq 0$

• $A_{5,1}$ integrable systems

$$\{J_3, J_5\} = J_1 \quad \{J_4, J_5\} = J_2 \quad (3.32)$$

$$\mathcal{H}^{(N)} = \mathcal{H} \left(\sum_{i=1}^N k_{1,i}, \sum_{i=1}^N k_{2,i}, \sum_{i=1}^N \left(k_{1,i} p_i + \frac{k_{3,i}}{k_{2,i}} \right), \sum_{i=1}^N k_{2,i} p_i, - \sum_{i=1}^N q_i \right) \quad (3.33)$$

$$\mathcal{C}_3^{(m)} = \sum_{i=1}^m k_{3,i} + \sum_{\substack{i,j=1 \\ i \neq j}}^m \frac{k_{3,i} k_{2,j}}{k_{2,i}} - \sum_{\substack{i,j=1 \\ i \neq j}}^m k_{2,i} k_{1,j} p_i + \sum_{\substack{i,j=1 \\ i \neq j}}^m k_{1,i} k_{2,j} p_i. \quad (3.34)$$

• $A_{5,3}$ integrable systems

$$\{J_3, J_4\} = J_2 \quad \{J_3, J_5\} = J_1 \quad \{J_4, J_5\} = J_3 \quad (3.35)$$

$$\mathcal{H}^{(N)} = \mathcal{H} \left(\sum_{i=1}^N k_{1,i}, \sum_{i=1}^N k_{2,i}, \sum_{i=1}^N p_i, - \sum_{i=1}^N \left(k_{2,i} q_i + \frac{k_3}{2k_{1,i}} \right), - \sum_{i=1}^N \left(k_{1,i} q_i + \frac{p^2}{2k_{2,i}} \right) \right) \quad (3.36)$$

$$\mathcal{C}_3^{(m)} = \sum_{i=1}^m k_{3,i} + \sum_{\substack{i,j=1 \\ i \neq j}}^m p_i p_j + \sum_{\substack{i,j=1 \\ i \neq j}}^m \frac{k_{3,i} k_{1,j}}{k_{1,i}} - \sum_{\substack{i,j=1 \\ i \neq j}}^m \frac{k_{2,j}}{k_{2,i}} p_i^2 + 2 \sum_{i,j=1}^m k_{1,i} k_{2,j} (q_j - q_i). \quad (3.37)$$

The N -particle Hamiltonian $\mathcal{H} = -J_5 + \mathcal{G}(-J_4)$ leads to

$$\mathcal{H}^{(N)} = \sum_{i=1}^N \frac{p_i^2}{2k_{2,i}} + \sum_{i=1}^N k_{1,i} q_i + \mathcal{G} \left(\sum_{i=1}^N k_{2,i} q_i + \sum_{i=1}^N \frac{k_{3,i}}{k_{1,i}} \right) \quad (3.38)$$

which is completely integrable for any choice of the function \mathcal{G} . In particular, a large family of ND integrable non-homogeneous polynomial potentials are included in this family.

3.2.2 Algebras with $R = 2$

Table 7. Symplectic realizations for 5D Lie-Poisson algebras with $R = 2$.

	$A_{5,2}$	$A_{5,8}^c$	$A_{5,10}$	$A_{5,14}^\alpha$
J_1	k_1	k_1	k_1	k_1
J_2	p	$k_1 p$	$k_1 p$	$k_1 p + \frac{k_1}{2\alpha} \log \left(\frac{k_2}{k_3} \right)$
J_3	$\frac{1}{2k_1} (p^2 - k_2)$	$k_3 e^p$	$\frac{k_1}{2} p^2 - \frac{k_2}{2k_1}$	$\sqrt{k_2} e^{\alpha p} \cos p$
J_4	$\frac{1}{6k_1^2} (p^3 - 3k_2 p + 2k_3)$	$\frac{k_3^c}{k_2} e^{cp}$	$k_3 e^p$	$\sqrt{k_2} e^{\alpha p} \sin p$
J_5	$-k_1 q$	$-q$	$-q$	$-q$
\mathcal{C}_2	$J_2^2 - 2J_1 J_3$	$\frac{J_3^c}{J_4}$	$J_2^2 - 2J_1 J_3$	$(J_3^2 + J_4^2) e^{-2\alpha \arctan \left(\frac{J_4}{J_3} \right)}$
\mathcal{C}_3	$J_2^3 + 3J_1^2 J_4 - J_1 J_2 J_3$	$J_3 e^{-\frac{J_2}{J_1}}$	$J_4 e^{-\frac{J_2}{J_1}}$	$(J_3^2 + J_4^2) e^{-2\alpha \frac{J_2}{J_1}}$
	$k_1 \neq 0$	$k_1, k_2, k_3 \neq 0$	$k_1, k_3 \neq 0$	$k_1 \neq 0, k_2, k_3 > 0$

• $A_{5,2}$ Integrable systems

$$\{J_2, J_5\} = J_1 \quad \{J_3, J_5\} = J_2 \quad \{J_4, J_5\} = J_3 \quad (3.39)$$

$$\mathcal{H}^{(N)} = \mathcal{H} \left(J_1^{(N)}, J_2^{(N)}, J_3^{(N)}, J_4^{(N)}, J_5^{(N)} \right) \quad (3.40)$$

$$\begin{aligned}
J_1^{(N)} &= \sum_{i=1}^N k_{1,i} & J_2^{(N)} &= \sum_{i=1}^N p_i \\
J_3^{(N)} &= \sum_{i=1}^N \frac{1}{2k_{1,i}} (p_i^2 - k_{2,i}) & J_4^{(N)} &= \sum_{i=1}^N \frac{1}{6k_{1,i}^2} (p_i^3 - 3k_{2,i}p_i + 2k_{3,i}) \\
J_5^{(N)} &= - \sum_{i=1}^N k_{1,i}q_i
\end{aligned} \tag{3.41}$$

$$\begin{aligned}
\mathcal{C}_2^{(m)} &= \sum_{i=1}^m k_{2,i} + \sum_{\substack{i,j=1 \\ i \neq j}}^m p_i p_j - \sum_{\substack{i,j=1 \\ i \neq j}}^m \frac{k_{1,j}}{k_{1,i}} (p_i^2 - k_{2,i}) \\
\mathcal{C}_3^{(m)} &= \left(\sum_{i=1}^m p_i \right)^3 + 3 \left(\sum_{i=1}^m k_{1,i} \right)^2 \left(\sum_{i=1}^m \frac{1}{6k_{1,i}^2} (p_i^3 - 3k_{2,i}p_i + 2k_{3,i}) \right) \\
&\quad - 3 \left(\sum_{i=1}^m k_{1,i} \right) \left(\sum_{i=1}^m p_i \right) \left(\sum_{i=1}^m \frac{1}{2k_{1,i}} (p_i^2 - k_{2,i}) \right).
\end{aligned} \tag{3.42}$$

- $A_{5,8}^c$ integrable systems ($0 < |c| \leq 1$)

$$\{J_2, J_5\} = J_1 \quad \{J_3, J_5\} = J_3 \quad \{J_4, J_5\} = c J_4 \tag{3.43}$$

$$\mathcal{H}^{(N)} = \mathcal{H} \left(\sum_{i=1}^N k_{1,i}, \sum_{i=1}^N k_{1,i}p_i, \sum_{i=1}^N k_{3,i}e^{p_i}, \sum_{i=1}^N \frac{k_{3,i}^c}{k_{2,i}} e^{cp_i}, - \sum_{i=1}^N q_i \right) \tag{3.44}$$

$$\mathcal{C}_2^{(m)} = \frac{\left(\sum_{i=1}^m k_{3,i} e^{p_i} \right)^c}{\left(\sum_{i=1}^m \frac{k_{3,i}^c}{k_{2,i}} e^{cp_i} \right)} \quad \mathcal{C}_3^{(m)} = \left(\sum_{i=1}^m k_{3,i} e^{p_i} \right) e^{-\frac{\left(\sum_{i=1}^m k_{1,i} p_i \right)}{\left(\sum_{i=1}^m k_{1,i} \right)}}. \tag{3.45}$$

- $A_{5,10}$ integrable systems

$$\{J_2, J_5\} = J_1 \quad \{J_3, J_5\} = J_2 \quad \{J_4, J_5\} = J_4 \tag{3.46}$$

$$\mathcal{H} = \mathcal{H} \left(J_1^{(N)}, J_2^{(N)}, J_3^{(N)}, J_4^{(N)}, J_5^{(N)} \right) \tag{3.47}$$

$$\begin{aligned}
J_1^{(N)} &= \sum_{i=1}^N k_{1,i} & J_2^{(N)} &= \sum_{i=1}^N k_{1,i}p_i \\
J_3^{(N)} &= \sum_{i=1}^N \left(\frac{k_{1,i}}{2} p_i^2 - \frac{k_{2,i}}{2k_{1,i}} \right) & J_4^{(N)} &= \sum_{i=1}^N k_{3,i}e^{p_i} \\
J_5^{(N)} &= - \sum_{i=1}^N q_i
\end{aligned} \tag{3.48}$$

$$\begin{aligned}
\mathcal{C}_2^{(m)} &= \sum_{i=1}^m k_{2,i} + \sum_{\substack{i,j=1 \\ i \neq j}}^m \left(\frac{k_{2,i}}{k_{1,i}} - k_{1,i} p_i^2 \right) k_{1,j} + \sum_{\substack{i,j=1 \\ i \neq j}}^m k_{1,i} k_{1,j} p_i p_j \\
\mathcal{C}_3^{(m)} &= \left(\sum_{i=1}^m k_{3,i} e^{p_i} \right) e^{-\frac{\left(\sum_{i=1}^m k_{1,i} p_i \right)}{\left(\sum_{i=1}^m k_{1,i} \right)}}.
\end{aligned} \tag{3.49}$$

• $A_{5,14}^\alpha$ integrable systems

$$\{J_2, J_5\} = J_1 \quad \{J_3, J_5\} = \alpha J_3 - J_4 \quad \{J_4, J_5\} = J_3 + \alpha J_4 \tag{3.50}$$

$$\mathcal{H}^{(N)} = \left(J_1^{(N)}, J_2^{(N)}, J_3^{(N)}, J_4^{(N)}, J_5^{(N)} \right) \tag{3.51}$$

$$\begin{aligned}
J_1^{(N)} &= \sum_{i=1}^N k_{1,i} & J_2^{(N)} &= \sum_{i=1}^N \left(k_{1,i} p_i + \frac{k_{1,i}}{2\alpha} \log \frac{k_{2,i}}{k_{3,i}} \right) \\
J_3^{(N)} &= \sum_{i=1}^N \sqrt{k_{2,i}} e^{\alpha p_i} \cos p_i & J_4^{(N)} &= \sum_{i=1}^N \sqrt{k_{2,i}} e^{\alpha p_i} \sin p_i \\
J_5^{(N)} &= - \sum_{i=1}^N q_i
\end{aligned} \tag{3.52}$$

$$\begin{aligned}
\mathcal{C}_2^{(m)} &= \left(\sum_{i=1}^m k_{2,i} e^{2\alpha p_i} + 2 \sum_{\substack{i,j=1 \\ i \neq j}}^m \sqrt{k_{2,i} k_{2,j}} e^{\alpha(p_i+p_j)} \cos(p_i - p_j) \right) e^{-2\alpha \arctan(\mathcal{F}(p_1, \dots, p_m))} \\
\mathcal{C}_3^{(m)} &= \left(\sum_{i=1}^m k_{2,i} e^{2\alpha p_i} + 2 \sum_{\substack{i,j=1 \\ i \neq j}}^m \sqrt{k_{2,i} k_{2,j}} e^{\alpha(p_i+p_j)} \cos(p_i - p_j) \right) e^{-2\alpha (\mathcal{G}(p_1, \dots, p_m))}
\end{aligned} \tag{3.53}$$

Where the functions \mathcal{F} and \mathcal{G} have the following expressions

$$\mathcal{F}(p_1, \dots, p_m) = \frac{\sum_{i=1}^m \sqrt{k_{2,i}} e^{\alpha p_i} \sin p_i}{\sum_{i=1}^m \sqrt{k_{2,i}} e^{\alpha p_i} \cos p_i} \quad \mathcal{G}(p_1, \dots, p_m) = \frac{\sum_{i=1}^m \left(k_{1,i} p_i + \frac{k_{1,i}}{2\alpha} \log \frac{k_{2,i}}{k_{3,i}} \right)}{\sum_{i=1}^m k_{1,i}}. \tag{3.54}$$

3.2.3 Algebras with $R = 3$ Table 8. Symplectic realizations for 5D Lie-Poisson algebras with $R = 3$.

	$A_{5,7}^{a,b,c}$	$A_{5,9}^{b,c}$	$A_{5,11}^c$	$A_{5,12}^c$	$A_{5,15}^c$
J_1	e^p	$k_3 e^p$	$k_2 e^p$	$k_1 e^p$	e^{-p}
J_2	$\frac{e^{\alpha p}}{k_1}$	$k_3 p e^p$	$k_2 p e^p$	$k_1 p e^p$	$(p - \log k_2) e^p$
J_3	$\frac{e^{bp}}{k_2}$	$\frac{k_3^b}{k_1} e^{bp}$	$\frac{k_2}{2} e^{bp} (p^2 + k_3)$	$\frac{k_1}{2} e^p (p^2 + k_2)$	$\frac{e^{\alpha p}}{k_1}$
J_4	$\frac{e^{cp}}{k_3}$	$\frac{k_3^c}{k_2} e^{cp}$	$\frac{k_2^c}{k_1} e^{cp}$	$\frac{k_1 e^p}{6} (2k_3 + 3k_2 p + p^3)$	$\frac{e^{\alpha p}}{\alpha k_1} (\alpha p - \log k_1 k_3)$
J_5	$-q$	$-q$	$-q$	$-q$	$-q$
	$k_1, k_2, k_3 \neq 0$	$k_3 \neq 0$	$k_2 \neq 0$	$k_1 \neq 0$	$k_2 > 0, \frac{k_1}{k_3} > 0$

- $A_{5,7}^{a,b,c}$ integrable systems ($abc \neq 0$) ($-1 \leq c \leq b \leq a \leq 1$)

$$\begin{aligned} \{J_1, J_5\} &= J_1 & \{J_2, J_5\} &= a J_2 \\ \{J_3, J_5\} &= b J_3 & \{J_4, J_5\} &= c J_4 \end{aligned} \quad (3.55)$$

$$\mathcal{H}^{(N)} = \mathcal{H} \left(\sum_{i=1}^N e^{p_i}, \sum_{i=1}^N \frac{e^{a p_i}}{k_{1,i}}, \sum_{i=1}^N \frac{e^{b p_i}}{k_{2,i}}, \sum_{i=1}^N \frac{e^{c p_i}}{k_{3,i}}, -\sum_{i=1}^N q_i \right). \quad (3.56)$$

The non-linear Casimirs for this algebra are

$$\mathcal{C}_1 = \frac{J_1^a}{J_2} \quad \mathcal{C}_2 = \frac{J_1^b}{J_3} \quad \mathcal{C}_3 = \frac{J_1^c}{J_4} \quad (3.57)$$

$$\mathcal{C}_1^{(m)} = \frac{\left(\sum_{i=1}^m e^{p_i} \right)^a}{\left(\sum_{i=1}^m \frac{e^{a p_i}}{k_{1,i}} \right)} \quad \mathcal{C}_2^{(m)} = \frac{\left(\sum_{i=1}^m e^{p_i} \right)^b}{\left(\sum_{i=1}^m \frac{e^{b p_i}}{k_{2,i}} \right)} \quad \mathcal{C}_3^{(m)} = \frac{\left(\sum_{i=1}^m e^{p_i} \right)^c}{\left(\sum_{i=1}^m \frac{e^{c p_i}}{k_{3,i}} \right)}. \quad (3.58)$$

- $A_{5,9}^{b,c}$ integrable systems ($0 \neq c \leq b$)

$$\begin{aligned} \{J_1, J_5\} &= J_1 & \{J_2, J_5\} &= J_1 + J_2 \\ \{J_3, J_5\} &= b J_3 & \{J_4, J_5\} &= c J_4 \end{aligned} \quad (3.59)$$

$$\mathcal{H}^{(N)} = \mathcal{H} \left(\sum_{i=1}^N k_{3,i} e^{p_i}, \sum_{i=1}^N k_{3,i} p_i e^{p_i}, \sum_{i=1}^N \frac{k_{3,i}^b}{k_{1,i}} e^{b p_i}, \sum_{i=1}^N \frac{k_{3,i}^c}{k_{2,i}} e^{c p_i}, -\sum_{i=1}^N q_i \right). \quad (3.60)$$

The non-linear Casimirs are

$$\mathcal{C}_1 = \frac{J_1^b}{J_3} \quad \mathcal{C}_2 = \frac{J_1^c}{J_4} \quad \mathcal{C}_3 = J_1 e^{-\frac{J_2}{J_1}} \quad (3.61)$$

$$\begin{aligned} \mathcal{C}_1^{(m)} &= \frac{\left(\sum_{i=1}^m k_{3,i} e^{p_i} \right)^b}{\left(\sum_{i=1}^m \frac{k_{3,i}^b}{k_{1,i}} e^{b p_i} \right)} & \mathcal{C}_2^{(m)} &= \frac{\left(\sum_{i=1}^m k_{3,i} e^{p_i} \right)^c}{\left(\sum_{i=1}^m \frac{k_{3,i}^c}{k_{2,i}} e^{c p_i} \right)} \\ \mathcal{C}_3^{(m)} &= \left(\sum_{i=1}^m k_{3,i} e^{p_i} \right) e^{-\frac{\left(\sum_{i=1}^m k_{3,i} p_i e^{p_i} \right)}{\left(\sum_{i=1}^m k_{3,i} e^{p_i} \right)}}. \end{aligned} \quad (3.62)$$

• $A_{5,11}^c$ integrable systems ($c \neq 0$)

$$\begin{aligned} \{J_1, J_5\} &= J_1 & \{J_2, J_5\} &= J_1 + J_2 \\ \{J_3, J_5\} &= J_2 + J_3 & \{J_4, J_5\} &= c J_4 \end{aligned} \quad (3.63)$$

$$\mathcal{H}^{(N)} = \mathcal{H} \left(\sum_{i=1}^N k_{2,i} e^{p_i}, \sum_{i=1}^N k_{2,i} p_i e^{p_i}, \sum_{i=1}^N \frac{k_{2,i}}{2} e^{p_i} (p_i^2 + k_{3,i}), \sum_{i=1}^N \frac{k_{2,i}^c}{k_{1,i}} e^{c p_i}, -\sum_{i=1}^N q_i \right). \quad (3.64)$$

Again, from the following non-linear Casimirs we get the constants of the motion for this system:

$$\mathcal{C}_1 = \frac{J_1^c}{J_4} \quad \mathcal{C}_2 = J_1 e^{-\frac{J_2}{J_1}} \quad \mathcal{C}_3 = \frac{2 J_3}{J_1} - \frac{J_2^2}{J_1^2} \quad (3.65)$$

$$\begin{aligned} \mathcal{C}_1^{(m)} &= \frac{\left(\sum_{i=1}^m k_{2,i} e^{p_i} \right)^c}{\left(\sum_{i=1}^m \frac{k_{2,i}^c}{k_{1,i}} e^{c p_i} \right)} & \mathcal{C}_2^{(m)} &= \left(\sum_{i=1}^m k_{2,i} e^{p_i} \right) e^{-\frac{\left(\sum_{i=1}^m k_{2,i} p_i e^{p_i} \right)}{\left(\sum_{i=1}^m k_{2,i} e^{p_i} \right)}} \\ \mathcal{C}_3^{(m)} &= \frac{\left(\sum_{i=1}^m k_{2,i} e^{p_i} (p_i^2 + k_{3,i}) \right)}{\left(\sum_{i=1}^m k_{2,i} e^{p_i} \right)} - \frac{\left(\sum_{i=1}^m k_{2,i} p_i e^{p_i} \right)^2}{\left(\sum_{i=1}^m k_{2,i} e^{p_i} \right)^2}. \end{aligned} \quad (3.66)$$

• $A_{5,12}$ integrable systems

$$\begin{aligned} \{J_1, J_5\} &= J_1 & \{J_2, J_5\} &= J_1 + J_2 \\ \{J_3, J_5\} &= J_2 + J_3 & \{J_4, J_5\} &= J_3 + J_4 \end{aligned} \quad (3.67)$$

$$\mathcal{H}^{(N)} = \mathcal{H}\left(J_1^{(N)}, J_2^{(N)}, J_3^{(N)}, J_4^{(N)}, J_5^{(N)}\right) \quad (3.68)$$

$$\begin{aligned} J_1^{(N)} &= \sum_{i=1}^N k_{1,i} e^{p_i} & J_2^{(N)} &= \sum_{i=1}^N k_{1,i} p_i e^{p_i} \\ J_3^{(N)} &= \sum_{i=1}^N \frac{k_{1,i}}{2} e^{p_i} (p_i^2 + k_{2,i}) & J_4^{(N)} &= \sum_{i=1}^N \frac{k_{1,i} e^{p_i}}{6} (2k_{3,i} + 3k_{2,i} p_i + p_i^3) \\ J_5^{(N)} &= -\sum_{i=1}^N q_i. \end{aligned} \quad (3.69)$$

The associated constants of the motion will be given by the m -th symplectic realizations of the non-linear Casimir functions for $A_{5,12}$, namely

$$\mathcal{C}_1 = J_1 e^{-\frac{J_2}{J_1}} \quad \mathcal{C}_2 = \frac{2J_3}{J_1} - \frac{J_2^2}{J_1^2} \quad \mathcal{C}_3 = \frac{3J_4}{J_1} - \frac{3J_2 J_3}{J_1^2} + \frac{J_2^3}{J_1^3} \quad (3.70)$$

$$\begin{aligned} \mathcal{C}_1^{(m)} &= \left(\sum_{i=1}^m k_{1,i} e^{p_i} \right) e^{-\frac{\left(\sum_{i=1}^m k_{1,i} p_i e^{p_i} \right)}{\left(\sum_{i=1}^m k_{1,i} e^{p_i} \right)}} \\ \mathcal{C}_2^{(m)} &= \frac{\left(\sum_{i=1}^m k_{1,i} e^{p_i} (p_i^2 + k_{2,i}) \right)}{\left(\sum_{i=1}^m k_{1,i} e^{p_i} \right)} - \frac{\left(\sum_{i=1}^m k_{1,i} p_i e^{p_i} \right)^2}{\left(\sum_{i=1}^m k_{1,i} e^{p_i} \right)^2} \\ \mathcal{C}_3^{(m)} &= \frac{1}{2} \frac{\left(\sum_{i=1}^m k_{1,i} e^{p_i} (2k_{3,i} + 3k_{2,i} p_i + p_i^3) \right)}{\left(\sum_{i=1}^m k_{1,i} e^{p_i} \right)} \\ &\quad - \frac{3}{2} \frac{\left(\sum_{i=1}^m k_{1,i} p_i e^{p_i} \right) \left(\sum_{i=1}^m k_{1,i} e^{p_i} (p_i^2 + k_{2,i}) \right)}{\left(\sum_{i=1}^m k_{1,i} e^{p_i} \right)^2} + \frac{\left(\sum_{i=1}^m k_{2,i} p_i e^{p_i} \right)^3}{\left(\sum_{i=1}^m k_{1,i} e^{p_i} \right)^3}. \end{aligned} \quad (3.71)$$

- $A_{5,15}^\alpha$ integrable systems ($|\alpha| \leq 1$)

$$\begin{aligned} \{J_1, J_5\} &= J_1 & \{J_2, J_5\} &= J_1 + J_2 \\ \{J_4, J_5\} &= J_3 + \alpha J_4 & \{J_3, J_5\} &= \alpha J_3 \end{aligned} \quad (3.72)$$

$$\mathcal{H}^{(N)} = \mathcal{H} \left(J_1^{(N)}, J_2^{(N)}, J_3^{(N)}, J_4^{(N)}, J_5^{(N)} \right) \quad (3.73)$$

$$\begin{aligned} J_1^{(N)} &= \sum_{i=1}^N e^{p_i} & J_2^{(N)} &= \sum_{i=1}^N (p_i - \log k_{2,i}) e^{p_i} \\ J_3^{(N)} &= \sum_{i=1}^N \frac{e^{\alpha p_i}}{k_{1,i}} & J_4^{(N)} &= \sum_{i=1}^N \frac{e^{\alpha p_i}}{\alpha k_{1,i}} (\alpha p_i - \log k_{1,i} k_{3,i}) \\ J_5^{(N)} &= - \sum_{i=1}^N q_i. \end{aligned} \quad (3.74)$$

Finally, the complete integrability is given again by the N -particle symplectic realizations of the following non-linear Casimirs:

$$\mathcal{C}_1 = \frac{J_1^\alpha}{J_3} \quad \mathcal{C}_2 = J_1 e^{-\frac{J_2}{J_1}} \quad \mathcal{C}_3 = J_3 e^{-\alpha \frac{J_4}{J_3}} \quad (3.75)$$

$$\begin{aligned} \mathcal{C}_1^{(m)} &= \frac{\left(\sum_{i=1}^m e^{p_i} \right)^\alpha}{\left(\sum_{i=1}^m \frac{e^{\alpha p_i}}{k_{1,i}} \right)} & \mathcal{C}_2^{(m)} &= \left(\sum_{i=1}^m e^{p_i} \right) e^{-\frac{\left(\sum_{i=1}^m (p_i - \log k_{2,i}) e^{p_i} \right)}{\left(\sum_{i=1}^m e^{p_i} \right)}} \\ \mathcal{C}_3^{(m)} &= \left(\sum_{i=1}^m \frac{e^{\alpha p_i}}{k_{1,i}} \right) e^{-\frac{\left(\sum_{i=1}^m \frac{e^{\alpha p_i}}{k_{1,i}} (\alpha p_i - \log k_{1,i} k_{3,i}) \right)}{\left(\sum_{i=1}^m \frac{e^{\alpha p_i}}{k_{1,i}} \right)}}. \end{aligned} \quad (3.76)$$

3.3 Integrable systems from 6D nilpotent coalgebras

The same approach can be applied to the study of the 22 nilpotent 6D algebras classified in [145]. Among them, only three fulfill the necessary integrability condition for the generic symplectic realization presented in Chapter 2. They are those algebras in which $r = 4$ and $R \geq 1$. Consequently,

$$s_m = \frac{6 - 4}{2} = 1 \quad (3.77)$$

and the existence of only one non-linear Casimir is enough to guarantee the complete integrability of the ND systems coming from them. For the remaining 19 nilpotent algebras, we have $r = 2$ with $R = 1$, so we have

$$s_m = \frac{6-2}{2} = 2 \quad (3.78)$$

and the integrability condition is not fulfilled.

In the sequel we present the coalgebra construction of the integrable systems arising from the three algebras with $r = 4$. As usual, we give the symplectic realization, the generic form of the integrable Hamiltonian and the non-linear casimirs whose coproduct will provide the integrals of the motion.

• $A_{6,3}$ integrable systems ($R = 1$)

$$\{J_1, J_2\} = J_6 \quad \{J_1, J_3\} = J_4 \quad \{J_2, J_3\} = J_5 \quad (3.79)$$

$$\begin{aligned} J_1 &= k_3 p \\ J_2 &= -q - \frac{k_4}{k_1} \\ J_3 &= -\frac{k_1}{k_3} q - k_2 p \\ J_4 &= k_1 \\ J_5 &= k_2 \\ J_6 &= k_3 \quad (k_1, k_2 \neq 0) \end{aligned} \quad (3.80)$$

$$\mathcal{H}^{(N)} = \mathcal{H} \left(\sum_{i=1}^N k_{3,i} p_i, -\sum_{i=1}^N \left(q_i + \frac{k_{4,i}}{k_{1,i}} \right), -\sum_{i=1}^N \left(\frac{k_{1,i}}{k_{3,i}} q_i + k_{2,i} p_i \right), \sum_{i=1}^N k_{1,i}, \sum_{i=1}^N k_{2,i}, \sum_{i=1}^N k_{3,i} \right). \quad (3.81)$$

The non-linear Casimir is

$$\mathcal{C}_4 = J_1 J_5 + J_3 J_6 - J_2 J_4 \quad (3.82)$$

$$\mathcal{C}_4^{(m)} = \sum_{i=1}^m k_{4,i} + \sum_{\substack{i,j=1 \\ i \neq j}}^m \frac{k_{4,i}}{k_{1,i}} k_{1,j} + \sum_{\substack{i,j=1 \\ i \neq j}}^m k_{3,i} k_{2,j} (p_i - p_j) - \sum_{\substack{i,j=1 \\ i \neq j}}^m \frac{k_{1,i} q_i}{k_{3,i}} k_{3,j} + \sum_{\substack{i,j=1 \\ i \neq j}}^m k_{1,j} q_i. \quad (3.83)$$

• $A_{6,1}$ integrable systems ($R = 2$)

$$\{J_1, J_2\} = J_3 \quad \{J_1, J_3\} = J_4 \quad \{J_1, J_5\} = J_6 \quad (3.84)$$

$$\begin{aligned}
J_1 &= q \\
J_2 &= \frac{k_1}{2}p^2 + \frac{k_4}{2k_1} \\
J_3 &= k_1p \\
J_4 &= k_1 \\
J_5 &= k_2p - \frac{k_3}{k_1} \\
J_6 &= k_2 \quad (k_1, k_2 \neq 0)
\end{aligned} \tag{3.85}$$

$$\mathcal{H}^{(N)} = \mathcal{H} \left(\sum_{i=1}^N q_i, \sum_{i=1}^N \left(\frac{k_{1,i}}{2} p_i^2 + \frac{k_{4,i}}{2k_{1,i}} \right), \sum_{i=1}^N k_{1,i} p_i, \sum_{i=1}^N k_{1,i}, \sum_{i=1}^N \left(k_{2,i} p_i - \frac{k_{3,i}}{k_{1,i}} \right), \sum_{i=1}^N k_{2,i} \right). \tag{3.86}$$

The non-linear Casimirs are

$$\begin{aligned}
\mathcal{C}_3 &= J_3 J_6 - J_4 J_5 \\
\mathcal{C}_4 &= 2J_2 J_4 - J_3^2.
\end{aligned} \tag{3.87}$$

The constants of the motion will be given by the m -th coproducts of the two non-linear Casimir functions:

$$\begin{aligned}
\mathcal{C}_3^{(m)} &= \sum_{i=1}^m k_{3,i} + \sum_{\substack{i,j=1 \\ i \neq j}}^m \frac{k_{3,i}}{k_{1,i}} k_{1,j} + \sum_{\substack{i,j=1 \\ i \neq j}}^m k_{1,i} k_{2,j} (p_i - p_j) \\
\mathcal{C}_4^{(m)} &= \sum_{i=1}^m k_{4,i} + \sum_{\substack{i,j=1 \\ i \neq j}}^m \frac{k_{4,i}}{k_{1,i}} k_{1,j} - \sum_{\substack{i,j=1 \\ i \neq j}}^m k_{1,i} k_{1,j} p_i p_j + \sum_{\substack{i,j=1 \\ i \neq j}}^m k_{1,i} k_{1,j} p_i^2.
\end{aligned} \tag{3.88}$$

• $A_{6,2}$ integrable systems ($R = 3$)

$$\begin{aligned}
\{J_1, J_2\} &= J_3 & \{J_1, J_3\} &= J_4 \\
\{J_1, J_4\} &= J_5 & \{J_1, J_5\} &= J_6
\end{aligned} \tag{3.89}$$

$$\begin{aligned}
J_1 &= k_1 q \\
J_2 &= \frac{p^4 - 6k_2 p^2 - 3k_2^2 + 12k_1^2 k_3 + 8k_4 p}{24k_1^3} \\
J_3 &= \frac{p^3 + 3k_2 p + 2k_4}{6k_1^2} \\
J_4 &= \frac{p^2 + k_2}{2k_1} \\
J_5 &= p \\
J_6 &= k_1 \quad (k_1 \neq 0).
\end{aligned} \tag{3.90}$$

Therefore the generic ND integrable Hamiltonian would be:

$$\mathcal{H}^{(N)} = \mathcal{H} \left(J_1^{(N)}, J_2^{(N)}, J_3^{(N)}, J_4^{(N)}, J_5^{(N)}, J_6^{(N)} \right) \quad (3.91)$$

where

$$\begin{aligned} J_1^{(N)} &= \sum_{i=1}^N k_{1,i} q_i & J_2^{(N)} &= \sum_{i=1}^N \left(\frac{p_i^4 - 6k_{2,i} p_i^2 - 3k_{2,i}^2 + 12k_{1,i}^2 k_{3,i} + 8p_i k_{4,i}}{24k_{1,i}^3} \right) \\ J_3^{(N)} &= \sum_{i=1}^N \left(\frac{p_i^3 + 3p_i k_{2,i} + 2k_{4,i}}{6k_{1,i}^2} \right) & J_4^{(N)} &= \sum_{i=1}^N \frac{p_i^2 + k_{2,i}}{2k_{1,i}} \\ J_5^{(N)} &= \sum_{i=1}^N p_i & J_6^{(N)} &= \sum_{i=1}^N k_{1,i} \end{aligned} \quad (3.92)$$

and the coproducts of the three non-linear Casimir functions

$$\begin{aligned} \mathcal{C}_2 &= 2J_4 J_6 - J_5^2 \\ \mathcal{C}_3 &= 2J_2 J_6 - 2J_3 J_5 + J_4^2 \\ \mathcal{C}_4 &= 3J_3 J_6^2 - 3J_4 J_5 J_6 + J_5^3 \end{aligned} \quad (3.93)$$

will give us the integrals of the motion by choosing once again, an appropriate set of $(N - 1)$ functionally independent quantities

$$\begin{aligned} \mathcal{C}_2^{(m)} &= \frac{1}{2} \sum_{i=1}^m (k_{2,i} - p_i^2) + \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^m \left(\frac{k_{2,i} + p_i^2}{k_{1,i}} \right) k_{1,j} - \sum_{\substack{i,j=1 \\ i \neq j}}^m p_i p_j \\ \mathcal{C}_3^{(m)} &= 2 \left(\sum_{i=1}^m \frac{p_i^4 - 6k_{2,i} p_i^2 - 3k_{2,i}^2 + 12k_{1,i}^2 k_{3,i} + 8p_i k_{4,i}}{24k_{1,i}^3} \right) \left(\sum_{i=1}^m k_{1,i} \right) \\ &\quad - 2 \left(\sum_{i=1}^m \frac{p_i^3 + 3p_i k_{2,i} + 2k_{4,i}}{6k_{1,i}^2} \right) \left(\sum_{i=1}^m p_i \right) + \left(\sum_{i=1}^m \frac{p_i^2 + k_{2,i}}{2k_{1,i}} \right)^2 \\ \mathcal{C}_4^{(m)} &= 3 \left(\sum_{i=1}^m \frac{p_i^3 + 3p_i k_{2,i} + 2k_{4,i}}{6k_{1,i}^2} \right) \left(\sum_{i=1}^m k_{1,i} \right)^2 \\ &\quad - 3 \left(\sum_{i=1}^m \frac{p_i^2 + k_{2,i}}{2k_{1,i}} \right) \left(\sum_{i=1}^m p_i \right) \left(\sum_{i=1}^m k_{1,i} \right) + \left(\sum_{i=1}^m p_i \right)^3. \end{aligned} \quad (3.94)$$

3.4 A family of algebras with arbitrary dimension

In view of the exhaustive description of the Lie algebras with dimensions 3, 4, 5, y 6 (the nilpotent ones) given in Chapters 2 and 3, it seems natural to wonder whether we can

identify some families of algebras that can be defined for any dimension and for which the integrability condition under the coalgebra approach holds. The answer to this question is affirmative, and in this Section we are going to show that the M -dimensional Lie algebra \mathfrak{g}_M defined by the non-vanishing Poisson brackets

$$\{J_i, J_M\} = J_{i-1} \quad \text{with } i = 2, \dots, M-1 \quad (3.95)$$

is a Lie algebra with $r = M-2$ and with $R = r-1 = M-3$, thus fulfilling the integrability criteria for $M \geq 4$, since in that case $s_m = (M-r)/2 = 1$. Moreover, we will explicitly compute a closed formula for the $(M-2)$ Casimir functions for \mathfrak{g}_M , as well as its generic symplectic realization.

The key observation for the construction of \mathfrak{g}_M is to realize that the following chain of inclusions among different algebras holds:

$$A_{3,1} \subset A_{4,1} \subset A_{5,2} \subset A_{6,2}. \quad (3.96)$$

In fact, the following isomorphisms of Lie algebras are easily proven through the appropriate changes of basis:

$$\mathfrak{g}_3 \equiv A_{3,1} \quad \mathfrak{g}_4 \equiv A_{4,1} \quad \mathfrak{g}_5 \equiv A_{5,2} \quad \mathfrak{g}_6 \equiv A_{6,2}. \quad (3.97)$$

Now, by making use of the expressions for the Casimirs of these four algebras given in this and the previous Chapter, we can obtain explicitly the Casimir functions for \mathfrak{g}_M in the cases $M = 3, 4, 5, 6$, which read:

- \mathfrak{g}_3 : $\mathcal{C}_1 = J_1$
- \mathfrak{g}_4 :
$$\begin{cases} \mathcal{C}_1 = J_1 \\ \mathcal{C}_2 = \frac{-J_2^2 + 2J_1J_3}{2J_1} \end{cases}$$
- \mathfrak{g}_5 :
$$\begin{cases} \mathcal{C}_1 = J_1 \\ \mathcal{C}_2 = \frac{-J_2^2 + 2J_1J_3}{2J_1} \\ \mathcal{C}_3 = \frac{J_2^3 - 3J_1J_2J_3 + 3J_1^2J_4}{3J_1^2} \end{cases}$$
- \mathfrak{g}_6 :
$$\begin{cases} \mathcal{C}_1 = J_1 \\ \mathcal{C}_2 = \frac{-J_2^2 + 2J_1J_3}{2J_1} \\ \mathcal{C}_3 = \frac{J_2^3 - 3J_1J_2J_3 + 3J_1^2J_4}{3J_1^2} \\ \mathcal{C}_4 = \frac{-J_2^4 + 4J_1J_2^2J_3 - 8J_1^2J_2J_4 + 8J_1^3J_5}{8J_1^3}. \end{cases}$$

In these expressions we can easily appreciate how the chain of algebra embeddings is constructed: as far as we add another generator, one more Casimir function appears, and it contains the J_{M-1} generator.

This construction can be generalized for the arbitrary \mathfrak{g}_M algebra, since the \mathcal{C}_r Casimir functions ($r = 1, \dots, M-2$) must fulfill (see [1, 59, 145])

$$\{J_i, \mathcal{C}_r\} = \sum_{j=1}^M \frac{\partial \mathcal{C}_r}{\partial J_j} \{J_i, J_j\} = 0 \quad \begin{cases} i = 1, 2, \dots, M \\ r = 1, 2, \dots, (M-2). \end{cases} \quad (3.98)$$

By taking into account the Poisson brackets (3.95), we get the following system of differential equations

$$J_{i-1} \frac{\partial \mathcal{C}_r}{\partial J_M} = 0 \quad \begin{cases} i = 2, 3, \dots, M \\ r = 1, 2, \dots, (M-2). \end{cases} \quad (3.99)$$

From this expression we see that

$$\mathcal{C}_{M-2} = \mathcal{C}_{M-2}(J_1, J_2, \dots, J_{M-1}) \quad (3.100)$$

and the general solution of the system can be found. It reads,

$$\bullet \mathfrak{g}_M \left\{ \begin{array}{l} \mathcal{C}_1 = J_1 \\ \mathcal{C}_2 = \frac{-J_2^2 + 2J_1 J_3}{2J_1} \\ \vdots \\ \mathcal{C}_i = \frac{(-1)^{i+1} J_2^i + \sum_{j=1}^{i-2} a_{j,i} J_1^j J_2^{(i-1-j)} J_{j+2} + i(i-2)! J_1^{i-1} J_{i+1}}{i(i-2)! J_1^{i-1}} \\ \mathcal{C}_i = \frac{(-1)^{i+1} J_2^i + \sum_{j=1}^{i-2} a_{j,i} J_1^j J_2^{(i-1-j)} J_{j+2} + i(i-2)! J_1^{i-1} J_{i+1}}{i(i-2)! J_1^{i-1}} \quad i = 3, \dots, M-2 \end{array} \right. \quad (3.101)$$

where the coefficients a_j are given by:

$$\begin{cases} a_{1,i} = (-1)^i i \\ a_{2,i} = (-1)^{i+1} i \cdot (i-2) \\ a_{3,i} = (-1)^{i+2} i \cdot (i-2) \cdot (i-3) \\ \vdots \\ a_{i-2,i} = (-1)^{i+i-3} i \cdot (i-2) \dots (i-(i-2)). \end{cases} \quad (3.102)$$

Finally, the generic symplectic realization for \mathfrak{g}_M can be found:

$$\begin{aligned}
J_1 &= k_1 \\
J_2 &= k_1 p \\
&\vdots \\
J_j &= k_1 \frac{p^{j-1}}{(j-1)!} + \sum_{i=1}^{j-2} k_{i+1} \frac{p^{j-2-i}}{(j-2-i)!} \quad j = 3, 4, \dots, (M-1) \\
J_M &= -q.
\end{aligned} \tag{3.103}$$

Note that this realization can be easily derived by taking into account that the commutation rules (3.95) imply that the bracket of J_i with J_M gives J_{i-1} . If we take $J_M = -q$ and the rest of generators J_j are taken as polynomials in p with degree $j-1$, this implies that

$$J_j = \int J_{j-1}(p) dp + k_{j-1} \quad j = 3, \dots, (M-1). \tag{3.104}$$

With this symplectic realization at hand, the construction of completely integrable systems on the \mathfrak{g}_M coalgebra is straightforward.

Finally, it is also interesting to stress that symplectic realizations that do not fulfill the integrability condition can also lead to interesting (but partially integrable) Hamiltonian models. This is the case of the ‘two-photon’ algebra h_6 , a 6D Lie algebra with $r = 2$ (therefore $s = 1, 2$) that admits an $s = 1$ symplectic realization for which, among the $2N$ integrals provided by the coalgebra, only $(2N - 5)$ of them are functionally independent and $(N - 2)$ are in involution. Hence, any Hamiltonian \mathcal{H} with h_6 -coalgebra symmetry is “almost” integrable (only one constant is left) to get complete integrability, and such a remaining integral does exist for some special choices of \mathcal{H} . In fact, the study of the integrable systems coming from this coalgebra will be the subject of the next two Chapters.

Chapter 4

ND integrability from two-photon coalgebra

*“It is difficult to make a prediction,
especially a future prediction.”*

N. Bohr

This Chapter is devoted to the application of the coalgebra symmetry method described in Chapter 2 to the Poisson version of the so-called two-photon/Schrödinger algebra h_6 [16, 46, 49, 50, 185]. As we shall show in the sequel, the h_6 -coalgebra symmetry turns out to be extremely powerful, since the N -particle framework provided by the coalgebra structure encompasses different N -body symmetries that underlie several useful and explicit integrability properties, leading to several families of new completely integrable Hamiltonians that will be further analysed in Chapter 5.

We stress, that in the coalgebra symmetry framework the dimension of the coalgebra and the number of its non-linear Casimirs are essential in order to elucidate whether the coproduct gives either complete or partial integrability under a chosen symplectic realization. This dimensionality problem is essential for coalgebras with dimension greater than three, and it has been fully analysed in [27]. In fact, the h_6 -coalgebra provides an explicit case of the coalgebra construction for a six-dimensional coalgebra with two Casimir functions in which one of them is a linear one (the generator M). Therefore we have $l = 6, r = 2, R = 1$ and $s_m = \frac{l-r}{2} = 2$, and the necessary integrability condition for the generic symplectic realization is not fulfilled. Nevertheless, we shall consider a non-generic symplectic realization with $s = 1$, and we shall see how

in this case the coalgebra approach leads to quasi-integrability. This is an essential feature of the h_6 -coalgebra that could not appear in any of the examples of the coalgebra construction considered so far (see, for instance, [9, 12, 14, 21, 24, 27, 93, 114]).

Therefore, as a consequence of the quasi-integrability of the h_6 -coalgebra symmetry, it becomes important to develop some additional algebraic machinery in order to investigate for which choices of the generic Hamiltonian (4.6) the complete integrability can be restored. This is the aim of Sections 3, 4 and 5 in which we present a novel algebraic completion of the coalgebra approach that provides a guide to obtain the additional independent integral of the motion \mathcal{I} by exploiting the rich subalgebra structure of h_6 . This approach is fully general and can be applied to any coalgebra with dimension greater than three. Moreover, having in mind further applications, particular choices for \mathcal{H} leading to ND natural systems, geodesic flows and static electromagnetic Hamiltonians will be explicitly identified.

In particular, in Section 3 we analyse two different possibilities in order to find an extra integral \mathcal{I} . We remark that, in both cases, \mathcal{I} will be always defined as a function of the generators of h_6 , since this guarantees its existence for any number of degrees of freedom. The first case arises if the chosen Hamiltonian can be defined on a subalgebra of h_6 , since then the N -particle Casimir of such subalgebra can be directly identified with the additional constant of the motion \mathcal{I} . This possibility is discussed in Section 4, where the rich subalgebra structure for h_6 is fully described. A second possibility is analysed in Section 5, where we introduce five new families of ND completely integrable systems that have one of the five non-central generators of h_6 as the remaining integral of motion \mathcal{I} . This latter construction turns out to be powerful, since all these systems depend on several arbitrary functions and provide a large number of new instances of interesting Hamiltonians, both from the mathematical and the physical viewpoints.

A third possibility is developed in Section 6, where the direct search of the remaining integral \mathcal{I} is performed through Poisson-algebra computations. In this way, several classes of new interesting completely integrable systems are obtained.

4.1 The two-photon/Schrödinger algebra h_6

Let us recall that the h_6 Lie–Poisson algebra is spanned by the six abstract generators $\{K, A_+, A_-, B_+, B_-, M\}$, whose one-particle symplectic realization is given by

$$\begin{aligned} A_+ &= \lambda_1 p_1 & A_- &= \lambda_1 q_1 & K &= q_1 p_1 - \frac{\lambda_1^2}{2} \\ B_+ &= p_1^2 & B_- &= q_1^2 & M &= \lambda_1^2 \end{aligned} \quad (4.1)$$

where λ_1 is a non-vanishing constant that labels the previous symplectic realization and where we have considered the usual Poisson bracket $\{q_1, p_1\} = 1$. The abstract Poisson brackets defining h_6 read

$$\begin{aligned}
\{K, A_+\} &= A_+ & \{K, A_-\} &= -A_- & \{A_-, A_+\} &= M \\
\{K, B_+\} &= 2B_+ & \{K, B_-\} &= -2B_- & \{B_-, B_+\} &= 4K + 2M \\
\{A_+, B_-\} &= -2A_- & \{A_+, B_+\} &= 0 & \{M, \cdot\} &= 0 \\
\{A_-, B_+\} &= 2A_+ & \{A_-, B_-\} &= 0. & &
\end{aligned} \tag{4.2}$$

A direct inspection of this algebra makes its rich subalgebra structure evident (for instance, the $gl(2)$ subalgebra $\{K, B_+, B_-, M\}$, the oscillator one $h_4 \equiv \{K, A_+, A_-, M\}$ and the Heisenberg one $h_3 \equiv \{A_+, A_-, M\}$ can be easily identified). This is one of the main features of h_6 , since this algebra generalizes many lower dimensional Lie symmetries in a transparent way, a fact that will be relevant in order to find new and more general integrability structures, as we shall see in Section 4.

As a consequence of (4.1), any quadratic Hamiltonian with one degree of freedom can always be written as a linear combination of the h_6 generators in the above representation. This is indeed the origin of the quantum mechanical relevance of the h_6 Lie algebra as a dynamical symmetry, since the quantum counterpart of such a quadratic Hamiltonian can be interpreted as a single-mode radiation field Hamiltonian including the number operator \hat{K} , creation and annihilation operators \hat{A}_+ and \hat{A}_- and two-photon creation and annihilation operators \hat{B}_+ and \hat{B}_- (the generator \hat{M} is a central one). In this context, different applications of the h_6 symmetry can be found in [46], and it is also interesting to recall that h_6 is isomorphic to the $(1+1)D$ centrally extended Schrödinger Lie algebra [10]. In fact, this isomorphism provides the two Casimir functions for h_6 , which are the central generator M and the fourth-order Casimir [145]

$$\mathcal{C} = (MB_+ - A_+^2)(MB_- - A_-^2) - (MK - A_-A_+ + M^2/2)^2 \tag{4.3}$$

that will play a relevant role hereafter. The function \mathcal{C} can be reduced to a third-order invariant by extracting M as a common factor:

$$\mathcal{C}_{h_6} = \mathcal{C}/M = MB_+B_- - B_+A_-^2 - B_-A_+^2 - M(K + M/2)^2 + 2A_-A_+(K + M/2). \tag{4.4}$$

Note that if we substitute the symplectic realization (4.1) within the two Casimir functions, the former is characterized by the values $M = \lambda_1^2$ and $\mathcal{C}_{h_6} = 0$.

Now, if we endow the h_6 Poisson algebra with a primitive coalgebra structure, *i.e.* with a two-body coproduct (which is a Poisson algebra homomorphism $\Delta : h_6 \rightarrow h_6 \otimes h_6$ [12, 16]) given by

$$\Delta(X) = X \otimes 1 + 1 \otimes X \quad X \in \{K, A_+, A_-, B_+, B_-, M\}, \tag{4.5}$$

then the one-particle dynamical symmetry given by (4.1) can be generalized for any number of degrees of freedom. As we shall see in Section 2, this gives rise to an infinite family of N -particle Hamiltonians defined as *any* smooth function \mathcal{H} of the generators of h_6

$$\mathcal{H} = \mathcal{H}(K, B_+, B_-, A_+, A_-, M) \quad (4.6)$$

and provided that \mathcal{H} is realized in the N -particle symplectic realization coming from the coproduct structure.

4.2 An infinite family of quasi-integrable systems

The integrability properties of the generic N -particle Hamiltonians with h_6 -coalgebra symmetry relies on the following result [28], which is based in the application of the coalgebra symmetry approach to a non-generic symplectic realization of the h_6 Poisson coalgebra.

In particular, let $\{\mathbf{q}, \mathbf{p}\} = \{(q_1, \dots, q_N), (p_1, \dots, p_N)\}$ be N pairs of canonical variables. The ND Hamiltonian

$$\mathcal{H} = \mathcal{H}(K, B_+, B_-, A_+, A_-, M) \quad (4.7)$$

defined as any smooth function $\mathcal{H} : \mathbb{R}^6 \rightarrow \mathbb{R}$ and

$$\begin{aligned} A_+ &= \sum_{i=1}^N \lambda_i p_i & A_- &= \sum_{i=1}^N \lambda_i q_i & K &= \sum_{i=1}^N \left(q_i p_i - \frac{\lambda_i^2}{2} \right) \\ B_+ &= \sum_{i=1}^N p_i^2 & B_- &= \sum_{i=1}^N q_i^2 & M &= \sum_{i=1}^N \lambda_i^2 \end{aligned} \quad (4.8)$$

where λ_i , are N arbitrary parameters, is quasi-integrable. The $(N - 2)$ functionally independent integrals of the motion for \mathcal{H} are

$$\mathcal{C}^{(m)} = \sum_{1 \leq i < j < k}^m (\lambda_i (p_j q_k - p_k q_j) + \lambda_j (p_k q_i - p_i q_k) + \lambda_k (p_i q_j - p_j q_i))^2 \quad (4.9)$$

where $m = 3, \dots, N$. These integrals are in involution and can be called ‘universal’ in the sense that they do not depend on the specific choice of the function \mathcal{H} .

The keystone to prove this result (see [28]) comes from the fact that (4.1) is a one-particle symplectic realization for the Poisson coalgebra (h_6, Δ) , labelled by the λ_1 parameter. Moreover, it can be easily checked that (4.8) is just the N -particle

symplectic realization of (h_6, Δ) that is obtained through the N -sites generalization of the coproduct (4.5):

$$\begin{aligned} \Delta^{(N)}(X) = & X \otimes 1 \otimes 1 \otimes \dots \otimes 1 \otimes 1 \\ & + 1 \otimes X \otimes 1 \otimes \dots \otimes 1 \otimes 1 + \dots \\ & + 1 \otimes 1 \otimes \dots \otimes 1 \otimes X. \end{aligned} \quad (4.10)$$

This means that the N -particle generators (4.8) fulfil the commutation rules (4.2) with respect to the canonical Poisson bracket $\{f, g\} = \sum_{i=1}^N \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i} \frac{\partial f}{\partial p_i} \right)$.

Moreover, such a coalgebra symmetry expressed through the symplectic realization implies that each of the N -particle generators (4.8) Poisson-commute with the $(N-1)$ functions $\mathcal{C}^{(m)}$ given by the m -th coproducts of the Casimir (4.9) with $m = 2, 3, \dots, N$ (see [12, 16] for details). However, in the case of the specific symplectic realization of h_6 given by (4.8), the $\mathcal{C}^{(2)}$ function vanishes (the two-body coproduct of the Casimir is zero [16]) and we are left with the set of $(N-2)$ integrals (4.9) that, also by construction, are functionally independent and Poisson-commuting. Therefore, any function \mathcal{H} (4.7) of the N -particle symplectic realization of the h_6 generators will be in involution with the set of integrals $\mathcal{C}^{(m)}$, which completes the proof.

This general result deserves the following remarks and comments:

- First of all, we have to characterize the symplectic realizations for h_6 in the framework of Chapter 2, by taking into account that $R = 1$, $r = 2$, $l = 6$. According to (2.19), the generic representation for h_6 would have

$$s_m = \frac{l-r}{2} = \frac{6-2}{2} = 2 \quad (4.11)$$

pairs of canonical variables. This generic realization would be ‘a Calogero-Moser’ realization, namely.

$$\begin{aligned} \mathcal{D}(A_+) &= \lambda_1 p_1 + \lambda_2 p_2 & \mathcal{D}(A_-) &= \lambda_1 q_1 + \lambda_2 q_2 \\ \mathcal{D}(K) &= q_1 p_1 + q_2 p_2 - \frac{(\lambda_1^2 + \lambda_2^2)}{2} & \mathcal{D}(B_+) &= p_1^2 + p_2^2 + \frac{b}{(\lambda_1 q_2 - \lambda_2 q_1)^2} \\ \mathcal{D}(B_-) &= q_1^2 + q_2^2 & \mathcal{D}(M) &= \lambda_1^2 + \lambda_2^2 \end{aligned} \quad (4.12)$$

In this case, the value for the non trivial Casimir operator of h_6 is

$$\mathcal{C}_{h_6} = b. \quad (4.13)$$

And concerning the necessary integrability condition expressed in (2.18) and (2.21) we would have that

$$s \leq R - \frac{(R-1)}{N} \rightarrow 2 \leq 1 \quad (4.14)$$

$$l \leq (2R + r) - \frac{2}{N}(R - 1) \rightarrow 6 \leq 4 \quad (4.15)$$

Obviously the former relations are not verified for the generic symplectic realization, and the construction of Hamiltonian systems from it does not lead to complete integrability. Moreover, the non-generic symplectic realization that we are using does fulfill the necessary integrability condition, but it shows that such condition is not sufficient: despite the dimensionality conditions are satisfied, the coalgebra framework does not provide complete integrability since one integral (the one coming from the first coproduct of the Casimir) vanishes.

- We can properly say that \mathcal{H} is a *quasi-integrable* Hamiltonian, since for *any dimension* N and *any choice of* \mathcal{H} there is only one integral left in order to get its complete integrability. Obviously, some specific choices for \mathcal{H} will lead to *completely integrable* Hamiltonians for which an additional integral does exist for any dimension N . The aim of this Chapter is just to find solutions to this problem.
- Within the coalgebra approach it is well-known that, in general, two different sets of integrals of the motion coming from ‘left’ and ‘right’ m -th coproducts of the Casimir can be obtained (see [21] for details). Indeed, this is also the case for the (h_6, Δ) coalgebra, where by making use of the ‘right’ m -th coproducts, the following alternative set of $(N - 2)$ integrals in involution $\mathcal{C}_{(m)}$ is obtained:

$$\mathcal{C}_{(m)} = \sum_{N-m+1 \leq i < j < k}^N (\lambda_i(p_j q_k - p_k q_j) + \lambda_j(p_k q_i - p_i q_k) + \lambda_k(p_i q_j - p_j q_i))^2. \quad (4.16)$$

This means that if we label the N sites on $h_6 \otimes h_6 \otimes \dots \otimes h_6$ by $1 \otimes 2 \otimes \dots \otimes N$, the ‘left’ Casimir $\mathcal{C}^{(m)}$ is defined on the sites $1 \otimes 2 \otimes \dots \otimes m$, while the ‘right’ one $\mathcal{C}_{(m)}$ is defined on $(N - m + 1) \otimes \dots \otimes (N - 1) \otimes N$. Moreover, it is straightforward to prove that the $(2N - 4)$ functions $\{\mathcal{C}^{(3)}, \mathcal{C}^{(4)}, \dots, \mathcal{C}^{(N)} \equiv \mathcal{C}_{(N)}, \mathcal{C}_{(N-1)}, \dots, \mathcal{C}_{(3)}, \mathcal{H}\}$ are functionally independent (assuming that \mathcal{H} is not a function of \mathcal{C} only) and the coalgebra symmetry ensures that each of the two subsets $\{\mathcal{C}^{(3)}, \dots, \mathcal{C}^{(N)}, \mathcal{H}\}$ or $\{\mathcal{C}_{(3)}, \dots, \mathcal{C}_{(N)}, \mathcal{H}\}$ is formed by $(N - 1)$ functions in involution [12, 21].

- As a consequence, in case that an additional integral \mathcal{I} is found for a given \mathcal{H} , this Hamiltonian will be not only integrable, but also *superintegrable* provided that the $(N - 3)$ ‘right’ constants $\mathcal{C}_{(m)}$ (with $m = 3, \dots, N - 1$) commute with \mathcal{I} (and this property will be ensured if \mathcal{I} is a function of the N -particle symplectic realization of the h_6 -coalgebra).
- The role of the λ_i parameters is essential in this approach, since they provide an N -parameter freedom for the Hamiltonian. From a co-algebraic viewpoint

these λ_i parameters can be neatly interpreted: each of them fixes the one-particle symplectic realization that we are using on the i -th site of the underlying h_6 symmetry lattice $h_6 \otimes h_6 \otimes \dots \otimes h_6$.

- We stress that the integrals (4.9) can be interpreted as sums of the squares of a linear combination (through the λ_i parameters) of ‘Euclidean angular momentum’ components J_{ij} . In particular, the $N(N-1)/2$ functions $J_{ij} = (q_i p_j - q_j p_i)$ with $i < j$ and $i, j = 1, \dots, N$ span an $so(N)$ Lie–Poisson algebra so that $\mathcal{C}^{(m)}$ can be read as

$$\mathcal{C}^{(m)} = \sum_{1 \leq i < j < k}^m (\lambda_i J_{jk} + \lambda_j J_{ki} + \lambda_k J_{ij})^2 \quad (4.17)$$

where $J_{ki} = -J_{ik}$. Hence each term in $\mathcal{C}^{(m)}$ is the square of an element of the Lie–Poisson algebra $so(3) = \{J_{ij}, J_{ik}, J_{jk}\}$ (and the same happens in $\mathcal{C}_{(m)}$). From this perspective, the h_6 -coalgebra symmetry can be interpreted as a ‘generalization’ of the spherical symmetry, which will be fully recovered when the Hamiltonian is defined on the $gl(2)$ Poisson subalgebra of h_6 .

- Although the central generator M is also a Casimir for the two-photon coalgebra, its N -th coproduct gives rise to N trivial integrals of the motion

$$M^{(m)} = \sum_{i=1}^m \lambda_i^2 \quad m = 1, \dots, N \quad (4.18)$$

that, do not provide any dynamical information. In this sense, M can be considered either as a generator on its own right or as a constant that depends on both the dimension N and the chosen symplectic realizations through the λ_i parameters.

Finally, is worth mentioning that the role of the h_6 algebra in the integrability properties of certain 3D Hamiltonian systems was already pointed out from a different viewpoint in [179]. Nevertheless, the introduction of a coalgebra structure in h_6 turns out to be essential in order to fully exploit its integrability information and to generalize it to arbitrary dimensions.

4.2.1 Some relevant quasi-integrable Hamiltonians

Among the bunch of ND quasi-integrable systems that are provided by (4.7), the following particular subclasses are physically outstanding.

Natural systems

The Hamiltonian

$$\mathcal{H} = \frac{1}{2}B_+ + \mathcal{F}(A_-, B_-) \quad (4.19)$$

where \mathcal{F} is a function playing the role of a potential, gives rise to the following quasi-integrable system on the ND Euclidean space \mathbb{E}^N :

$$\mathcal{H}^{(N)} = \sum_{i=1}^N \frac{p_i^2}{2} + \mathcal{F}\left(\sum_{i=1}^N \lambda_i q_i, \sum_{i=1}^N q_i^2\right). \quad (4.20)$$

Notice that central potentials (endowed with spherical symmetry) directly arise whenever \mathcal{F} does not depend on A_- , since the Hamiltonian is then defined on the $gl(2)$ subalgebra. Thus in the case with generic $\mathcal{F}(A_-, B_-)$, the spherical symmetry is broken and its associated (super)integrability is, in principle, reduced to quasi-integrability. Nevertheless, the h_6 -coalgebra symmetry of (4.20) reduces the ND integrability problem to the search for only one additional integral, whose existence and explicit form will depend on the particular choice of the function \mathcal{F} .

Electromagnetic Hamiltonians

The most general ND quasi-integrable Hamiltonian including linear terms in the momenta is given by

$$\mathcal{H} = \frac{1}{2}B_+ + K \mathcal{F}(A_-, B_-) + A_+ \mathcal{G}(A_-, B_-) + \mathcal{R}(A_-, B_-) \quad (4.21)$$

where \mathcal{F} , \mathcal{G} and \mathcal{R} are smooth functions. In terms of canonical variables, it reads

$$\begin{aligned} \mathcal{H}^{(N)} = & \sum_{i=1}^N \frac{p_i^2}{2} + \left(\sum_{i=1}^N \left(q_i p_i - \frac{\lambda_i^2}{2} \right) \right) \mathcal{F}\left(\sum_{i=1}^N \lambda_i q_i, \sum_{i=1}^N q_i^2 \right) \\ & + \left(\sum_{i=1}^N \lambda_i p_i \right) \mathcal{G}\left(\sum_{i=1}^N \lambda_i q_i, \sum_{i=1}^N q_i^2 \right) + \mathcal{R}\left(\sum_{i=1}^N \lambda_i q_i, \sum_{i=1}^N q_i^2 \right). \end{aligned} \quad (4.22)$$

In 3D, this Hamiltonian describes the motion of a particle on \mathbb{E}^3 under the action of a static electromagnetic field which is determined by the functions \mathcal{F} , \mathcal{G} and \mathcal{R} . Namely, if we compare (4.22) with the 3D electromagnetic Hamiltonian

$$\mathcal{H}_{em} = \frac{1}{2} \left(\vec{p} - e\vec{A} \right)^2 + e\psi \quad (4.23)$$

where e is the electric charge, \vec{A} is the vector potential and ψ is the scalar one, we get

$$A_i = -\frac{q_i}{e}\mathcal{F}(A_-, B_-) - \frac{\lambda_i}{e}\mathcal{G}(A_-, B_-) \quad i = 1, 2, 3 \quad (4.24)$$

$$\begin{aligned} \psi = & \frac{1}{e}\mathcal{R}(A_-, B_-) - \frac{1}{2e}M\mathcal{F}(A_-, B_-) \\ & - \frac{1}{2e}\left[B_-\mathcal{F}(A_-, B_-)^2 + 2A_-\mathcal{F}(A_-, B_-)\mathcal{G}(A_-, B_-) + M\mathcal{G}(A_-, B_-)^2\right]. \end{aligned} \quad (4.25)$$

Note the relevant role that the λ_i parameters play in the definition of the electromagnetic field. Recall also that ND superintegrable electromagnetic systems have been recently obtained in [24] by making use of an $sl(2, \mathbb{R})$ -coalgebra symmetry. In fact, as we shall show in Section 4.4, the latter systems are a particular subfamily of (4.22) since $sl(2, \mathbb{R})$ is a sub-coalgebra of h_6 . We also recall that only low-dimensional integrable Euclidean Hamiltonians with velocity-dependent potentials have been previously studied in [35, 42, 78, 106, 108, 124, 152, 153, 167].

Geodesic flow Hamiltonians

A third family of relevant systems is the one given by ND quasi-integrable Hamiltonians of the type

$$\mathcal{H}^{(N)} = \sum_{i,j=1}^N g^{ij}(q_1, \dots, q_N) p_i p_j \quad (4.26)$$

that are obtained by considering

$$\begin{aligned} \mathcal{H} = & B_+\mathcal{F}(A_-, B_-) + A_+^2\mathcal{G}(A_-, B_-) + \left(K + \frac{M}{2}\right)^2 \mathcal{R}(A_-, B_-) \\ & + \left(K + \frac{M}{2}\right) A_+ \mathcal{S}(A_-, B_-) \end{aligned} \quad (4.27)$$

since for any choice of the functions \mathcal{F} , \mathcal{G} , \mathcal{R} and \mathcal{S} we obtain a Hamiltonian which is a quadratic homogeneous function in the momenta. Explicitly,

$$\begin{aligned} \mathcal{H}^{(N)} = & \left(\sum_{i=1}^N \frac{p_i^2}{2}\right) \mathcal{F}\left(\sum_{i=1}^N \lambda_i q_i, \sum_{i=1}^N q_i^2\right) + \left(\sum_{i=1}^N \lambda_i p_i\right)^2 \mathcal{G}\left(\sum_{i=1}^N \lambda_i q_i, \sum_{i=1}^N q_i^2\right) \\ & + \left(\sum_{i=1}^N q_i p_i\right)^2 \mathcal{R}\left(\sum_{i=1}^N \lambda_i q_i, \sum_{i=1}^N q_i^2\right) \\ & + \left(\sum_{i=1}^N q_i p_i\right) \left(\sum_{i=1}^N \lambda_i p_i\right) \mathcal{S}\left(\sum_{i=1}^N \lambda_i q_i, \sum_{i=1}^N q_i^2\right). \end{aligned} \quad (4.28)$$

We stress that the specific form of the metric g^{ij} is determined by the \mathcal{F} , \mathcal{G} , \mathcal{R} and \mathcal{S} functions which, in general, gives rise to an ND space of *nonconstant* curvature. In any case, the set of constants of motion (4.9) is once again universal and does not depend on the specific choice of the functions in the Hamiltonian. Moreover, additional potentials on these h_6 -coalgebra spaces can be naturally considered by adding functions such as, e.g., $\mathcal{U}(A_-, B_-)$ to the free Hamiltonian (4.27). In this way the Euclidean natural systems (4.19) can be generalized to the curved spaces defined through (4.27) without breaking the quasi-integrability of the geodesic flow Hamiltonian.

In this respect, we recall that the complete integrability of a free Hamiltonian on a curved space is a rather nontrivial property which is connected with geometric and topological features of the underlying manifold [25, 110, 115, 146]. From the physical viewpoint, the study of integrable geodesic flows in arbitrary dimension is becoming increasingly popular in general relativity, supergravity and superstring theories, where the explicit knowledge of the Stäckel–Killing integrals of motion for the geodesic flows of ND curved spaces provides very useful information (see, for instance, [70, 116, 143, 174]). Since integrable examples of such ND curved geodesic flows are quite scarce, the search of new completely integrable instances among the Hamiltonians (4.28) is thus meaningful.

4.3 Complete integrability

At this point the main problem to be faced is the characterization of those Hamiltonians \mathcal{H} for which an additional integral \mathcal{I} does exist *for any dimension* N , thus providing their complete ND integrability.

In order to ensure the existence of \mathcal{I} for any dimension N , we shall assume that this additional integral is also h_6 -coalgebra invariant, which means that it can be written as a function

$$\mathcal{I} = \mathcal{I}(K, B_+, B_-, A_+, A_-, M) \quad (4.29)$$

where the h_6 generators are written in their N -particle symplectic realization (4.8). In this way, if \mathcal{I} is functionally independent with respect to both the h_6 Casimir (4.4) and the Hamiltonian \mathcal{H} , the coalgebra symmetry ensures –by construction– the involutivity of \mathcal{I} with respect to the $(N - 2)$ ‘left’ integrals $\mathcal{C}^{(m)}$ ($m = 3, \dots, N$) and its functional independence with respect to them. And the very same result holds for the $(N - 3)$ ‘right’ integrals $\mathcal{C}_{(m)}$ where $m = 3, \dots, N - 1$ (we recall that $\mathcal{C}_{(N)} = \mathcal{C}^{(N)}$).

This means that if \mathcal{I} does exist in the form (4.29), then \mathcal{H} will be not only a completely integrable system but also a *superintegrable* one, since a total number of $(N - 2) + (N - 3) + 1 = (2N - 4)$ functionally independent constants of motion for

\mathcal{H} has been explicitly found. Nevertheless, even in this superintegrable case, \mathcal{H} is not a *maximally superintegrable* Hamiltonian, since *two* more independent constants of the motion would be needed to get the maximum possible total number of $(2N - 2)$ independent integrals. Again, these two remaining integrals could exist for some very particular choices for \mathcal{H} , but in any case neither their existence nor their explicit form can be derived from the h_6 -coalgebra symmetry.

The rest of this Chapter is devoted to show the search for the additional \mathcal{I} (4.29) by different algebraic strategies within h_6 . In particular, we shall consider two different situations in which the existence of \mathcal{I} is guaranteed by construction:

A) If the Hamiltonian \mathcal{H} is defined within a subalgebra of h_6 that has a non-linear Casimir invariant, the N -particle realization of the Casimir of the subalgebra provides the integral \mathcal{I} . This *subalgebra integrability* approach will be analysed in the next Section (4.4), in which the subalgebra structure for h_6 will be fully described.

B) Let X be a fixed generator of h_6 . The N -particle symplectic realization of X will Poisson commute with any N -particle Hamiltonian \mathcal{H}_X defined as a function of all the remaining h_6 generators commuting with X and of the Casimirs of all the subalgebras containing the given generator X . Under such hypotheses, \mathcal{H}_X is completely integrable since the generator X is just the additional constant of motion \mathcal{I} . We have five relevant generators $\{K, B_+, B_-, A_+, A_-\}$ (the central generator M would give no dynamical information), so this *generator integrability* procedure will give rise to five families of completely integrable systems that will be studied in detail in Section 4.5.

Finally, we stress that if a given Hamiltonian does not fit within the two previous approaches, the search for the remaining integral \mathcal{I} –in case it does exist– have to be performed by using direct methods. Indeed, some particular solutions can be found, and a several examples will be given in the final Section. This direct search method will be also the cornerstone for all the results presented in Chapter 5.

4.4 Subalgebra integrability

The subalgebras of h_6 with a non-trivial (*i.e.* linear) Casimir function are summarized in Table 9 together with their 1D symplectic realization. They are:

- Two ‘book’ algebras \mathcal{D}_+ and \mathcal{D}_- generated by a dilation plus two translations.
- The harmonic oscillator algebra h_4 .
- Two centrally extended $(1 + 1)$ D Galilean algebras $\bar{\mathcal{G}}_+$ and $\bar{\mathcal{G}}_-$.

- A centrally extended 2D Euclidean algebra $\bar{\mathcal{E}}$ (where μ and ν are non-zero real parameters).
- The $gl(2)$ algebra.

More details on these subalgebras and on their associated Lie–Poisson structures can be found in [16]. Clearly the Heisenberg–Weyl algebra $h_3 = \{A_+, A_-, M\}$ is a subalgebra of h_4 , and $gl(2)$ contains an $sl(2, \mathbb{R})$ subalgebra (by mapping $K \rightarrow K + M/2$), so we have the following subalgebra embeddings:

$$h_3 \subset h_4 \subset h_6 \quad \bar{\mathcal{G}}_{\pm} \subset h_6 \quad sl(2, \mathbb{R}) \subset gl(2) \subset h_6. \quad (4.30)$$

Notice also that $\bar{\mathcal{E}}$ is a proper Euclidean subalgebra whenever μ and ν have the same sign; on the contrary, $\bar{\mathcal{E}}$ is in fact a centrally extended (1 + 1)D Poincaré subalgebra. In the sequel we do not distinguish the two real forms as the resulting expressions for $\bar{\mathcal{E}}$ will be globally parametrized through μ and ν .

Table 9. Relevant subalgebras of h_6 .

Subalgebra	Generators	Symplectic realization	Casimir function
\mathcal{D}_+	K, A_+, B_+	$qp - \frac{\lambda^2}{2}, \lambda p, p^2$	A_+^2/B_+
\mathcal{D}_-	K, A_-, B_-	$qp - \frac{\lambda^2}{2}, \lambda q, q^2$	A_-^2/B_-
h_4	K, A_-, A_+, M	$qp - \frac{\lambda^2}{2}, \lambda q, \lambda p, \lambda^2$	$M(K + \frac{1}{2}M) - A_-A_+$
$\bar{\mathcal{G}}_+$	B_+, A_-, A_+, M	$p^2, \lambda q, \lambda p, \lambda^2$	$MB_+ - A_+^2$
$\bar{\mathcal{G}}_-$	B_-, A_-, A_+, M	$q^2, \lambda q, \lambda p, \lambda^2$	$MB_- - A_-^2$
$\bar{\mathcal{E}}$	$\mu B_+ + \nu B_-,$ A_-, A_+, M	$\mu p^2 + \nu q^2,$ $\lambda q, \lambda p, \lambda^2$	$M(\mu B_+ + \nu B_-)$ $-\mu A_+^2 - \nu A_-^2$
$gl(2)$	K, B_-, B_+, M	$qp - \frac{\lambda^2}{2}, q^2, p^2, \lambda^2$	$B_-B_+ - (K + \frac{1}{2}M)^2$

As we have pointed out in the previous Section, any Hamiltonian \mathcal{H}_g defined on one of the abovementioned subalgebras g is completely integrable by construction, since the N -th coproduct of the Casimir \mathcal{C}_g provides the extra integral \mathcal{I} , which completes the set of $(N - 2)$ left integrals $\mathcal{C}^{(m)}$ coming from the h_6 -coalgebra. Note that \mathcal{I} is a function of the h_6 generators and, as a consequence, is in involution with each of the $\mathcal{C}^{(m)}$ integrals. Therefore, we can state that the following ND Hamiltonians define

Table 10. *ND symplectic realization of the Casimir of each of the sub-coalgebras of (h_6, Δ) given in Table 9.*

Sub-coalgebra	Integrals of motion
(\mathcal{D}_+, Δ)	$\mathcal{C}_{\mathcal{D}_+}^{(N)} = \left(\sum_{i=1}^N \lambda_i p_i \right)^2 / \left(\sum_{j=1}^N p_j^2 \right)$
(\mathcal{D}_-, Δ)	$\mathcal{C}_{\mathcal{D}_-}^{(N)} = \left(\sum_{i=1}^N \lambda_i q_i \right)^2 / \left(\sum_{j=1}^N q_j^2 \right)$
(h_4, Δ)	$\mathcal{C}_{h_4}^{(N)} = \sum_{1 \leq i < j}^N (\lambda_j p_i - \lambda_i p_j)(\lambda_j q_i - \lambda_i q_j)$
$(\bar{\mathcal{G}}_+, \Delta)$	$\mathcal{C}_{\bar{\mathcal{G}}_+}^{(N)} = \sum_{1 \leq i < j}^N (\lambda_j p_i - \lambda_i p_j)^2$
$(\bar{\mathcal{G}}_-, \Delta)$	$\mathcal{C}_{\bar{\mathcal{G}}_-}^{(N)} = \sum_{1 \leq i < j}^N (\lambda_j q_i - \lambda_i q_j)^2$
$(\bar{\mathcal{E}}, \Delta)$	$\mathcal{C}_{\bar{\mathcal{E}}}^{(N)} = \sum_{1 \leq i < j}^N \left\{ \mu(\lambda_j p_i - \lambda_i p_j)^2 + \nu(\lambda_j q_i - \lambda_i q_j)^2 \right\}$
$(gl(2), \Delta)$	$\mathcal{C}_{gl(2)}^{(N)} = \sum_{1 \leq i < j}^N (q_j p_i - q_i p_j)^2$

completely integrable systems:

$$\begin{aligned}
\mathcal{H}_{\mathcal{D}_+} &= \mathcal{H}_{\mathcal{D}_+}(K, A_+, B_+) \\
\mathcal{H}_{\mathcal{D}_-} &= \mathcal{H}_{\mathcal{D}_-}(K, A_-, B_-) \\
\mathcal{H}_{h_4} &= \mathcal{H}_{h_4}(K, A_-, A_+, M) \\
\mathcal{H}_{\bar{\mathcal{G}}_+} &= \mathcal{H}_{\bar{\mathcal{G}}_+}(B_+, A_-, A_+, M) \\
\mathcal{H}_{\bar{\mathcal{G}}_-} &= \mathcal{H}_{\bar{\mathcal{G}}_-}(B_-, A_-, A_+, M) \\
\mathcal{H}_{\bar{\mathcal{E}}} &= \mathcal{H}_{\bar{\mathcal{E}}}(\mu B_+ + \nu B_-, A_-, A_+, M) \\
\mathcal{H}_{gl(2)} &= \mathcal{H}_{gl(2)}(K, B_-, B_+, M)
\end{aligned} \tag{4.31}$$

where the h_6 generators are taken through their *ND* symplectic realization (4.8) and the Hamiltonian functions are any smooth functions of the corresponding arguments. Moreover, all these systems are superintegrable, since the h_6 -coalgebra provides $(N-3)$

additional and functionally independent integrals of the motion given by the ‘right’ integrals $\mathcal{C}_{(m)}$.

The explicit form of the Casimir invariant $\mathcal{C}_g \equiv \mathcal{I}$ of each subalgebra g is given in Table 10 in terms of the N -particle symplectic realization (4.8). It is interesting to remark that the two-photon Casimir (4.4) can be expressed in terms of the Casimirs of the four subalgebras $h_3, h_4, \bar{\mathcal{G}}_+, \bar{\mathcal{G}}_-$ as

$$\mathcal{C}_{h_6} = \frac{1}{\mathcal{C}_{h_3}} \left(\mathcal{C}_{\bar{\mathcal{G}}_+} \mathcal{C}_{\bar{\mathcal{G}}_-} - \mathcal{C}_{h_4}^2 \right). \quad (4.32)$$

Let us finally comment that all of these subalgebras are also sub-coalgebras since the same primitive coproduct (4.5) holds for all of them as Lie–Poisson algebras. In fact, an alternative approach to the integrability of the systems (4.31) would be to consider directly the coalgebra construction for the subalgebra in which \mathcal{H}_g is defined, thus forgetting about the whole h_6 scheme. In that case, the integrals of motion would be given by the ‘left’ and ‘right’ m -th coproducts of the Casimir of the subalgebra \mathcal{C}_g , say $\mathcal{C}_g^{(m)}$ and $\mathcal{C}_{g,(m)}$ ($m = 2, \dots, N$), respectively.

In this way, by taking into account that $\mathcal{C}_g^{(2)}$ and $\mathcal{C}_{g,(2)}$ do not vanish in the subalgebra, we would obtain (in principle) a maximum number of $(2N - 3)$ independent constants of motion for \mathcal{H}_g , and each set of N functions $\{\mathcal{C}_g^{(m)}, \mathcal{H}_g\}$ or $\{\mathcal{C}_{g,(m)}, \mathcal{H}_g\}$ would be in involution. However, in the case of $\mathcal{D}_+, \mathcal{D}_-$ and the two centrally extended $(1 + 1)$ D Galilean algebras $\bar{\mathcal{G}}_+$ and $\bar{\mathcal{G}}_-$, the right integrals $\mathcal{C}_{g,(m)}$ turn out to be functionally dependent with respect to the left ones, and in these cases the superintegrability of the associated systems can be only derived by making use of the h_6 -coalgebra construction.

Some of these subalgebras have been considered previously from the coalgebra method, so that we refer to the various papers on the subject [14, 24, 27]. Nevertheless it is worthy to point out that in the present h_6 framework the integrals of motion coming from $gl(2)$ just provide the ones coming from the spherical symmetry (see Table 10). In terms of the $so(N)$ generators J_{ij} , (see Section 4.2) these read

$$\mathcal{C}_{gl(2)}^{(m)} = \sum_{1 \leq i < j}^m J_{ij}^2 \quad \mathcal{C}_{(m),gl(2)} = \sum_{N-m+1 \leq i < j}^m J_{ij}^2.$$

Therefore when a Hamiltonian of the type $\mathcal{H}_{gl(2)}$ is considered the spherical symmetry and its associated superintegrability is recovered as a particular case of the more general h_6 -coalgebra setting.

4.5 Generator integrability

Now, let us choose a given generator X of h_6 . If we look for all the generators X_j ($j = 1, \dots, l$) commuting with X and we look for all the subalgebras g_k ($k = 1, \dots, t$) containing X as generator, it becomes obvious that the Hamiltonian constructed through any function of the type

$$\mathcal{H}_X = \mathcal{H}_X(\mathcal{C}_{g_1}, \dots, \mathcal{C}_{g_t}, X, X_1, \dots, X_l), \quad (4.33)$$

where \mathcal{C}_{g_k} is the Casimir function of the subalgebra g_k , verifies that

$$\{\mathcal{H}_X, X\} = 0. \quad (4.34)$$

Moreover, the N -th particle symplectic realization of both X and \mathcal{H}_X will Poisson-commute with the two sets of integrals $\mathcal{C}^{(m)}$ (4.9) and $\mathcal{C}_{(m)}$ (4.16). Therefore, the N -th symplectic realization of \mathcal{H}_X is a completely integrable ND Hamiltonian system (in fact, superintegrable with a total number of $(2N - 4)$ integrals of the motion).

As we shall see in the sequel, by taking into account the information concerning the h_6 subalgebras that is contained in Tables 9 and 10, this result provides in a straightforward way a bunch of new ND integrable systems. We stress that particular integrable systems belonging to the three classes of the generically quasi-integrable Hamiltonians (4.20), (4.22) and (4.28) can be straightforwardly identified.

4.5.1 Hamiltonians in involution with K

Let us start by considering $X \equiv K$. It is immediate to check that the only generator that Poisson-commutes with K is M , the central one. On the other hand, K is contained in the subalgebras \mathcal{D}_+ , \mathcal{D}_- , h_4 and $gl(2)$. Therefore, the most general Hamiltonian with h_6 -coalgebra symmetry and in involution with K is

$$\mathcal{H}_K = \mathcal{H}_K(\mathcal{C}_{\mathcal{D}_+}, \mathcal{C}_{\mathcal{D}_-}, \mathcal{C}_{h_4}, \mathcal{C}_{gl(2)}, K, M) \quad (4.35)$$

where the smooth function $\mathcal{H}_K : \mathbb{R}^6 \rightarrow \mathbb{R}$. Now, if we take the ND symplectic realization of \mathcal{H}_K , we obtain an ND integrable Hamiltonian with $(N - 1)$ integrals of motion in involution given by (4.9) together with

$$\mathcal{I} \equiv K = \sum_{i=1}^N \left(q_i p_i - \frac{\lambda_i^2}{2} \right). \quad (4.36)$$

Next, in order to classify the type of Hamiltonian systems that can be constructed from \mathcal{H}_K , we have to realize that the symplectic realization of K and \mathcal{C}_{h_4} are linear in

the momenta, while $\mathcal{C}_{gl(2)}$ is quadratic in p . On the other hand, \mathcal{C}_{D_+} is rational in p (for arbitrary λ_i) and \mathcal{C}_{D_-} is a rational function in the canonical coordinates q . With these ingredients in mind and by considering the three families of systems given in Section 4.2.1, a family of *completely integrable geodesic flows* on ND curved spaces is obtained through a choice of the Hamiltonian \mathcal{H}_K leading to a quadratic homogeneous function in the momenta. Namely, the most general possibility of this type turns out to be

$$\mathcal{H}_K = \mathcal{C}_{gl(2)}\mathcal{F}(\mathcal{C}_{D_-}) + \left(K + \frac{M}{2}\right)^2 \mathcal{G}(\mathcal{C}_{D_-}) + \mathcal{C}_{h_4}^2 \mathcal{R}(\mathcal{C}_{D_-}) + \left(K + \frac{M}{2}\right) \mathcal{C}_{h_4} \mathcal{S}(\mathcal{C}_{D_-}) \quad (4.37)$$

where \mathcal{F} , \mathcal{G} , \mathcal{R} and \mathcal{S} are smooth functions. When this Hamiltonian is written in terms of canonical coordinates we get

$$\begin{aligned} \mathcal{H}_K^{(N)} = & \left(\sum_{1 \leq i < j}^N (q_j p_i - q_i p_j)^2 \right) \mathcal{F} \left(\frac{\left(\sum_{i=1}^N \lambda_i q_i \right)^2}{\sum_{i=1}^N q_i^2} \right) + \left(\sum_{i=1}^N q_i p_i \right)^2 \mathcal{G} \left(\frac{\left(\sum_{i=1}^N \lambda_i q_i \right)^2}{\sum_{i=1}^N q_i^2} \right) \\ & + \left(\sum_{1 \leq i < j}^N (\lambda_j p_i - \lambda_i p_j) (\lambda_j q_i - \lambda_i q_j) \right)^2 \mathcal{R} \left(\frac{\left(\sum_{i=1}^N \lambda_i q_i \right)^2}{\sum_{i=1}^N q_i^2} \right) \\ & + \left(\sum_{1 \leq i < j}^N (\lambda_j p_i - \lambda_i p_j) (\lambda_j q_i - \lambda_i q_j) \right) \left(\sum_{i=1}^N q_i p_i \right) \mathcal{S} \left(\frac{\left(\sum_{i=1}^N \lambda_i q_i \right)^2}{\sum_{i=1}^N q_i^2} \right) \quad (4.38) \end{aligned}$$

which is an ND integrable geodesic flow that depends on four smooth functions and N free parameters λ_i .

4.5.2 Hamiltonians in involution with A_+

When the generator A_+ is considered, we find that both B_+ and M commute with it. On the other hand, A_+ belongs to the subalgebras \mathcal{D}_+ , h_4 , $\overline{\mathcal{G}}_+$, $\overline{\mathcal{G}}_-$ and $\overline{\mathcal{E}}$. Both facts lead to the completely integrable Hamiltonian

$$\mathcal{H}_{A_+} = \mathcal{H}_{A_+} \left(\mathcal{C}_{\mathcal{D}_+}, \mathcal{C}_{h_4}, \mathcal{C}_{\overline{\mathcal{G}}_+}, \mathcal{C}_{\overline{\mathcal{G}}_-}, \mathcal{C}_{\overline{\mathcal{E}}}, A_+, B_+, M \right) \quad (4.39)$$

whatever the function \mathcal{H}_{A_+} be. In this case, the appearance of B_+ (the Euclidean kinetic energy term) allows for a wider set of possibilities. In particular, all the following

types of integrable Hamiltonians can be considered as specific cases of \mathcal{H}_{A_+} for which the remaining integral is $\mathcal{I} \equiv A_+ = \sum_{i=1}^N \lambda_i p_i$. This integral can be interpreted as a λ_i -generalization of the translational symmetry.

- *Natural Hamiltonians.* The only possibility is

$$\mathcal{H}_{A_+} = \frac{1}{2} B_+ + \mathcal{F}(\mathcal{C}_{\bar{\mathcal{G}}_-}) = \sum_{i=1}^N \frac{p_i^2}{2} + \mathcal{F}\left(\sum_{1 \leq i < j}^N (\lambda_j q_i - \lambda_i q_j)^2\right). \quad (4.40)$$

Note that this Hamiltonian is *not* defined within the $\bar{\mathcal{G}}_-$ subalgebra. In the $N = 2$ case, the Calogero–Moser systems [53, 127, 177] arise as particular choices for \mathcal{F} .

- *Electromagnetic Hamiltonians.* We can add linear terms in the momenta to the previous Hamiltonian leading to

$$\begin{aligned} \mathcal{H}_{A_+} &= \frac{1}{2} B_+ + c_{h_4} \mathcal{G}(\mathcal{C}_{\bar{\mathcal{G}}_-}) + A_+ \mathcal{R}(\mathcal{C}_{\bar{\mathcal{G}}_-}) + \mathcal{F}(\mathcal{C}_{\bar{\mathcal{G}}_-}) \quad (4.41) \\ \mathcal{H}_{A_+}^{(N)} &= \sum_{i=1}^N \frac{p_i^2}{2} + \left(\sum_{1 \leq i < j}^N (\lambda_j p_i - \lambda_i p_j)(\lambda_j q_i - \lambda_i q_j)\right) \mathcal{G}\left(\sum_{1 \leq i < j}^N (\lambda_j q_i - \lambda_i q_j)^2\right) \\ &\quad + \left(\sum_{i=1}^N \lambda_i p_i\right) \mathcal{R}\left(\sum_{1 \leq i < j}^N (\lambda_j q_i - \lambda_i q_j)^2\right) + \mathcal{F}\left(\sum_{1 \leq i < j}^N (\lambda_j q_i - \lambda_i q_j)^2\right). \quad (4.42) \end{aligned}$$

- *Geodesic flows.* The most general expression coming from \mathcal{H}_{A_+} and with homogeneous quadratic dependence in the momenta is given by

$$\begin{aligned}
\mathcal{H}_{A_+} &= \mathcal{C}_{h_4}^2 \mathcal{F}(\mathcal{C}_{\bar{g}_-}) + \mathcal{C}_{\bar{g}_+} \mathcal{G}(\mathcal{C}_{\bar{g}_-}) + B_+ \mathcal{R}(\mathcal{C}_{\bar{g}_-}) + A_+^2 \mathcal{S}(\mathcal{C}_{\bar{g}_-}) + A_+ \mathcal{C}_{h_4} \mathcal{T}(\mathcal{C}_{\bar{g}_-}) \\
\mathcal{H}_{A_+}^{(N)} &= \left(\sum_{1 \leq i < j}^N (\lambda_j p_i - \lambda_i p_j) (\lambda_j q_i - \lambda_i q_j) \right)^2 \mathcal{F} \left(\sum_{1 \leq i < j}^N (\lambda_j q_i - \lambda_i q_j)^2 \right) \\
&\quad + \left(\sum_{1 \leq i < j}^N (\lambda_j p_i - \lambda_i p_j)^2 \right) \mathcal{G} \left(\sum_{1 \leq i < j}^N (\lambda_j q_i - \lambda_i q_j)^2 \right) \\
&\quad + \left(\sum_{i=1}^N p_i^2 \right) \mathcal{R} \left(\sum_{1 \leq i < j}^N (\lambda_j q_i - \lambda_i q_j)^2 \right) + \left(\sum_{i=1}^N \lambda_i p_i \right)^2 \mathcal{S} \left(\sum_{1 \leq i < j}^N (\lambda_j q_i - \lambda_i q_j)^2 \right) \\
&\quad + \left(\sum_{i=1}^N \lambda_i p_i \right) \left(\sum_{1 \leq i < j}^N (\lambda_j p_i - \lambda_i p_j) (\lambda_j q_i - \lambda_i q_j) \right) \mathcal{T} \left(\sum_{1 \leq i < j}^N (\lambda_j q_i - \lambda_i q_j)^2 \right). \quad (4.43)
\end{aligned}$$

4.5.3 Hamiltonians in involution with either A_- , B_- or B_+

To end with we present jointly these tree types of Hamiltonians, since all of them provide new examples of ND geodesic flows.

- *A_- -Hamiltonians.* If we consider that $\mathcal{I} \equiv A_- = \sum_{i=1}^N \lambda_i q_i$, is straightforward to prove that the most general integrable A_- -Hamiltonian reads

$$\mathcal{H}_{A_-} = \mathcal{H}_{A_-}(\mathcal{C}_{\mathcal{D}_-}, \mathcal{C}_{h_4}, \mathcal{C}_{\bar{g}_+}, \mathcal{C}_{\bar{g}_-}, \mathcal{C}_{\bar{g}}, A_-, B_-, M). \quad (4.44)$$

In this case, geodesic flow Hamiltonians are available through the particular choice

$$\mathcal{H}_{A_-} = \mathcal{C}_{h_4}^2 \mathcal{F}(\mathcal{C}_{\mathcal{D}_-}, \mathcal{C}_{\bar{g}_-}, A_-, B_-) + \mathcal{C}_{\bar{g}_+} \mathcal{G}(\mathcal{C}_{\mathcal{D}_-}, \mathcal{C}_{\bar{g}_-}, A_-, B_-) \quad (4.45)$$

and its N -particle symplectic realization can be immediately obtained.

- *B_- -Hamiltonians.* A similar situation is encountered when $\mathcal{I} \equiv B_- = \sum_{i=1}^N q_i^2$ is considered. In this case we have that

$$\mathcal{H}_{B_-} = \mathcal{H}_{B_-}(\mathcal{C}_{\mathcal{D}_-}, \mathcal{C}_{\bar{g}_-}, \mathcal{C}_{gl(2)}, B_-, A_-, M) \quad (4.46)$$

and since $\mathcal{C}_{gl(2)}$ is the only term quadratic in the momenta, we are led to the integrable geodesic flow given by

$$\mathcal{H}_{B_-} = \mathcal{C}_{gl(2)} \mathcal{F}(\mathcal{C}_{\mathcal{D}_-}, \mathcal{C}_{\bar{g}_-}, B_-, A_-). \quad (4.47)$$

• *B₊-Hamiltonians.* Finally, the last possibility is given by $\mathcal{I} \equiv B_+ = \sum_{i=1}^N p_i^2$. Now, the most general integrable Hamiltonian is given by

$$\mathcal{H}_{B_+} = \mathcal{H}_{B_+} \left(\mathcal{C}_{\mathcal{D}_+}, \mathcal{C}_{\bar{\mathcal{G}}_+}, \mathcal{C}_{gl(2)}, B_+, A_+, M \right). \quad (4.48)$$

All the variables for \mathcal{H}_{B_+} (except M) depend on the momenta. Therefore, in this case the only integrable geodesic flow can be obtained through

$$\mathcal{H}_{B_+} = \alpha B_+ + \beta A_+^2 + \gamma \mathcal{C}_{gl(2)} + \delta \mathcal{C}_{\bar{\mathcal{G}}_+} \quad (4.49)$$

where α, β, γ and δ are constants.

4.6 Direct search of integrable Hamiltonians

As we mentioned in Section 4.3, a third possibility to show the complete integrability for a given Hamiltonian \mathcal{H} (4.7) with h_6 -coalgebra symmetry is the direct search for an additional integral \mathcal{I} , that can be assumed to be an unknown function of the h_6 generators. An example for this type of construction is given by the following examples of natural, electromagnetic and geodesic flow Hamiltonians, thus showing the versatility of the coalgebra construction.

4.6.1 Natural systems: the Garnier Hamiltonian

If we consider the following Hamiltonian

$$\mathcal{H} = \frac{1}{2} B_+ + \delta_1 B_- + \delta_2 A_-^2 + \delta_3 B_-^2 \quad (4.50)$$

where $\delta_1, \delta_2, \delta_3$ are arbitrary constants, a direct computation by making use of the Poisson algebra (4.2) shows that

$$\mathcal{I} = \delta_3 \mathcal{C}_{gl(2)} - \delta_2 (A_+^2 + 2A_-^2 (M \delta_2 + \delta_1 + \delta_3 B_-)) \quad (4.51)$$

is in involution with \mathcal{H} . Here is worth mentioning that if $\delta_2 = 0$, we are falling into the $gl(2)$ -subalgebra, and also the fact that the only allowed perturbation to the radial symmetry is coming from the term A_-^2 . In this sense, the h_6 coalgebra can be again interpreted as the algebraic tool to express the spherical symmetry breaking of a given model.

Let us consider the 2D and 3D symplectic realizations for (4.50). The general expression of our system in 2D reads

$$\mathcal{H}^{(2)} = \frac{1}{2}(p_1^2 + p_2^2) + \delta_1(q_1^2 + q_2^2) + \delta_2(\lambda_1^2 q_1^2 + \lambda_2^2 q_2^2 + 2\lambda_1 \lambda_2 q_1 q_2) + \delta_3(q_1^4 + q_2^4 + 2q_1^2 q_2^2). \quad (4.52)$$

By taking $\lambda_1 = 1$ and $\lambda_2 = 0$ we get

$$\mathcal{H}^{(2)} = \frac{1}{2}(p_1^2 + p_2^2) + Aq_1^2 + Bq_2^2 + \delta_3(q_1^2 + q_2^2)^2 \quad (4.53)$$

where

$$A = \delta_1 + \delta_2 \quad B = \delta_1 \quad (4.54)$$

which is the 2D Garnier system. By substituting the previous values of the λ parameters, the invariant becomes

$$\mathcal{I}^{(2)} = \delta_3(p_2 q_1 - p_1 q_2)^2 - \delta_2(p_1^2 + 2q_1^2(\delta_1 + \delta_2 + \delta_3(q_1^2 + q_2^2))). \quad (4.55)$$

The 3D realization of (4.50) gives

$$\begin{aligned} \mathcal{H}^{(3)} &= \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + \delta_1(q_1^2 + q_2^2 + q_3^2) + \delta_2(\lambda_1 q_1 + \lambda_2 q_2 + \lambda_3 q_3)^2 \\ &\quad + \delta_3(q_1^2 + q_2^2 + q_3^2)^2 \end{aligned} \quad (4.56)$$

and if we choose $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = 0$ we get

$$\mathcal{H}^{(3)} = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + (\delta_1 + \delta_2)q_1^2 + \delta_1 q_2^2 + \delta_1 q_3^2 + \delta_3(q_1^2 + q_2^2 + q_3^2)^2. \quad (4.57)$$

In this case the invariants have the following expressions:

$$\begin{aligned} \mathcal{C}_{h_6}^{(3)} &= (q_2 p_3 - q_3 p_2)^2 \\ \mathcal{I}^{(3)} &= \delta_3(p_2^2 q_1^2 + p_3^2 q_1^2 - 2p_1 p_2 q_1 q_2 + p_1^2 q_2^2 + p_3^2 q_2^2 - 2p_1 p_3 q_1 q_3 - 2p_2 p_3 q_2 q_3 + p_1^2 q_3^2 \\ &\quad + p_2^2 q_3^2 - 2\delta_2(q_1^4 + q_1^2 q_2^2 + q_1^2 q_3^2)) - \delta_2(p_1^2 + 2\delta_1 q_1^2 + 2\delta_2 q_1^2). \end{aligned} \quad (4.58)$$

Note that in the 3D case the oscillator term is non isotropic only in the q_1 coordinate, and this feature is preserved in any dimension. Again, if δ_2 vanishes, the radial symmetry of the system is recovered and, from the algebraic viewpoint, the Hamiltonian is defined on $gl(2)$.

4.6.2 An electromagnetic Hamiltonian

Now we consider the following Hamiltonian, which represents a isotropic oscillator plus linear terms in the momenta encoded within K and A_+ :

$$\mathcal{H} = \frac{B_+}{2} + \delta_1 B_- + \delta_5 K A_- + \delta_6 A_+. \quad (4.59)$$

A long but straightforward computation shows that this Hamiltonian commutes with the following polynomial object defined on h_6 :

$$\begin{aligned} \mathcal{I} = & -6\delta_1(A_+^2 + 2\delta_1 A_-^2) + 2\delta_5 A_+(B_+ M - A_+^2) \\ & + 4\delta_1 \delta_5 (-3A_+ A_-^2 + K A_- M - A_+ B_- M + 2A_- M^2) \\ & + \delta_5^2 (-3A_+^2 A_-^2 + 3A_-^2 B_+ M - A_+^2 B_- M + B_+ B_- M^2 + A_+ A_- M^2 - K M^3) \\ & - 4\delta_6 (3\delta_1 A_+ M + \delta_5 A_+^2 M + 2\delta_1 \delta_5 B_- M^2 + 2\delta_5^2 K A_- M + 2\delta_5 \delta_6 A_+ M^2). \end{aligned} \quad (4.60)$$

Note that this integral is quadratic in the momenta.

In 3D the Hamiltonian (4.59) can be interpreted as the one describing an electromagnetic field, provided we write it in the form

$$\mathcal{H} = \frac{1}{2} (\vec{p} - e\vec{A})^2 + e\psi \quad (4.61)$$

where e is the electric charge, \vec{A} is the vector potential, and ψ the scalar one. Now, by considering the 3D symplectic realization of h_6 with arbitrary λ parameters we get that (4.59) corresponds to the motion of a charged particle under the the following vector and scalar potentials:

$$A_i = -\frac{\delta_5}{e} q_i (\lambda_1 q_1 + \lambda_2 q_2 + \lambda_3 q_3) - \frac{\delta_6}{e} \lambda_i \quad i = 1, 2, 3 \quad (4.62)$$

$$\psi = -\frac{\delta_5}{2e} M - \frac{e}{2} \sum_{i=1}^3 A_i^2 + \frac{\delta_1}{e} \sum_{i=1}^3 q_i^2. \quad (4.63)$$

From them, the magnetic $(\Omega_1, \Omega_2, \Omega_3)$ and electric field components (E_1, E_2, E_3) would be

$$\begin{aligned} \Omega_1 &= \frac{\delta_5}{e} (\lambda_3 q_2 - \lambda_2 q_3) \\ \Omega_2 &= \frac{\delta_5}{e} (\lambda_1 q_3 - \lambda_3 q_1) \\ \Omega_3 &= \frac{\delta_5}{e} (\lambda_2 q_1 - \lambda_1 q_2) \end{aligned} \quad (4.64)$$

$$E_i = -\partial_{q_i}\psi \quad i = 1, 2, 3. \quad (4.65)$$

Therefore, we can say that this static electromagnetic field generates integrable motions in 3D. It is worth stressing that by switching off the harmonic oscillator term ($\delta_1 \rightarrow 0$) the integral of the motion is significantly simplified, although the magnetic field components Ω_i do not change.

4.6.3 Geodesic flows

A third type of interesting examples is obtained by considering geodesic flow Hamiltonians. We will present here four examples for which the additional integral that guarantees the complete integrability of the associated ND geodesic flow can be explicitly found as a function of the h_6 coalgebra generators. These four examples do not exhaust the list of completely integrable geodesic flows that can be obtained in this approach through more complicated choices of the Hamiltonian function.

Example I

Let us consider the Hamiltonian

$$\mathcal{H} = B_+ (\alpha_1 A_- + \alpha_2 B_- + \alpha_3) \quad (4.66)$$

where $\alpha_1, \alpha_2, \alpha_3$ are arbitrary constants. Despite its simplicity, this quadratic h_6 Hamiltonian neither lives in any h_6 subalgebra nor can be included within the cases studied in the previous Section. However, the following additional (and functionally independent) constant of motion can be found by direct computation

$$\mathcal{I} = 4\alpha_1\alpha_2 A_+ \left(K + \frac{M}{2} \right) + 4\alpha_2\alpha_3 B_+ + 4\alpha_2^2 K(K + M) - \alpha_1^2 \mathcal{C}_{\overline{\mathcal{G}}_+}. \quad (4.67)$$

This integral provides the complete integrability of the system (4.66) for any dimension N . Note that in the limit $\alpha_2 \rightarrow 0$ the Hamiltonian (4.66) belongs to the subalgebra $\overline{\mathcal{G}}_+$, and in that case \mathcal{I} is just the Casimir function for such a subalgebra, as it should be.

Although we will not study in detail here the geometric content of the spaces defined by the integrable geodesic flows (4.66), we can say that they are spaces with non-constant curvature. This can be easily proven by taking the 2D symplectic realization of (4.66) and by computing the 2D Gaussian curvature K of the associated metrics. A long but straightforward computation shows that

$$\mathcal{K} = \frac{4\alpha_2\alpha_3 - \alpha_1^2(\lambda_1^2 + \lambda_2^2)}{2 [\alpha_2(q_1^2 + q_2^2) + \alpha_3 + \alpha_1(\lambda_1 q_1 + \lambda_2 q_2)]}. \quad (4.68)$$

Note that either if $\alpha_1 = \alpha_2 = 0$ or $\alpha_1 = \alpha_3 = 0$ the space is flat, but 2D spaces with constant (but non-vanishing) curvature are not included in (4.66).

Example II

Another interesting example is provided by the following Hamiltonian

$$\mathcal{H} = B_+(\alpha_1 A_-^2 + \alpha_2 B_- + \alpha_3). \quad (4.69)$$

A direct computation shows that the extra invariant is now given by

$$\mathcal{I} = \alpha_1 \alpha_2 K M (K + M) + \alpha_2^2 \left(K + \frac{M}{2} \right)^2 + \alpha_1 \alpha_3 \mathcal{C}_{\bar{g}_+} + \alpha_2 \alpha_3 B_+. \quad (4.70)$$

In 2D, the Gaussian curvature of the spaces defined through the flow (4.69) is

$$\mathcal{K} = - \frac{\alpha_1 \sum_{i=1}^2 \lambda_i^2 \left(-\alpha_3 + \alpha_1 \left\{ \sum_{i=1}^2 \lambda_i q_i \right\}^2 \right) + \alpha_2 \left\{ -2\alpha_3 + \alpha_1 \left[\left(\sum_{i=1}^2 \lambda_i q_i \right)^2 - (\lambda_2 q_1 - \lambda_1 q_2)^2 \right] \right\}}{\left[\alpha_2 \sum_{i=1}^2 q_i^2 + \alpha_3 + \alpha_1 \left(\sum_{i=1}^2 \lambda_i q_i \right)^2 \right]} \quad (4.71)$$

and again we have obtained a family of integrable spaces with, in general, non-constant curvature. We also remark that the $\alpha_1 \rightarrow 0$ limit of Examples I and II leads to the same system.

Example III

A third example would be

$$\mathcal{H} = B_+(\alpha_1 A_- B_- + \alpha_2) \quad (4.72)$$

for which the extra invariant is

$$\begin{aligned} \mathcal{I} = & \alpha_1 (-4K^2 A_-^2 + 4K A_+ A_- B_- - A_+^2 B_-^2 - 4K A_-^2 M + 2A_- A_+ B_- M + B_+ B_-^2 M) \\ & - \alpha_1 A_-^2 M^2 + \alpha_2 (8K A_+ - 8A_- B_+ + 4A_+ M) \end{aligned} \quad (4.73)$$

and the 2D Gaussian curvature reads

$$\mathcal{K} = - \frac{\alpha_1 \left[-8\alpha_2 \sum_{i=1}^2 \lambda_i q_i + \alpha_1 \left\{ \sum_{i=1}^2 q_i^2 \right\}^2 \left(\sum_{i=1}^2 \lambda_i^2 \right) \right]}{2 \left(\alpha_2 + \alpha_1 \left\{ \sum_{i=1}^2 q_i^2 \right\} \left(\sum_{i=1}^2 \lambda_i^2 \right) \right)} \quad (4.74)$$

where again the flat limit is given by $\alpha_1 \rightarrow 0$.

Example IV

And finally we consider the geodesic flow defined by

$$\mathcal{H} = B_+(\alpha_1 A_-^2 + \alpha_2 B_-^2 + \alpha_3) \quad (4.75)$$

for which the extra invariant becomes much more complicated, namely

$$\begin{aligned} \mathcal{I} = & \alpha_1^2 (-2KA_+A_-M + A_+^2B_-M + K^2M^2 - A_+A_-M^2 - B_+B_-M^2 + KM^3) \\ & + \alpha_1\alpha_2 (-4K^2A_-^2 + 4KA_+A_-B_- - A_+^2B_-^2 - 4KA_-^2M - A_-^2M^2) \\ & + 2\alpha_1\alpha_2A_+A_-B_-M - \alpha_1\alpha_3A_+^2 - 4\alpha_2\alpha_3(K^2 - B_+B_- + KM). \end{aligned} \quad (4.76)$$

In this case, the 2D gaussian curvature is

$$\mathcal{K} = \frac{-\alpha_1 \sum_{i=1}^2 \lambda_i^2 \left[-\alpha_3 + \alpha_1 \left(\sum_{i=1}^2 \lambda_i q_i \right)^2 \right] + \alpha_2 \sum_{i=1}^2 q_i^2 \left\{ 8\alpha_3 + \alpha_1 \left(\sum_{i=1}^2 q_i^2 \right) \left(\sum_{i=1}^2 \lambda_i^2 \right) \right\}}{\alpha_2 \left(\sum_{i=1}^2 q_i^2 \right)^2 + \alpha_3 + \alpha_1 \left(\sum_{i=1}^2 \lambda_i q_i \right)^2}. \quad (4.77)$$

Obviously, the $\alpha_2 \rightarrow 0$ limit of Examples II and IV leads to the same system.

We stress that this direct search for the remaining integral can be indeed very useful, since it can be quite easily computerized. In fact, the integral \mathcal{I} can be searched among h_6 functions with cubic or higher dependence on the momenta (note that all the integrals that we have presented so far are, at most, quadratic in the momenta).

Moreover, nothing prevents that, although for a certain \mathcal{H} defined on h_6 the additional integral does exist, such \mathcal{I} cannot be written as a function (4.29) of the h_6 generators (*i.e.*, \mathcal{I} would not be coalgebra-invariant). This implies that in this case all the previous methods are not applicable and the explicit form for \mathcal{I} has to be found for each dimension N , which constitutes a much more cumbersome task. Nevertheless, we would like to emphasize that any h_6 -coalgebra invariant Hamiltonian \mathcal{H} of the form (4.7) is only one integral away from being completely integrable, and the search of the complete list of integrable choices for \mathcal{H} is certainly interesting.

Chapter 5

Integrable perturbations of the ND oscillator

*“Rather than trust our own wisdom,
one should ultimately rely upon algebraic calculations.”*

L. Euler

The harmonic oscillator is indeed the paradigm of integrable Hamiltonian dynamics on the N -dimensional (ND) Euclidean space, and its superintegrability properties are very well established through group-theoretic methods (see, for instance, [4, 87, 147]). Therefore, the construction of non-trivial integrable perturbations/deformations of the ND harmonic oscillator (understood in a broad sense as the addition of any ND potential term -not necessarily small- that preserves the complete integrability of the system) are interesting from both physical and mathematical viewpoints.

In this respect, although certain 2D and 3D perturbations have been thoroughly studied (see, for instance [47, 78, 83, 95, 96, 100, 101, 105, 106, 121, 122, 144, 154, 176, 179] and references therein) the number of ND results is more limited. Among them we can quote the search for coupled polynomial perturbations [48, 72, 117, 97, 155], Garnier systems [90], the Smorodinski-Winternitz Hamiltonian [81, 89], Calogero-Moser systems [53, 127] and coupled oscillators coming from symmetric spaces [85]. In a more generic sense, integrable oscillators on ND curved spaces can also be considered as deformations of the ND Euclidean one in which the kinetic energy term is also perturbed through functions of the canonical coordinates containing a curvature parameter (see [24, 26, 29] and references therein).

In this Chapter we use the h_6 coalgebra machinery introduced in the previous

Chapter, in order to demonstrate the complete integrability of the following N -dimensional natural Hamiltonian system

$$\mathcal{H}^{(N)} = \frac{1}{2} \sum_{i=1}^N p_i^2 + \delta_1 \sum_{i=1}^N q_i^2 + \mathcal{F} \left(\sum_{1 \leq i < j}^N (\lambda_j q_i - \lambda_i q_j)^2 \right) + \mathcal{G} \left(\sum_{i=1}^N \lambda_i q_i \right) \quad (5.1)$$

where \mathcal{F} and \mathcal{G} are arbitrary functions and δ_1 and λ_i ($i = 1, \dots, N$) are $(N + 1)$ free parameters. Obviously, the first term in the potential is just the ND isotropic harmonic oscillator, and the second one ($\mathcal{F} + \mathcal{G}$) can be interpreted as two independent families of integrable deformations/perturbations whose superposition preserves the complete integrability of the whole system. Moreover, since the expressions for the integrals of the motion are analytic in δ_1 , the oscillator term can also be removed without altering the integrability properties of the system.

In Section 5.1 we shall prove the complete integrability of \mathcal{H} by giving explicitly a set of $(N - 1)$ functionally independent integrals of the motion for \mathcal{H} , which are obtained by making use of the underlying h_6 -Poisson coalgebra symmetry of the Hamiltonian. In fact, such symmetry implies that this Hamiltonian has two different sets of $(N - 1)$ independent integrals of the motion, and among both sets there will be a total number of $(2N - 4)$ independent constants of the motion in involution with \mathcal{H} , which implies that (5.1) is a superintegrable system (albeit not maximally superintegrable). We shall also prove that no oscillator anisotropies can be obtained by including quadratic terms in the canonical coordinates in either \mathcal{F} or \mathcal{G} .

As a consequence, by considering appropriate specializations of the dimension N , the arbitrary functions \mathcal{F} and \mathcal{G} and the $(N + 1)$ free parameters, we can obtain many different integrable systems. In particular, in Section 5.2 we shall present a number of integrable polynomial perturbations that can be easily obtained from \mathcal{H} .

Moreover, another family of integrable perturbations of the ND oscillator is introduced in Section 5.3, namely

$$\mathcal{H}_r^{(N)} = \frac{1}{2} \sum_{i=1}^N p_i^2 + \delta_1 \sum_{i=1}^N q_i^2 + \mathcal{V}_{-2} \left(\sum_{i=1}^N \lambda_i q_i, \sum_{i=1}^N q_i^2 \right) + \mathcal{J} \left(\sum_{i=1}^N q_i^2 \right) \quad (5.2)$$

where \mathcal{V}_{-2} is any homogeneous function of degree -2 in the canonical coordinates such that it can be written in terms of the variables $\sum_{i=1}^N \lambda_i q_i$ and $\sum_{i=1}^N q_i^2$. Moreover, \mathcal{J} is an arbitrary radial perturbation and δ_1 and λ_i ($i = 1, \dots, N$) are again $(N + 1)$ free parameters. The (super)integrability of this family of ND Hamiltonian systems will be proven by restoring to their h_6 -Poisson coalgebra symmetry.

We would like to mention that in the 3D case Hamiltonians (5.1) and (5.2) are subclasses of the systems considered in [179] through a 3D realization of a so-called ‘extended’ $sl(2, \mathbb{R})$ Poisson-Lie algebra (which is just isomorphic to h_6). However, the ND integrable generalization here presented is only possible by endowing h_6 with the appropriate underlying coalgebra structure and by imposing the full Hamiltonian to be h_6 -coalgebra invariant.

5.1 A new ND integrable Hamiltonian

In order to prove the complete integrability of the ND Hamiltonian (5.1) we consider the functions

$$\begin{aligned} A_+ &= \sum_{i=1}^N \lambda_i p_i & A_- &= \sum_{i=1}^N \lambda_i q_i & K &= \sum_{i=1}^N \left(q_i p_i - \frac{\lambda_i^2}{2} \right) \\ B_+ &= \sum_{i=1}^N p_i^2 & B_- &= \sum_{i=1}^N q_i^2 & M &= \sum_{i=1}^N \lambda_i^2 \end{aligned} \quad (5.3)$$

where λ_i are arbitrary parameters and N fixes the number of degrees of freedom. These functions provide a N -dimensional symplectic realization of the h_6 -Poisson coalgebra with Lie-Poisson brackets given by [16, 28, 92]

$$\begin{aligned} \{K, A_+\} &= A_+ & \{K, A_-\} &= -A_- & \{A_-, A_+\} &= M \\ \{K, B_+\} &= 2B_+ & \{K, B_-\} &= -2B_- & \{B_-, B_+\} &= 4K + 2M \\ \{A_+, B_-\} &= -2A_- & \{A_+, B_+\} &= 0 & \{M, \cdot\} &= 0 \\ \{A_-, B_+\} &= 2A_+ & \{A_-, B_-\} &= 0. \end{aligned} \quad (5.4)$$

The Casimir functions for this algebra [145] are given by the central generator M and by

$$\mathcal{C} = (MB_+ - A_+^2)(MB_- - A_-^2) - (MK - A_-A_+ + M^2/2)^2. \quad (5.5)$$

Thus, we can say that the Hamiltonian \mathcal{H} (5.1) is h_6 -coalgebra invariant, since it can be written as the following function of the h_6 -coalgebra generators:

$$\mathcal{H} = \frac{1}{2}B_+ + \delta_1 B_- + \mathcal{F}(\mathcal{C}_{\bar{g}_-}) + \mathcal{G}(A_-) \quad (5.6)$$

provided they are realized in the ND symplectic realization (5.3) and where

$$\mathcal{C}_{\bar{g}_-} = MB_- - A_-^2 = \sum_{i < j}^N (\lambda_j q_i - \lambda_i q_j)^2 \quad (5.7)$$

is just the Casimir for a (1+1)-Galilei subalgebra of h_6 spanned by $\{B_-, A_-, A_+, M\}$. This invariance implies that, the following $(N - 2)$ functionally independent functions

$$\mathcal{C}_{h_6}^{(m)} = \sum_{i < j < k}^m (\lambda_i(p_j q_k - p_k q_j) + \lambda_j(p_k q_i - p_i q_k) + \lambda_k(p_i q_j - p_j q_i))^2 \quad m = 3, 4, \dots, N \quad (5.8)$$

Poisson-commute with \mathcal{H} and among themselves. In fact, a given integral $\mathcal{C}_{h_6}^{(m)}$ is just the Casimir function (5.5) divided by M and expressed in terms of the m -particle realization of the h_6 algebra.

Thus there is only one integral left in order to prove the complete integrability of \mathcal{H} . Such an integral \mathcal{I} can be also found in an h_6 -invariant form through direct computation and by making use of the Poisson brackets (5.4), and reads:

$$\mathcal{I} = A_+^2 + 2 \delta_1 A_-^2 + 2 \mathcal{G}(A_-) M. \quad (5.9)$$

In terms of the ND symplectic realization, this last integral is explicitly written as

$$\mathcal{I}^{(N)} = \left(\sum_{i=1}^N \lambda_i p_i \right)^2 + 2 \delta_1 \left(\sum_{i=1}^N \lambda_i q_i \right)^2 + 2 \left(\sum_{i=1}^N \lambda_i^2 \right) \mathcal{G} \left(\sum_{i=1}^N \lambda_i q_i \right) \quad (5.10)$$

and note that, \mathcal{I} does not depend on the choice of the function \mathcal{F} .

We stress that the h_6 -coalgebra symmetry ensures that \mathcal{I} Poisson-commutes with all the $\mathcal{C}_{h_6}^{(m)}$ integrals, and the fact that $\{\mathcal{H}, \mathcal{I}\} = 0$ is easily proven by direct computation. The functional independence of \mathcal{I} with respect to the remaining integrals is proven by taking $\delta_1 = 0$ and $\mathcal{G} = 0$. In this case $\mathcal{I} = A_+^2$ depends only on the momenta, and is indeed functionally independent of $\mathcal{C}_{h_6}^{(m)}$ (that does not depend on δ_1 , \mathcal{F} and \mathcal{G}) and of the Hamiltonian. This functional independence will be maintained under the ‘deformations’ $\delta_1 \neq 0$ and $\mathcal{G} \neq 0$.

Moreover, the h_6 -coalgebra symmetry implies that the Hamiltonian \mathcal{H} is not only integrable but superintegrable, since the set of integrals (5.8) can be replaced by the alternative one

$$\mathcal{C}_{h_6, (m)} = \sum_{N-m+1 \leq i < j < k}^N (\lambda_i(p_j q_k - p_k q_j) + \lambda_j(p_k q_i - p_i q_k) + \lambda_k(p_i q_j - p_j q_i))^2 \quad (5.11)$$

where $m = 3, \dots, N$. Note that $\mathcal{C}_{h_6, (N)} = \mathcal{C}_{h_6}^{(N)}$, so \mathcal{H} has a total number of $(2N - 4)$ functions in involution (including \mathcal{I}). We recall that the ND isotropic oscillator (without any perturbations) has a total number of $(2N - 2)$ independent integrals, that makes it

a maximally superintegrable system. From this perspective, the perturbation given by \mathcal{F} and \mathcal{G} strongly preserves the integrability properties of the oscillator Hamiltonian, since only two integrals of the motion are lost under perturbation, which is certainly a consequence of imposing the h_6 invariance of the latter.

Example

If we consider $\mathcal{F}(x) = \alpha/x$ we get

$$\mathcal{H}^{(N)} = \frac{1}{2} \sum_{i=1}^N p_i^2 + \delta_1 \sum_{i=1}^N q_i^2 + \frac{\alpha}{\sum_{1 \leq i < j}^N (\lambda_j q_i - \lambda_i q_j)^2} + \mathcal{G} \left(\sum_{i=1}^N \lambda_i q_i \right) \quad (5.12)$$

which in the case $N = 2$ is just the rational Calogero-Moser (CM) system plus an arbitrary function \mathcal{G} :

$$\mathcal{H}^{(2)} = \frac{1}{2} (p_1^2 + p_2^2) + \delta_1 (q_1^2 + q_2^2) + \frac{\alpha}{(\lambda_2 q_1 - \lambda_1 q_2)^2} + \mathcal{G}(\lambda_1 q_1 + \lambda_2 q_2).$$

The integral of the motion in this case is (5.10) with $N = 2$, since the integrals (5.8) and (5.11) only exist from $N = 3$. Note that \mathcal{I} does not depend on the CM potential given by \mathcal{F} . We stress that the ND generalization (5.12) obtained from (5.1) is not the CM Hamiltonian, but it is indeed completely integrable in any dimension.

5.1.1 The isotropy problem

Before entering into more detailed examples, it is convenient to study whether anisotropic oscillators can be ‘hidden’ in the general result (5.1) through quadratic contributions coming from the perturbation term

$$\phi = \mathcal{F}(x) + \mathcal{G}(y). \quad (5.13)$$

Since the only terms that can give rise to a homogeneous quadratic contribution in the canonical coordinates are $\alpha_1 x$ and $\beta_1 y^2$, we write such perturbations in the form

$$\mathcal{F}(x) = \alpha_1 x + f(x) \quad \mathcal{G}(y) = \beta_1 y^2 + g(y) \quad (5.14)$$

where f and g are generic functions with no linear and quadratic term, respectively. In this way we rewrite the Hamiltonian as

$$\mathcal{H}^{(N)} = \frac{1}{2} \sum_{i=1}^N p_i^2 + \delta_1 \sum_{i=1}^N q_i^2 + \alpha_1 \left(\sum_{i < j}^N (\lambda_j q_i - \lambda_i q_j)^2 \right) + \beta_1 \left(\sum_{i=1}^N \lambda_i q_i \right)^2$$

$$+f \left(\sum_{i<j}^N (\lambda_j q_i - \lambda_i q_j)^2 \right) + g \left(\sum_{i=1}^N \lambda_i q_i \right). \quad (5.15)$$

Now, if we impose that all the coefficients of the crossed quadratic terms of the type $q_i q_j$ vanish, we get only two possible solutions:

- A) $\alpha_1 = \beta_1$. In this case a straightforward computation shows that the Hamiltonian (5.15) is again a ND isotropic oscillator perturbation given by

$$\mathcal{H}^{(N)} = \frac{1}{2} \sum_{i=1}^N p_i^2 + \omega^2 \sum_{i=1}^N q_i^2 + f \left(\sum_{i<j}^N (\lambda_j q_i - \lambda_i q_j)^2 \right) + g \left(\sum_{i=1}^N \lambda_i q_i \right) \quad (5.16)$$

with $\omega^2 = \delta_1 + \alpha_1 \sum_{i=1}^N \lambda_i^2$.

- B) For a fixed i , another solution is given by $\lambda_i = \lambda$ and $\lambda_j = 0$ for any $j \neq i$. With no loss of generality, we thus consider the case $\lambda_1 = \lambda$, and the Hamiltonian (5.15) reads

$$\mathcal{H}^{(N)} = \frac{1}{2} p_1^2 + \omega_1^2 q_1^2 + g(\lambda q_1) + \frac{1}{2} \sum_{i=2}^N p_i^2 + \omega_2^2 \sum_{i=2}^N q_i^2 + f \left(\lambda^2 \sum_{i=2}^N q_i^2 \right) \quad (5.17)$$

with $\omega_1^2 = \lambda^2 \beta_1 + \delta_1$ and $\omega_2^2 = \lambda^2 \alpha_1 + \delta_1$. This Hamiltonian is indeed separable as the sum of two Hamiltonians: the first one depends only on q_1 and the second one is a radial deformation of the $(N-1)$ -dimensional isotropic oscillator with frequency ω_2 . In this case the integrals of the motion read

$$\mathcal{C}_{h_6}^{(m)} = \lambda^2 \sum_{2 \leq j < k}^m (p_j q_k - p_k q_j)^2 \quad m = 3, 4, \dots, N \quad (5.18)$$

$$\mathcal{I}^{(N)} = \lambda^2 (p_1^2 + 2\omega_1^2 q_1^2 + 2g(\lambda q_1)). \quad (5.19)$$

In fact, the first set of integrals reflects the axial symmetry of the ND Hamiltonian, and the last one is directly related to the abovementioned separability.

As a consequence of this analysis, and with no loss of generality, in the rest of the Chapter we will only consider the ND isotropic case (5.16).

5.2 ND coupled nonlinear oscillators

As an immediate application of our general result, we will now consider in detail integrable polynomial perturbations of the ND isotropic oscillator given by the Hamiltonian

(5.16). Based on the previous discussion, we put $\alpha_1 = \beta_1 = 0$. Afterwards, if we write the perturbation functions $\mathcal{F} \equiv f$ and $\mathcal{G} \equiv g$ in the form

$$f(x) = \sum_{j=2}^k \alpha_j x^j \quad g(y) = \sum_{j=3}^l \beta_{j-1} y^j \quad (5.20)$$

we will obtain a superposition of a polynomial deformation of degree $2k$ in the coordinates (the argument $x \equiv \mathcal{C}_{\mathcal{G}_-}$ is quadratic in q) and another of degree l (the argument $y \equiv A_-$ is linear in q).

Since the integrability of the perturbation is preserved by the superposition of any of the monomials given in (5.20) (the only thing we have to do is to include the appropriate function $\mathcal{G}(A_-) = g(A_-)$ in the invariant \mathcal{I}), we can analyse separately the homogeneous perturbations with a fixed degree m . Note that for odd m only $g(A_-)$ can contribute, which means that the richest cases will be obtained for m even. Therefore, in the sequel we will analyse in detail the cases $m = 4$ and $m = 6$.

5.2.1 Homogeneous quartic perturbations

The homogeneous quartic case is obtained by taking

$$\phi = f(\mathcal{C}_{\mathcal{G}_-}) + g(A_-) = \alpha_2 \mathcal{C}_{\mathcal{G}_-}^2 + \beta_3 A_-^4 = \alpha_2 \left(\sum_{i < j}^N (\lambda_j q_i - \lambda_i q_j)^2 \right)^2 + \beta_3 \left(\sum_{i=1}^N \lambda_i q_i \right)^4. \quad (5.21)$$

Let us describe more explicitly the 2D and 3D cases.

2D examples

The generic 2D Hamiltonian coming from (5.21) is

$$\mathcal{H}^{(2)} = \frac{1}{2}(p_1^2 + p_2^2) + \delta_1(q_1^2 + q_2^2) + \alpha_2 (\lambda_2 q_1 - \lambda_1 q_2)^4 + \beta_3 (\lambda_1 q_1 + \lambda_2 q_2)^4 \quad (5.22)$$

and the following cases can be distinguished.

- If $\lambda_1 = 1, \lambda_2 = 0$ we obtain the separable Hamiltonian

$$\mathcal{H}^{(2)} = \frac{1}{2}(p_1^2 + p_2^2) + \delta_1(q_1^2 + q_2^2) + \beta_3 q_1^4 + \alpha_2 q_2^4 \quad (5.23)$$

whose invariant (5.10) reflects such separability since $\mathcal{I}^{(2)} = p_1^2 + 2(\delta_1 q_1^2 + \beta_3 q_1^4)$. Note that the separability of the generic 2D Hamiltonian (5.22) in terms of the $(\lambda_2 q_1 - \lambda_1 q_2)$ and $(\lambda_1 q_1 + \lambda_2 q_2)$ coordinates with $\lambda_1^2 + \lambda_2^2 = 1$ is straightforward.

- On the other hand, by taking $\lambda_1 = 1, \lambda_2 = 1, \alpha_2 = \beta_3 = \epsilon/2$ we recover the well-known 1:6:1 integrable quartic perturbation (case (4)3' in [105] with $E = 0$)

$$\mathcal{H}^{(2)} = \frac{1}{2} (p_1^2 + p_2^2) + \delta_1 (q_1^2 + q_2^2) + \epsilon (q_1^4 + 6q_1^2 q_2^2 + q_2^4) \quad (5.24)$$

and the corresponding integral is obtained through direct substitution onto \mathcal{I} :

$$\mathcal{I}^{(2)} = (p_1 + p_2)^2 + 2(q_1 + q_2)^2 (\delta_1 + \epsilon(q_1 + q_2)^2). \quad (5.25)$$

This system is known to be connected with the Hirota-Satsuma coupled KdV system (see [7]).

3D examples

The 3D realization of (5.21) for arbitrary values of the parameters reads:

$$\begin{aligned} \mathcal{H}^{(3)} = & \frac{1}{2} (p_1^2 + p_2^2 + p_3^2) + \delta_1 (q_1^2 + q_2^2 + q_3^2) + (\beta_3 \lambda_1^4 + \alpha_2 \lambda_2^4 + 2\alpha_2 \lambda_2^2 \lambda_3^2 + \alpha_2 \lambda_3^4) q_1^4 \\ & + (\alpha_2 \lambda_1^4 + \beta_3 \lambda_2^4 + 2\alpha_2 \lambda_1^2 \lambda_3^2 + \alpha_2 \lambda_3^4) q_2^4 + (\alpha_2 \lambda_1^4 + 2\alpha_2 \lambda_1^2 \lambda_2^2 + \alpha_2 \lambda_2^4 + \beta_3 \lambda_3^4) q_3^4 \\ & - 4\lambda_1 \lambda_3 (\alpha_2 (\lambda_1^2 + \lambda_2^2) - \beta_3 \lambda_3^2) q_1 q_3^3 - 4\lambda_1 \lambda_2 (\alpha_2 (\lambda_1^2 + \lambda_3^2) - \beta_3 \lambda_2^2) q_1 q_2^3 \\ & + 4\lambda_1 \lambda_2 (\beta_3 \lambda_1^2 - \alpha_2 (\lambda_2^2 + \lambda_3^2)) q_1^3 q_2 + 4\lambda_1 \lambda_3 (\beta_3 \lambda_1^2 - \alpha_2 (\lambda_2^2 + \lambda_3^2)) q_1^3 q_3 \\ & - 4\lambda_2 \lambda_3 (\alpha_2 (\lambda_1^2 + \lambda_2^2) - \beta_3 \lambda_3^2) q_2 q_3^3 + 4\lambda_2 \lambda_3 (\beta_3 \lambda_2^2 - \alpha_2 (\lambda_1^2 + \lambda_2^2)) q_2^3 q_3 \\ & + 2 \left\{ 3\beta_3 \lambda_1^2 \lambda_2^2 + \alpha_2 \left[\lambda_3^2 (\lambda_2^2 + \lambda_3^2) + \lambda_1^2 (3\lambda_2^2 + \lambda_3^2) \right] \right\} q_1^2 q_2^2 \\ & + 2 \left\{ 3\beta_3 \lambda_2^2 \lambda_3^2 + \alpha_2 \left[\lambda_1^4 + 3\lambda_2^2 \lambda_3^2 + \lambda_1^2 (\lambda_2^2 + \lambda_3^2) \right] \right\} q_2^2 q_3^2 \\ & + 2 \left\{ 3\beta_3 \lambda_1^2 \lambda_2^2 + \alpha_2 \left[\lambda_2^2 (\lambda_2^2 + \lambda_3^2) + \lambda_1^2 (\lambda_2^2 + 3\lambda_3^2) \right] \right\} q_1^2 q_3^2 \\ & - 4\lambda_1 \lambda_2 \left\{ -3\beta_3 \lambda_3^2 + \alpha_2 (\lambda_1^2 + \lambda_2^2 - 2\lambda_3^2) \right\} q_1 q_2 q_3^2 \\ & - 4\lambda_1 \lambda_3 \left\{ -3\beta_3 \lambda_2^2 + \alpha_2 (\lambda_1^2 - 2\lambda_2^2 + \lambda_3^2) \right\} q_1 q_2^2 q_3 \\ & - 4\lambda_2 \lambda_3 \left\{ -3\beta_3 \lambda_1^2 + \alpha_2 (-2\lambda_1^2 + \lambda_2^2 + \lambda_3^2) \right\} q_1^2 q_2 q_3 \end{aligned} \quad (5.26)$$

and the two invariants for this Hamiltonian are (5.10) and the ‘universal’ integral

$$\mathcal{C}_{h_6}^{(3)} = [\lambda_1 (p_2 q_3 - p_3 q_2) + \lambda_2 (p_3 q_1 - p_1 q_3) + \lambda_3 (p_1 q_2 - p_2 q_1)]^2. \quad (5.27)$$

Two representative cases would be:

- $\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = 0$, which again provides the separability of the q_1 coordinate

$$\mathcal{H}^{(3)} = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + \delta_1(q_1^2 + q_2^2 + q_3^2) + \alpha_2(q_2^2 + q_3^2)^2 + \beta_3 q_1^4 \quad (5.28)$$

that is also reflected in the invariant $\mathcal{I}^{(3)} = p_1^2 + 2(\delta_1 q_1^2 + \beta_3 q_1^4)$. Moreover, the second invariant reads $\mathcal{C}_{h_6}^{(3)} = (p_2 q_3 - p_3 q_2)^2$, which indicates the axial symmetry with respect to the coordinates (q_2, q_3) . This is the case (1) of Table 1 in [78].

- If we take $\lambda_1 = \lambda_2 = \lambda_3 = 1$ we get

$$\begin{aligned} \mathcal{H}^{(3)} = & \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + \delta_1(q_1^2 + q_2^2 + q_3^2) \\ & + 2\alpha_2((q_2 - q_1)^4 + (q_3 - q_1)^4 + (q_3 - q_2)^4) + \beta_3(q_1 + q_2 + q_3)^4 \end{aligned} \quad (5.29)$$

which is just the 3D Chudnowski potential plus an additional quartic term (see [71, 179]). The specific integral is

$$\mathcal{I}^{(3)} = (p_1 + p_2 + p_3)^2 + 2(q_1 + q_2 + q_3)^2 (\delta_1 + 3\beta_3(q_1 + q_2 + q_3)^2). \quad (5.30)$$

5.2.2 Homogeneous sextic perturbations

In the same way, the generic homogeneous sextic case is obtained by taking

$$\phi = f(\mathcal{C}_{\bar{g}_-}) + g(A_-) = \alpha_3 \mathcal{C}_{\bar{g}_-}^3 + \beta_5 A_-^6 = \alpha_3 \left(\sum_{i < j}^N (\lambda_j q_i - \lambda_i q_j)^2 \right)^3 + \beta_5 \left(\sum_{i=1}^N \lambda_i q_i \right)^6. \quad (5.31)$$

Let us also analyse the 2D and 3D cases.

2D examples

The generic 2D integrable sextic perturbation reads

$$\mathcal{H}^{(2)} = \frac{1}{2}(p_1^2 + p_2^2) + \delta_1(q_1^2 + q_2^2) + \alpha_3(\lambda_2 q_1 - \lambda_1 q_2)^6 + \beta_5(\lambda_1 q_1 + \lambda_2 q_2)^6. \quad (5.32)$$

- Once again, the separability in terms of the $(\lambda_2 q_1 - \lambda_1 q_2)$ and $(\lambda_1 q_1 + \lambda_2 q_2)$ coordinates with $\lambda_1^2 + \lambda_2^2 = 1$ is straightforward. In particular, if $\lambda_1 = 1, \lambda_2 = 0$ we have

$$\mathcal{H}^{(2)} = \frac{1}{2}(p_1^2 + p_2^2) + \delta_1(q_1^2 + q_2^2) + \beta_5 q_1^6 + \alpha_3 q_2^6 \quad (5.33)$$

and the integral is $\mathcal{I}^{(2)} = p_1^2 + 2(\delta_1 q_1^2 + \beta_5 q_1^6)$.

- The case $\lambda_1 = \lambda_2 = 1$ with $\alpha_2 = \beta_5 = \frac{b}{2}$ gives the known coupled sextic oscillator (see [117], case (i) of Table 5.3.):

$$\mathcal{H}^{(2)} = \frac{1}{2} (p_1^2 + p_2^2) + \delta_1 (q_1^2 + q_2^2) + b (q_1^6 + q_2^6 + 15 q_1^2 q_2^2 (q_1^2 + q_2^2)) \quad (5.34)$$

and we also recover from (5.10) the right expression for the invariant:

$$\mathcal{I}^{(2)} = (p_1 + p_2)^2 + 2(q_1 + q_2)^2 (\delta_1 + b(q_1 + q_2)^4). \quad (5.35)$$

3D examples

In 3D the most general integrable homogeneous sextic perturbation coming from (5.1) reads

$$\begin{aligned} \mathcal{H}^{(3)} = & \frac{1}{2} (p_1^2 + p_2^2 + p_3^2) + \delta_1 (q_1^2 + q_2^2 + q_3^2) + \beta_5 (\lambda_1 q_1 + \lambda_2 q_2 + \lambda_3 q_3)^6 \\ & + \alpha_3 \left[(\lambda_2 q_1 - \lambda_1 q_2)^2 + (\lambda_3 q_1 - \lambda_1 q_3)^2 + (\lambda_3 q_2 - \lambda_2 q_3)^2 \right]^3 \end{aligned} \quad (5.36)$$

which, by construction, commutes the corresponding (5.10) and the ‘universal’ integral (5.27). Similarly to the quartic oscillator case we can mention two particular cases:

- The choice $\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = 0$, which gives

$$\mathcal{H}^{(3)} = \frac{1}{2} (p_1^2 + p_2^2 + p_3^2) + \delta_1 (q_1^2 + q_2^2 + q_3^2) + \alpha_3 (q_2^2 + q_3^2)^3 + \beta_5 q_1^6 \quad (5.37)$$

which is again separable in q_1 with associated invariant $\mathcal{I}^{(3)} = p_1^2 + 2(\delta_1 q_1^2 + \beta_5 q_1^6)$ and axial symmetry in (q_2, q_3) is also recovered.

- An the case $\lambda_1 = \lambda_2 = \lambda_3 = 1$ in (5.36),

$$\begin{aligned} \mathcal{H}^{(3)} = & \frac{1}{2} (p_1^2 + p_2^2 + p_3^2) + \delta_1 (q_1^2 + q_2^2 + q_3^2) \\ & + 8 \alpha_3 (q_1^2 + q_2^2 + q_3^2 - q_2 q_3 - q_1 q_2 - q_1 q_3)^3 + \beta_5 (q_1 + q_2 + q_3)^6 \end{aligned} \quad (5.38)$$

whose invariant reads

$$\mathcal{I}^{(3)} = (p_1 + p_2 + p_3)^2 + 2(q_1 + q_2 + q_3)^2 (\delta_1 + 3\beta_5 (q_1 + q_2 + q_3)^4). \quad (5.39)$$

In order to show the complexity of this kind of systems we expand (5.36).

$$\begin{aligned}
\mathcal{H}^{(3)} &= \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + \delta_1(q_1^2 + q_2^2 + q_3^2) \\
&+ (\beta_5 \lambda_1^6 + \alpha_3 \{\lambda_2^2 + \lambda_3^2\}^3) q_1^6 + (\beta_5 \lambda_2^6 + \alpha_3 \{\lambda_1^2 + \lambda_3^2\}^3) q_2^6 + (\beta_5 \lambda_3^6 + \alpha_3 \{\lambda_1^2 + \lambda_2^2\}^3) q_3^6 \\
&- 6\lambda_1 \lambda_2 (-\beta_5 \lambda_1^4 + \alpha_3 \{\lambda_2^2 + \lambda_3^2\}^2) q_1^5 q_2 - 6\lambda_1 \lambda_2 (-\beta_5 \lambda_2^4 + \alpha_3 \{\lambda_1^2 + \lambda_3^2\}^2) q_1 q_2^5 \\
&- 6\lambda_2 \lambda_3 (-\beta_5 \lambda_2^4 + \alpha_3 \{\lambda_1^2 + \lambda_2^2\}^2) q_2^5 q_3 - 6\lambda_2 \lambda_3 (-\beta_5 \lambda_3^4 + \alpha_3 \{\lambda_1^2 + \lambda_2^2\}^2) q_2 q_3^5 \\
&- 6\lambda_1 \lambda_3 (-\beta_5 \lambda_3^4 + \alpha_3 \{\lambda_1^2 + \lambda_2^2\}^2) q_1 q_3^5 - 6\lambda_1 \lambda_3 (-\beta_5 \lambda_1^4 + \alpha_3 \{\lambda_2^2 + \lambda_3^2\}^2) q_1^5 q_3 \\
&+ 3 \left(5\beta_5 \lambda_1^4 \lambda_2^2 + \alpha_3 \left\{ \lambda_2^2 + \lambda_3^2 \right\} \left[\lambda_3^2 (\lambda_2^2 + \lambda_3^2) + \lambda_1^2 (5\lambda_2^2 + \lambda_3^2) \right] \right) q_1^4 q_2^2 \\
&+ 3 \left(5\beta_5 \lambda_1^2 \lambda_2^4 + \alpha_3 \left\{ \lambda_1^2 + \lambda_3^2 \right\} \left[\lambda_3^2 (\lambda_2^2 + \lambda_3^2) + \lambda_1^2 (5\lambda_2^2 + \lambda_3^2) \right] \right) q_1^2 q_2^4 \\
&+ 3 \left(5\beta_5 \lambda_2^2 \lambda_3^4 + \alpha_3 \left\{ \lambda_1^2 + \lambda_2^2 \right\} \left[\lambda_1^2 (\lambda_2^2 + \lambda_3^2) + \lambda_1^4 + 5\lambda_2^2 \lambda_3^2 \right] \right) q_2^2 q_3^4 \\
&+ 3 \left(5\beta_5 \lambda_2^4 \lambda_3^2 + \alpha_3 \left\{ \lambda_1^2 + \lambda_3^2 \right\} \left[\lambda_1^2 (\lambda_2^2 + \lambda_3^2) + \lambda_1^4 + 5\lambda_2^2 \lambda_3^2 \right] \right) q_2^4 q_3^2 \\
&+ 3 \left(5\beta_5 \lambda_1^2 \lambda_3^4 + \alpha_3 \left\{ \lambda_1^2 + \lambda_2^2 \right\} \left[\lambda_2^2 (\lambda_2^2 + \lambda_3^2) + \lambda_1^2 (\lambda_2^2 + 5\lambda_3^2) \right] \right) q_1^2 q_3^4 \\
&+ 3 \left(5\beta_5 \lambda_1^4 \lambda_3^2 + \alpha_3 \left\{ \lambda_2^2 + \lambda_3^2 \right\} \left[\lambda_2^2 (\lambda_2^2 + \lambda_3^2) + \lambda_1^2 (\lambda_2^2 + 5\lambda_3^2) \right] \right) q_1^4 q_3^2 \\
&- 4\lambda_2 \lambda_3 \left(-5\beta_5 \lambda_2^2 \lambda_3^2 + \alpha_3 \left[3\lambda_1^4 + 5\lambda_2 \lambda_3^2 + 3\lambda_1^2 \left\{ \lambda_2^2 + \lambda_3^2 \right\} \right] \right) q_2^3 q_3^3 \\
&- 4\lambda_1 \lambda_3 \left(-5\beta_5 \lambda_1^2 \lambda_3^2 + \alpha_3 \left[3\lambda_2^2 (\lambda_2^2 + \lambda_3^2) + \lambda_1^2 \left\{ 3\lambda_2^2 + 5\lambda_3^2 \right\} \right] \right) q_1^3 q_3^3 \\
&- 4\lambda_1 \lambda_2 \left(-5\beta_5 \lambda_1^2 \lambda_2^2 + \alpha_3 \left[3\lambda_3^2 (\lambda_2^2 + \lambda_3^2) + \lambda_1^2 \left\{ 5\lambda_2^2 + 3\lambda_3^2 \right\} \right] \right) q_1^3 q_2^3 \\
&- 6\lambda_1 \lambda_2 \left(-5\beta_5 \lambda_3^4 + \alpha_3 \left\{ \lambda_1^2 + \lambda_2^2 \right\} \left[\lambda_1^2 + \lambda_2^2 - 4\lambda_3^2 \right] \right) q_1 q_2 q_3^4 \\
&- 6\lambda_1 \lambda_3 \left(-5\beta_5 \lambda_2^4 + \alpha_3 \left\{ \lambda_1^2 + \lambda_3^2 \right\} \left[\lambda_1^2 - 4\lambda_2^2 + \lambda_3^2 \right] \right) q_1 q_2^4 q_3 \\
&- 6\lambda_2 \lambda_3 \left(-5\beta_5 \lambda_1^4 - \alpha_3 \left\{ \lambda_2^2 + \lambda_3^2 \right\} \left[4\lambda_1^2 - \lambda_2^2 - \lambda_3^2 \right] \right) q_1^4 q_2 q_3 \\
&- 12\lambda_1 \lambda_3 \left(-5\beta_5 \lambda_2^2 \lambda_3^2 + \alpha_3 \left[\lambda_1^4 - 2\lambda_2^4 + 3\lambda_2^2 \lambda_3^2 + \lambda_1^2 \left\{ -\lambda_2^2 + \lambda_3^2 \right\} \right] \right) q_1 q_2^2 q_3^3 \\
&- 12\lambda_1 \lambda_2 \left(-5\beta_5 \lambda_2^2 \lambda_3^2 + \alpha_3 \left[\lambda_1^4 + 3\lambda_2^2 \lambda_3^2 - 2\lambda_3^4 + \lambda_1^2 \left\{ \lambda_2^2 - \lambda_3^2 \right\} \right] \right) q_1 q_2^3 q_3^2 \\
&- 12\lambda_1 \lambda_2 \left(-5\beta_5 \lambda_1^2 \lambda_3^2 + \alpha_3 \left[\lambda_2^4 - \lambda_2^2 \lambda_3^2 - 2\lambda_3^4 + \lambda_1^2 \left\{ \lambda_2^2 + 3\lambda_3^2 \right\} \right] \right) q_1^3 q_2 q_3^2 \\
&12\lambda_2 \lambda_3 \left(5\beta_5 \lambda_1^2 \lambda_3^2 + \alpha_3 \left[2\lambda_2^4 + \lambda_1^2 \left\{ \lambda_2^2 - 3\lambda_3^2 \right\} - \lambda_2^2 \left\{ \lambda_2^2 + \lambda_3^2 \right\} \right] \right) q_1^2 q_2 q_3^3 \\
&12\lambda_2 \lambda_3 \left(5\beta_5 \lambda_1^2 \lambda_2^2 + \alpha_3 \left[2\lambda_1^4 + \lambda_1^2 \left\{ -3\lambda_2^2 + \lambda_3^2 \right\} - \lambda_3^2 \left\{ \lambda_2^2 + \lambda_3^2 \right\} \right] \right) q_1^2 q_2^3 q_3 \\
&- 12\lambda_1 \lambda_3 \left(-5\beta_5 \lambda_1^2 \lambda_2^2 + \alpha_3 \left[-2\lambda_2^4 - \lambda_2^2 \lambda_3^2 + \lambda_3^4 + \lambda_1^2 \left\{ 3\lambda_2^2 + \lambda_3^2 \right\} \right] \right) q_1^3 q_2^2 q_3 \\
&+ 18 \left(5\beta_5 \lambda_1^2 \lambda_2^2 \lambda_3^2 + \alpha_3 \left[\lambda_1^2 \left\{ \lambda_2^2 - \lambda_3^2 \right\}^2 + \lambda_1^4 \left\{ \lambda_2^2 + \lambda_3^2 \right\} + \lambda_2^2 \lambda_3^2 \left\{ \lambda_2^2 + \lambda_3^2 \right\} \right] \right) q_1^2 q_2^2 q_3^2 \quad (5.40)
\end{aligned}$$

5.3 Another ND integrable perturbation

The second main result of this Chapter is the following. Let

$$\mathcal{V}_{-2}(A_-, B_-) = \mathcal{V}_{-2}\left(\sum_{i=1}^N \lambda_i q_i, \sum_{i=1}^N q_i^2\right) \quad (5.41)$$

be a homogeneous function of degree -2 in the canonical coordinates. Then the ND Hamiltonian

$$\begin{aligned} \mathcal{H}_r &= \frac{1}{2}B_+ + \delta_1 B_- + \mathcal{V}_{-2}(A_-, B_-) + \mathcal{J}(B_-) \\ &= \frac{1}{2}\sum_{i=1}^N p_i^2 + \delta_1 \sum_{i=1}^N q_i^2 + \mathcal{V}_{-2}\left(\sum_{i=1}^N \lambda_i q_i, \sum_{i=1}^N q_i^2\right) + \mathcal{J}\left(\sum_{i=1}^N q_i^2\right) \end{aligned} \quad (5.42)$$

where \mathcal{J} is an arbitrary radial function and $\delta_1, \lambda_i (i = 1, \dots, N)$ are free parameters, is completely integrable.

The $(N - 1)$ functionally independent integrals of the motion in involution with \mathcal{H}_r are the $(N - 2)$ ‘universal’ ones coming from the h_6 -coalgebra symmetry (5.8) and the additional integral

$$\mathcal{I}_r = B_- B_+ - \left(K + \frac{1}{2}M\right)^2 + 2B_- \mathcal{V}_{-2}(A_-, B_-) \quad (5.43)$$

that written in terms of the canonical coordinates reads:

$$\mathcal{I}_r^{(N)} = \sum_{1 \leq i < j}^N (q_j p_i - q_i p_j)^2 + 2 \left(\sum_{i=1}^N q_i^2\right) \mathcal{V}_{-2}\left(\sum_{i=1}^N \lambda_i q_i, \sum_{i=1}^N q_i^2\right). \quad (5.44)$$

Again, the involutivity of \mathcal{H}_r and \mathcal{I}_r with respect to the integrals (5.5) is a direct consequence of the h_6 -coalgebra symmetry of both functions and the bracket $\{\mathcal{H}_r, \mathcal{I}_r\} = 0$ is easily proven by direct computation. Finally, the functional independence of \mathcal{I}_r can be easily proven in the harmonic oscillator case $\mathcal{J} = \mathcal{V}_{-2} = 0$ with $\lambda_i = \delta_i^j$ for a given j . Therefore, it follows for the generic case with arbitrary λ_i and non-vanishing \mathcal{J} and \mathcal{V}_{-2} functions.

Note that in $N = 3$ the Hamiltonian (5.42) is a particular subclass of the Hamiltonians presented in [179]. This is due to the fact that in our construction we have imposed \mathcal{V}_{-2} to be h_6 -coalgebra invariant, which is an additional constraint with respect to [179], where the only condition imposed on \mathcal{V}_{-2} was to be a homogeneous function of degree -2 in the 3D canonical coordinates.

5.3.1 A new ND integrable model

If we take

$$\mathcal{V}_{-2}(A_-, B_-) = \varepsilon \frac{A_-^2 - B_-}{B_- (MB_- - A_-^2)} \quad (5.45)$$

and $\forall i, \lambda_i = 1$ we obtain the Hamiltonian

$$\mathcal{H}_r^{(N)} = \frac{1}{2} \sum_{i=1}^N p_i^2 + \delta_1 \sum_{i=1}^N q_i^2 + \varepsilon \frac{2 \sum_{1 \leq i < j}^N q_i q_j}{\left(\sum_{i=1}^N q_i^2 \right) \left(\sum_{1 \leq i < j}^N (q_i - q_j)^2 \right)} + \mathcal{J} \left(\sum_{i=1}^N q_i^2 \right) \quad (5.46)$$

in which the function $\mathcal{V}_{-2}(A_-, B_-)$ gives rise a homogeneous function of degree -2 in the canonical coordinates. In the $N = 2$ case (and with the radial perturbation $\mathcal{J} = 0$) the quantum mechanical analogue of (5.46) has been studied in [76] as an exactly solvable ‘generalization’ of the 2D CM model. Therefore, presumably (5.46) provides a good candidate for a new exactly solvable model in arbitrary dimension, whose exact solvability should be indeed algebraically related with its underlying h_6 coalgebra symmetry.

Chapter 6

Propagating integrability from 2D to ND

*“The good Christian should beware the mathematician
and all those who make empty prophecies.
The danger already exists that the mathematicians
have made a covenant with the devil
to darken the spirit and to confine man in the bonds of hell.”*

St. Augustine

The aim of this Chapter is to present yet another application of coalgebra symmetry in order to introduce a general procedure that allows the construction of a ND completely integrable system from any 2D integrable potential of the form $\mathcal{V}(x^2, y)$. We show that such potentials can be always defined as a precise $N = 2$ symplectic realization of the direct-sum Poisson algebra $sl(2, \mathbb{R}) \oplus h_6$. This implies that by using the coalgebra approach in the $sl(2, \mathbb{R})$ sector we can straightforwardly obtain a N -dimensional integrable generalization of all these systems. This construction can be interpreted as the proper Poisson-algebraic generalization for the procedure given in [97, 98] for natural systems, and the application of this method to electromagnetic systems and geodesic flows would follow the same lines. Moreover, this coalgebra symmetry opens the possibility to introduce integrable deformations of all the systems presented in this Chapter by making use, for instance, of the q -Poisson $sl(2, \mathbb{R})$ algebra presented in Chapter 2.

As outstanding examples of this construction we shall generalize the three integrable cases of the 2D Hénon-Heiles Hamiltonian, originally introduced in order to model

the dynamics of a star when orbiting around an axially symmetric galaxy (see for instance [38, 39, 86, 100, 102]). Also, the Ramani series of 2D integrable polynomial perturbations of the harmonic oscillator [154] will be considered. As a consequence, we will obtain ND generalizations of all these systems, and the coupling between different orders in the polynomial perturbations will be studied. Moreover, we will show that all these results can be generalized by preserving their complete integrability through the addition of $(N - 1)$ centrifugal terms, and the $sl(2, \mathbb{R}) \oplus h_6$ symmetry will also give us automatically the extra contributions in the integrals of the motion coming from them.

Finally, we will also show that this approach can be generalized to consider those 2D integrable Hamiltonians with potentials of the form $\mathcal{V}(x^2, y^2)$. In this case, the underlying Poisson algebra that allows the construction of the ND integrable generalization of such systems is just $sl(2, \mathbb{R}) \oplus sl(2, \mathbb{R})$. In this case our construction will allow the introduction of N centrifugal terms, and two examples of this type of systems will be explicitly given.

6.1 From 2D to ND through axial symmetry

Let us firstly recall the key observation given in [97]. Let us consider any 2D integrable natural Hamiltonian system in the form

$$\mathcal{H}^{(2)} = \frac{1}{2}(p_x^2 + p_y^2) + \mathcal{V}(x, y) + \frac{\lambda}{x^2} \quad (6.1)$$

i.e., we select those 2D integrable potentials $\mathcal{V}(x, y)$ that can be perturbed through a centrifugal term in at least one of the variables without destroying its integrability. In that case, it is clear that $\mathcal{H}^{(2)}$ can be interpreted as the radial reduction of the following axially symmetric 3D Hamiltonian

$$\mathcal{H}^{(3)} = \frac{1}{2}(p_{x_1}^2 + p_{x_2}^2 + p_y^2) + \mathcal{V}(\sqrt{x_1^2 + x_2^2}, y) \quad (6.2)$$

through the change of variables $x = \sqrt{x_1^2 + x_2^2}$. The centrifugal term λ/x^2 is then the usual kinetic energy term coming from the conserved angular momentum λ on the (x_1, x_2) plane. Therefore, the authors of [97] stated that the ND Hamiltonian

$$\mathcal{H}^{(N)} = \frac{1}{2} \sum_{i=1}^{N-1} p_{x_i}^2 + \frac{1}{2} p_y^2 + \mathcal{V} \left(\sqrt{\sum_{i=1}^{N-1} x_i^2}, y \right) \quad (6.3)$$

is a completely integrable one, and gave explicitly its $(N - 1)$ integrals of the motion: one of them is the ND generalization of the 2D integral in involution with (6.1), and the

remaining ones arise from the spherical symmetry in the $(x_1, x_2, \dots, x_{N-1})$ variables. Surprisingly enough, in [106], all the potentials that admit a centrifugal term of the type (6.1) are functions of x^2 . This observation opens the path to the algebraic formulation of this construction in terms of Poisson coalgebras, that we shall describe in the sequel.

6.2 Hamiltonian systems with $sl(2, \mathbb{R}) \oplus h_6$ symmetry

The essential algebraic object for our construction will be the 9-dimensional Poisson algebra $sl(2, \mathbb{R}) \oplus h_6$, with generators $\{J_+, J_-, J_3, K, A_+, A_-, B_+, B_-, M\}$ and Poisson brackets given by

$$\{J_3, J_+\} = 2J_+ \quad \{J_3, J_-\} = -2J_- \quad \{J_-, J_+\} = 4J_3 \quad (6.4)$$

$$\begin{aligned} \{K, A_+\} &= A_+ & \{K, A_-\} &= -A_- & \{A_-, A_+\} &= M \\ \{K, B_+\} &= 2B_+ & \{K, B_-\} &= -2B_- & \{B_-, B_+\} &= 4K + 2M \\ \{A_+, B_-\} &= -2A_- & \{A_+, B_+\} &= 0 & \{M, \cdot\} &= 0 \\ \{A_-, B_+\} &= 2A_+ & \{A_-, B_-\} &= 0. \end{aligned} \quad (6.5)$$

(all the brackets between generators of $sl(2, \mathbb{R})$ and h_6 vanish). Obviously, this Poisson algebra has three Casimir functions: the $sl(2, \mathbb{R})$ Casimir (2.27) and the two h_6 ones (4.3) and M .

Now we consider the following ND symplectic realization of $sl(2, \mathbb{R}) \oplus h_6$, in which the first set of $(N - 1)$ degrees of freedom is associated to $sl(2, \mathbb{R})$, while the last one is just the usual one-particle h_6 realization with $\lambda = 1$:

$$J_+ = \sum_{i=1}^{N-1} p_i^2 \quad J_- = \sum_{i=1}^{N-1} q_i^2 \quad J_3 = \sum_{i=1}^{N-1} q_i p_i \quad (6.6)$$

$$\begin{aligned} A_+ &= p_N & A_- &= q_N & K &= q_N p_N - \frac{1}{2} \\ B_+ &= p_N^2 & B_- &= q_N^2 & M &= 1. \end{aligned} \quad (6.7)$$

Note that we can say that this is the $(0, 0, \dots, 0) \oplus (1, 0)$ symplectic realization of $sl(2, \mathbb{R}) \oplus h_6$.

Now, it is straightforward to realize that any ND Hamiltonian constructed through the previous symplectic realization as

$$\mathcal{H}^{(N)} = \mathcal{H}(J_+, J_-, J_3, K, B_+, B_-, A_+, A_-, M) \quad (6.8)$$

is *quasi-integrable*, since the following $(N - 2)$ functions

$$\mathcal{C}^{(m)} = \sum_{1 \leq i < j}^m (q_i p_j - q_j p_i)^2 \quad m = 2, \dots, N - 1 \quad (6.9)$$

are constants of the motion for \mathcal{H} . In fact (6.9) are just the $sl(2, \mathbb{R})$ -coalgebra integrals for the $sl(2, \mathbb{R})$ -sector of the Hamiltonian, and any contribution coming from the last degree of freedom (q_N, p_N) will obviously Poisson-commute with them.

Therefore, in case that for a certain choice of \mathcal{H} we are able to find just one additional integral of the form

$$\mathcal{I} = \mathcal{I}(J_+, J_-, J_3, K, B_+, B_-, A_+, A_-, M) \quad (6.10)$$

and we realize it under the ND symplectic realization (6.6) and (6.7), the complete integrability of $\mathcal{H}^{(N)}$ will be obtained. Moreover, the coalgebra symmetry of the $sl(2, \mathbb{R})$ sector of the Hamiltonian will imply that there does exist an additional set of $(N - 2)$ integrals $\mathcal{C}_{(m)}$ (2.30, $b_i = 0$), thus making $\mathcal{H}^{(N)}$ a (non-maximally) superintegrable system.

As we shall see in the following Section, suitable candidates for completely integrable \mathcal{H} are obtained from certain known 2D integrable Hamiltonians, since the additional integral \mathcal{I} can be proposed by working out the $sl(2, \mathbb{R}) \oplus h_6$ symmetry of the 2D system.

6.3 Complete integrability

Let us consider any integrable 2D Hamiltonian of the form

$$\mathcal{H}^{(2)} = \frac{1}{2}(p_x^2 + p_y^2) + \mathcal{V}(x^2, y) \quad (6.11)$$

whose integral of the motion $\mathcal{I}^{(2)}$ can be written as a function of the form

$$\mathcal{I}^{(2)} = \mathcal{I}^{(2)}(p_x^2, x^2, x p_x, y p_y, p_y^2, y^2, p_y, y). \quad (6.12)$$

Under these hypotheses it becomes clear that $\mathcal{H}^{(2)}$ is $sl(2, \mathbb{R}) \oplus h_6$ symmetric, since it would be just the $N = 2$ case of the generic Hamiltonian

$$\mathcal{H} = \frac{1}{2}(J_+ + B_+) + \mathcal{V}(J_-, A_-) \quad (6.13)$$

in which we have identified $x \equiv q_1$ and $y \equiv q_N = q_2$. This is tantamount to say that we have assigned the x degree of freedom with $sl(2, \mathbb{R})$ (note that $\mathcal{H}^{(2)}$ depends on x^2) and y corresponds to the degree of freedom associated to h_6 .

Moreover, the integral $\mathcal{I}^{(2)}$ gives rise to a generic $sl(2, \mathbb{R}) \oplus h_6$ integral through the substitution

$$\mathcal{I}^{(2)}(p_x^2, x^2, x p_x, y p_y, p_y^2, y^2, p_y, y) \rightarrow \mathcal{I}(J_+, J_-, J_3, K, B_+, B_-, A_+, A_-). \quad (6.14)$$

As we shall see in practical examples, the substitution (6.14) could be performed in different ways. As a rule, the most general one has to be considered, and afterwards one has to check that the object so obtained does Poisson commutes with (6.13) by using the algebra (6.4)-(6.5). In this way the true ‘abstract’ $sl(2, \mathbb{R}) \oplus h_6$ integral is obtained, and its ND realization provides the remaining constant of the motion.

6.4 ND Hénon-Heiles systems

In this Section, by using the abovementioned symmetry we will construct the ND generalization of the three integrable cases among the family of 2D Hénon-Heiles Hamiltonians

$$\mathcal{H}^{(2)} = \frac{1}{2}(p_1^2 + p_2^2) + \delta q_1^2 + (\delta + \Omega) q_2^2 + \alpha (q_1^2 q_2 + \beta q_2^3). \quad (6.15)$$

It is well-known [2, 41, 68, 95, 104, 156] that the only integrable members of this class of Hamiltonians are given by the following choices of the parameters, where we will use the terminology of [121] and the results given in [84, 138]:

- The Sawada-Kotera case: $\beta = 1/3$ and $\Omega = 0$.
- The KdV case: $\beta = 2$ and Ω arbitrary.
- The Kaup-Kuperschmit case: $\beta = 16/3$ and $\Omega = 15 \delta$.

We recall that in the KdV case the integral of the motion is quadratic in the momenta, while in the Sawada-Kotera and Kaup-Kuperschmit case is a quartic one. Nevertheless, the separability of the latter Hamiltonian has been shown through a very involved canonical transformation in [161].

6.4.1 The Sawada-Kotera case

The 2D Hamiltonian is

$$\mathcal{H}^{(2)} = \frac{1}{2}(p_1^2 + p_2^2) + \delta(q_1^2 + q_2^2) + \alpha (q_1^2 q_2 + \frac{1}{3} q_2^3). \quad (6.16)$$

If we consider the $N = 2$ realization (6.6) and (6.7) of the algebra $sl(2, \mathbb{R}) \oplus h_6$ we can write:

$$\mathcal{H} = \frac{1}{2}(J_+ + B_+) + \delta(J_- + A_-^2) + \alpha \left(J_- A_- + \frac{1}{3} A_-^3 \right). \quad (6.17)$$

On the other hand, the integral of the motion for (6.16) is given by [117]

$$\begin{aligned}
\mathcal{I}^{(2)} &= -\frac{1}{4}(p_1^4 + p_2^4) - (p_1^2 + p_2^2)(\delta q_1^2 + \delta q_2^2) - \delta^2(q_1^4 + q_2^4) \\
&\quad - \alpha q_1^2 q_2(p_1^2 + p_2^2) - \alpha \frac{1}{3} q_2^3(p_1^2 + p_2^2) - \alpha \delta \frac{2}{3} q_2^3(q_1^2 + q_2^2) - \alpha \delta \frac{4}{3} q_1^4 q_2 \\
&\quad + \alpha^2 \frac{1}{9} \left(\frac{1}{2} q_1^6 - q_2^6 \right) - \alpha^2 q_1^2 q_2^2 \left(\frac{2}{3} q_1^2 + \frac{1}{6} q_2^2 \right) + 2\delta p_1 p_2 q_1 q_2 \\
&\quad + \alpha p_1 p_2 q_1 \left(\frac{1}{3} q_1^2 + q_2^2 \right)
\end{aligned} \tag{6.18}$$

which can also be written in terms of the coalgebra generators as

$$\begin{aligned}
\mathcal{I} &= -2\alpha A_- J_3^2 - 3\delta A_+^2 A_-^2 - \alpha A_+^2 A_-^3 - 3\delta^2 A_-^4 - 2\alpha \delta A_-^5 - \frac{\alpha^2}{3} A_-^6 \\
&\quad - 3\delta A_+^2 J_- - 3\alpha A_+^2 A_- J_- - 2\alpha \delta A_-^3 J_- - \frac{\alpha^2}{2} A_-^4 J_- \\
&\quad - 3\delta^2 J_-^2 - 4\alpha \delta A_- J_-^2 - 2\alpha^2 A_-^2 J_-^2 + \frac{\alpha^2}{6} J_-^3 \\
&\quad + 6\delta A_+ A_- J_3 + 3\alpha A_+ A_-^2 J_3 + \alpha A_+ J_- J_3 \\
&\quad - J_+(A_-^2 + J_-)(3\delta + \alpha A_-) - \frac{3}{4} J_+^2 - \frac{3}{4} A_+^4.
\end{aligned} \tag{6.19}$$

By making use of the Poisson algebra (6.4) and (6.5), a straightforward computation shows that (6.17) and (6.19) are in involution, and this will be true for any symplectic realization of the $sl(2, \mathbb{R}) \oplus h_6$ Poisson algebra that we could consider. Therefore, we take (6.17) as the algebraic definition of the Sawada-Kotera Hamiltonian, and the specialization of (6.17) and (6.19) through the symplectic realization (6.6)-(6.7) will provide the explicit form of the ND generalization of this Hamiltonian.

Let us explicitly write the 3D and ND systems:

- **The 3D case.** By taking $N = 3$ in (6.6)-(6.7) and, afterwards, by substituting these expressions into (6.17)-(6.19) we get:

$$\mathcal{H}^{(3)} = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + \delta(q_1^2 + q_2^2 + q_3^2) + \alpha q_3(q_1^2 + q_2^2) + \alpha \frac{1}{3} q_3^3. \tag{6.20}$$

Note that (and this is a general fact in the formalism here presented) that we are obtaining a ‘radial’ generalization of the Sawada-Kotera potential in the (q_1, q_2) subspace.

- **The ND case.** The same construction leads to the Hamiltonian

$$\mathcal{H}^{(N)} = \sum_{i=1}^N \frac{1}{2} p_i^2 + \delta \sum_{i=1}^N q_i^2 + \alpha \left(q_N \sum_{i=1}^{N-1} q_i^2 + \frac{1}{3} q_N^3 \right) \tag{6.21}$$

and to the following integral of the motion coming from (6.19) is now

$$\begin{aligned}
\mathcal{I}^{(N)} &= -2\alpha q_N \left(\sum_{i=1}^{N-1} p_i q_i \right)^2 - p_N^2 q_N^2 (3\delta + \alpha q_N) - q_N^4 (3\delta^2 + 2\alpha\delta q_N) \\
&\quad - \frac{\alpha^2}{3} q_N^6 - 3\delta p_N^2 \left(\sum_{i=1}^{N-1} q_i^2 \right) - 3\alpha p_N^2 q_N \left(\sum_{i=1}^{N-1} q_i^2 \right) \\
&\quad - 2\alpha\delta q_N^3 \left(\sum_{i=1}^{N-1} q_i^2 \right) - \frac{\alpha^2}{2} q_N^4 \left(\sum_{i=1}^{N-1} q_i^2 \right) - 3\delta^2 \left(\sum_{i=1}^{N-1} q_i^2 \right)^2 \\
&\quad - 4\alpha\delta q_N \left(\sum_{i=1}^{N-1} q_i^2 \right)^2 - 2\alpha^2 q_N^2 \left(\sum_{i=1}^{N-1} q_i^2 \right)^2 + \frac{\alpha^2}{6} \left(\sum_{i=1}^{N-1} q_i^2 \right)^3 \\
&\quad - \left(\sum_{i=1}^{N-1} p_i^2 \right) (3\delta + \alpha q_N) \left[q_N^2 + \left(\sum_{i=1}^{N-1} q_i^2 \right) \right] - \frac{3}{4} \left(\sum_{i=1}^{N-1} p_i^2 \right)^2 - \frac{3}{4} p_N^4.
\end{aligned} \tag{6.22}$$

6.4.2 The KdV case

In this case the Hénon-Heiles Hamiltonian is given by

$$\mathcal{H}^{(2)} = \frac{1}{2}(p_1^2 + p_2^2) + \delta q_1^2 + (\delta + \Omega) q_2^2 + \alpha (q_1^2 q_2 + 2 q_2^3) \tag{6.23}$$

and its integral of the motion is given by [117]

$$\mathcal{I}^{(2)} = p_1^2 \left(\frac{3}{2}\delta - \frac{\Omega}{2} - \alpha q_2 \right) + \alpha p_2 p_1 q_1 + 3\delta^2 q_1^2 + \frac{\alpha^2}{4} q_1^4 + 2\alpha\delta q_2 q_1^2 + \alpha^2 q_2^2 q_1^2 - \delta\Omega q_1^2. \tag{6.24}$$

This system can be written in a $sl(2) \otimes h_6$ invariant form as

$$\mathcal{H} = \frac{1}{2}(J_+ + B_+) + \delta(J_- + A_-^2) + \Omega A_-^2 + \alpha(J_- A_- + 2A_-^3) \tag{6.25}$$

together with the following integral of the motion

$$\mathcal{I} = J_+ \left(\frac{3}{2}\delta - \frac{\Omega}{2} - \alpha A_- \right) + \alpha J_3 A_+ + 3\delta^2 J_- + \frac{\alpha^2}{4} J_-^2 + 2\alpha\delta J_- A_- + \alpha^2 J_- A_-^2 - \delta\Omega J_- \tag{6.26}$$

whose involutivity with respect to (6.25) is easily shown by using the abstract Poisson brackets for this algebra.

Now, the symplectic realization (6.6)-(6.7) provides the expression for the ND system ($N \geq 2$):

$$\mathcal{H}^{(N)} = \sum_{i=1}^{N-1} \left(\frac{p_i^2}{2} + \delta q_i^2 \right) + \left(\frac{p_N^2}{2} + (\delta + \Omega) q_N^2 \right) + \alpha \left(q_N \sum_{i=1}^{N-1} q_i^2 + 2 q_N^3 \right) \quad (6.27)$$

and the integral of the motion reads

$$\begin{aligned} \mathcal{I}^{(N)} &= \left(\sum_{i=1}^{N-1} p_i^2 \right) \left(\frac{3}{2} \delta - \frac{\Omega}{2} - \alpha q_N \right) + \alpha p_N \sum_{i=1}^{N-1} p_i q_i + 3 \delta^2 \left(\sum_{i=1}^{N-1} q_i^2 \right) + \frac{\alpha^2}{4} \left(\sum_{i=1}^{N-1} q_i^2 \right)^2 \\ &\quad + 2 \alpha \delta q_N \left(\sum_{i=1}^{N-1} q_i^2 \right) + \alpha^2 q_N^2 \left(\sum_{i=1}^{N-1} q_i^2 \right) - \delta \Omega \left(\sum_{i=1}^{N-1} q_i^2 \right). \end{aligned} \quad (6.28)$$

This, together with the coalgebra integrals (6.9), prove the complete integrability of the system.

6.4.3 The Kaup-Kuperschmit case

Finally, we consider the 2D Hamiltonian

$$\mathcal{H}^{(2)} = \frac{1}{2} (p_1^2 + p_2^2) + \delta q_1^2 + 16 \delta q_2^2 + \alpha \left(q_1^2 q_2 + \frac{16}{3} q_2^3 \right) \quad (6.29)$$

whose invariant reads [117]

$$\begin{aligned} \mathcal{I}^{(2)} &= -\frac{3}{4} p_1^4 - 3 \delta p_1^2 q_1^2 + \alpha p_1 p_2 q_1^3 - 3 \delta^2 q_1^4 + \frac{\alpha^2}{6} q_1^6 - 3 \alpha p_1^2 q_1^2 q_2 \\ &\quad + 2 \alpha \delta q_1^4 q_2 + \alpha^2 q_1^4 q_2^2. \end{aligned} \quad (6.30)$$

The very same procedure gives us the $sl(2, \mathbb{R}) \oplus h_6$ invariant object:

$$\mathcal{H} = \frac{1}{2} (J_+ + B_+) + \delta (J_- + 16 A_-^2) + \alpha \left(J_- A_- + \frac{16}{3} A_-^3 \right) \quad (6.31)$$

together with

$$\begin{aligned} \mathcal{I} &= -J_+ \left(\frac{3}{4} J_+ + 3 \delta J_- + \alpha A_- J_- \right) + J_-^2 (-3 \delta^2 + 2 \alpha \delta A_- + \alpha^2 A_-^2) \\ &\quad + \frac{\alpha^2}{6} J_-^3 + \alpha J_3 (A_+ J_- - 2 \alpha A_- J_3). \end{aligned} \quad (6.32)$$

Starting from the latter expressions, the ND generalization of the Kaup-Kuperschmidt Hamiltonian reads ($N \geq 2$)

$$\mathcal{H}^{(N)} = \frac{1}{2} \sum_{i=1}^N p_i^2 + \delta \sum_{i=1}^{N-1} q_i^2 + 16 \delta q_N^2 + \left(\alpha q_N \sum_{i=1}^{N-1} q_i^2 + \frac{16\alpha}{3} q_N^3 \right) \quad (6.33)$$

$$\begin{aligned} \mathcal{I}^{(N)} = & - \left(\sum_{i=1}^{N-1} p_i^2 \right) \left\{ \frac{3}{4} \sum_{i=1}^{N-1} p_i^2 + 3\delta \sum_{i=1}^{N-1} q_i^2 + \alpha q_N \sum_{i=1}^{N-1} q_i^2 \right\} \\ & + \left(\sum_{i=1}^{N-1} q_i^2 \right)^2 \left\{ -3\delta^2 + 2\alpha\delta q_N + \alpha^2 q_N^2 \right\} + \frac{\alpha^2}{6} \left(\sum_{i=1}^{N-1} q_i^2 \right)^3 \\ & + \alpha \left(\sum_{i=1}^{N-1} p_i q_i \right) \left\{ p_N \sum_{i=1}^{N-1} q_i^2 - 2q_N \sum_{i=1}^{N-1} p_i q_i \right\}. \end{aligned} \quad (6.34)$$

6.5 ND Ramani potentials

An interesting family of examples is given by any of the Ramani potentials $\mathcal{V}_M(x^2, y)$, which are homogeneous polynomial potentials with degree M and given by [106, 154]

$$\mathcal{V}_M(x, y) = \sum_{k=0}^{\lfloor \frac{M}{2} \rfloor} 2^{M-2k} \binom{M-k}{k} x^{2k} y^{M-2k}. \quad (6.35)$$

The first ones are

$$\mathcal{V}_1 = 2y \quad (6.36)$$

$$\mathcal{V}_2 = 4y^2 + x^2 \quad (6.37)$$

$$\mathcal{V}_3 = 8y^3 + 4x^2y \quad (6.38)$$

$$\mathcal{V}_4 = 16y^4 + 12x^2y^2 + x^4 \quad (6.39)$$

$$\mathcal{V}_5 = 32y^5 + 32x^2y^3 + 6x^4y \quad (6.40)$$

$$\mathcal{V}_6 = 64y^6 + 80x^2y^4 + 24x^4y^2 + x^6. \quad (6.41)$$

Note that \mathcal{V}_2 is just the 2D harmonic oscillator with frequency ratio 1 : 4 and \mathcal{V}_3 is proportional to the KdV Hénon-Heiles potential. In fact, if we write the \mathcal{V}_M Ramani Hamiltonian as

$$\mathcal{H}^{(2)} = \frac{1}{2} (p_1^2 + p_2^2) + \mathcal{V}_M(q_1^2, q_2) = \frac{1}{2} (p_1^2 + p_2^2) + \sum_{k=0}^{\lfloor \frac{M}{2} \rfloor} 2^{M-2k} \binom{M-k}{k} q_1^{2k} q_2^{M-2k} \quad (6.42)$$

its integral of the motion (for $M \geq 2$) is given by [106]

$$\mathcal{I}^{(2)} = -q_2 p_1^2 + q_1 p_1 p_2 + q_1^2 \mathcal{V}_{M-1} \quad (6.43)$$

and for $M = 1$ the Hamiltonian is directly separable in the (x, y) coordinates. Moreover, it is well known all these potentials can be superposed freely without destroying their integrability (see [106]). In particular, it can be proven that the Hamiltonian with (non homogeneous) polynomial potential of degree R given by

$$\mathcal{H}_R^{(2)} = \frac{1}{2} (p_1^2 + p_2^2) + \sum_{i=2}^R \alpha_i \mathcal{V}_i(q_1^2, q_2) \quad (6.44)$$

with α_i being arbitrary constants, Poisson-commutes with

$$\mathcal{I}_R^{(2)} = -q_2 p_1^2 + q_1 p_1 p_2 + q_1^2 \sum_{i=2}^R \alpha_i \mathcal{V}_{i-1}(q_1^2, q_2). \quad (6.45)$$

The Hamiltonian \mathcal{H}_R (6.44) can be immediately written in terms of $sl(2, \mathbb{R}) \oplus h_6$ as follows:

$$\mathcal{H}_R = \frac{1}{2} (J_+ + B_+) + \sum_{i=2}^R \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} \alpha_i 2^{i-2k} \binom{i-k}{k} J_-^k A_-^{i-2k}. \quad (6.46)$$

and in terms of the algebra generators we will have the following integral

$$\mathcal{I}_R = -A_- J_+ + J_3 A_+ + J_- \sum_{i=2}^R \sum_{k=0}^{\lfloor \frac{i-1}{2} \rfloor} \alpha_i 2^{i-1-2k} \binom{i-1-k}{k} J_-^k A_-^{i-1-2k} \quad (6.47)$$

whose Poisson involutivity with respect to (6.46) can be proven by direct computation.

Therefore, the ND generalization of the Ramani Hamiltonian will be given by taking the corresponding ND symplectic realization of (6.46), which reads

$$\mathcal{H}_R^{(N)} = \frac{1}{2} \sum_{i=1}^N p_i^2 + \sum_{i=2}^R \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} \alpha_i 2^{i-2k} \binom{i-k}{k} \left(\sum_{j=1}^{N-1} q_j^2 \right)^k q_N^{i-2k}. \quad (6.48)$$

This is indeed a new completely integrable system, since it commutes by construction with the $(N-2)$ integrals (6.9) and with the ND realization of (6.47), namely

$$\begin{aligned} \mathcal{I}_R^{(N)} = & -q_N \left(\sum_{i=1}^{N-1} p_i^2 \right) + p_N \left(\sum_{i=1}^{N-1} p_i q_i \right) \\ & + \left(\sum_{l=1}^{N-1} q_l^2 \right) \left(\sum_{i=2}^R \sum_{k=0}^{\lfloor \frac{i-1}{2} \rfloor} \alpha_i 2^{i-1-2k} \binom{i-1-k}{k} \left(\sum_{j=1}^{N-1} q_j^2 \right)^k q_N^{i-1-2k} \right). \end{aligned} \quad (6.49)$$

6.5.1 The ND Ramani Hamiltonian \mathcal{H}_6

In order to illustrate the previous result, we shall consider in the sequel the $R = 6$ case, that can be considered as an integrable sextic deformation of the KdV Hénon-Heiles system. In this case the initial 2D Ramani system (6.44) will be given by

$$\begin{aligned} \mathcal{H}_6^{(2)} &= \frac{1}{2}(p_1^2 + p_2^2) + \alpha_2(q_1^2 + 4q_2^2) + \alpha_3(4q_1^2q_2 + 8q_2^3) \\ &\quad + \alpha_4(q_1^4 + 12q_1^2q_2^2 + 16q_2^4) + \alpha_5(6q_1^4q_2 + 32q_1^2q_2^3 + 32q_2^5) \\ &\quad + \alpha_6(q_1^6 + 24q_1^4q_2^2 + 80q_1^2q_2^4 + 64q_2^6) \end{aligned} \quad (6.50)$$

and its invariant (6.45) is

$$\begin{aligned} \mathcal{I}^{(2)} &= p_2p_1q_1 - q_2(p_1^2 - 2\alpha_2q_1^2) + \alpha_3q_1^4 + 4\alpha_3q_1^2q_2^2 + 4\alpha_4q_1^4q_2 + 8\alpha_4q_1^2q_2^3 \\ &\quad + \alpha_5q_1^6 + 12\alpha_5q_1^4q_2^2 + 16\alpha_5q_1^2q_2^4 + 6\alpha_6q_1^6q_2 + 32\alpha_6q_1^4q_2^3 + 32\alpha_6q_1^2q_2^5. \end{aligned} \quad (6.51)$$

A straightforward computation shows that the corresponding algebraic expressions in terms of the $sl(2, \mathbb{R}) \oplus h_6$ Poisson algebra generators read

$$\begin{aligned} \mathcal{H}_6 &= \frac{1}{2}(J_+ + B_+) + \alpha_2(J_- + 4A_-^2) + 4\alpha_3(J_-A_- + 2A_-^3) \\ &\quad + \alpha_4(J_-^2 + 12J_-A_-^2 + 16A_-^4) + \alpha_5(6J_-^2A_- + 32J_-A_-^3 + 32A_-^5) \\ &\quad + \alpha_6(J_-^3 + 24J_-^2A_-^2 + 80J_-A_-^4 + 64A_-^6) \end{aligned} \quad (6.52)$$

$$\begin{aligned} \mathcal{I}_6 &= A_+J_3 - A_-J_+ + 2\alpha_2A_-J_- + \alpha_3J_-^2 + 4\alpha_3J_-A_-^2 + 4\alpha_4J_-^2A_- \\ &\quad + 8\alpha_4J_-A_-^3 + \alpha_5J_-^3 + 12\alpha_5J_-^2A_-^2 + 16\alpha_5J_-A_-^4 \\ &\quad + 6\alpha_6J_-^3A_- + 32\alpha_6J_-^2A_-^3 + 32\alpha_6J_-A_-^5. \end{aligned} \quad (6.53)$$

By using these expressions it is immediate to write the 3D generalization of this system as

$$\begin{aligned} \mathcal{H}_6^{(3)} &= \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + \alpha_2(q_1^2 + q_2^2 + 4q_3^2) \\ &\quad + 4\alpha_3(q_3(q_1^2 + q_2^2) + 2q_3^3) + \alpha_4(q_1^4 + 2q_1^2q_2^2 + q_2^4 + 12q_1^2q_3^2 + 12q_2^2q_3^2 + 16q_3^4) \\ &\quad + \alpha_5(6q_1^4q_3 + 12q_1^2q_2^2q_3 + 6q_2^4q_3 + 32q_1^2q_3^3 + 32q_2^2q_3^3 + 32q_3^5) \\ &\quad + \alpha_6(q_1^6 + 3q_1^4q_2^2 + 3q_1^2q_2^4 + q_2^6 + 48q_1^2q_2^2q_3^2 + 24q_2^4q_3^2 + 24q_1^2q_3^2) \\ &\quad + \alpha_6(80q_1^2q_3^4 + 80q_2^2q_3^4 + 64q_3^6) \end{aligned} \quad (6.54)$$

$$\begin{aligned}
\mathcal{I}_6^{(3)} &= p_3(p_1q_1 + p_2q_2) - q_3(p_1^2 - 2\alpha_2q_1^2 + p_2^2 - 2\alpha_2q_2^2) \\
&\quad + \alpha_3(q_1^4 + q_2^4) + 2\alpha_3q_1^2q_2^2 + 4\alpha_3q_3(q_1^2 + q_2^2) + 4\alpha_4q_3(q_1^4 + q_2^4) \\
&\quad + 8\alpha_4q_3^3(q_1^2 + q_2^2) + 8\alpha_4q_1^2q_2^2q_3 + \alpha_5(q_1^6 + q_2^6) + 3\alpha_5q_1^4q_2^2 + 3\alpha_5q_1^2q_2^4 \\
&\quad + 12\alpha_5q_3^2(q_1^4 + q_2^4) + 16\alpha_5q_3^4(q_1^2 + q_2^2) + 24\alpha_5q_1^2q_2^2q_3^2 + 64\alpha_6q_1^2q_2^2q_3^3 \\
&\quad + 6\alpha_6q_3(q_1^6 + q_2^6) + 18\alpha_6q_3(q_1^2q_2^4 + q_1^4q_2^2) + 32\alpha_6q_3^3(q_1^4 + q_2^4) \\
&\quad + 32\alpha_6q_3^5(q_1^2 + q_2^2). \tag{6.55}
\end{aligned}$$

And the ND case would be

$$\begin{aligned}
\mathcal{H}_6^{(N)} &= \frac{1}{2} \sum_{i=1}^N p_i^2 + \alpha_2 \left(\sum_{i=1}^{N-1} q_i^2 + 4q_N^2 \right) + 4\alpha_3 \left(q_N \sum_{i=1}^{N-1} q_i^2 + 2q_N^3 \right) \\
&\quad + \alpha_4 \left(\left\{ \sum_{i=1}^{N-1} q_i^2 \right\}^2 + 12q_N^2 \sum_{i=1}^{N-1} q_i^2 + 16q_N^4 \right) \\
&\quad + \alpha_5 \left(6q_N \left\{ \sum_{i=1}^{N-1} q_i^2 \right\}^2 + 32q_N^3 \sum_{i=1}^{N-1} q_i^2 + 32q_N^5 \right) \\
&\quad + \alpha_6 \left(\left\{ \sum_{i=1}^{N-1} q_i^2 \right\}^3 + 24q_N^2 \left\{ \sum_{i=1}^{N-1} q_i^2 \right\}^2 + 80q_N^4 \sum_{i=1}^{N-1} q_i^2 + 64q_N^6 \right) \tag{6.56}
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_6^{(N)} &= p_N \sum_{i=1}^{N-1} p_i q_i - q_N \sum_{i=1}^{N-1} p_i^2 + 2\alpha_2 q_N \sum_{i=1}^{N-1} q_i^2 + \alpha_3 \left(\sum_{i=1}^{N-1} q_i^2 \right)^2 + 4\alpha_3 q_N^2 \left(\sum_{i=1}^{N-1} q_i^2 \right) \\
&\quad + 4\alpha_4 q_N \left(\sum_{i=1}^{N-1} q_i^2 \right)^2 + 8\alpha_4 q_N^3 \left(\sum_{i=1}^{N-1} q_i^2 \right) + \alpha_5 \left(\sum_{i=1}^{N-1} q_i^2 \right)^3 + 12\alpha_5 q_N^2 \left(\sum_{i=1}^{N-1} q_i^2 \right)^2 \\
&\quad + 16\alpha_5 q_N^4 \left(\sum_{i=1}^{N-1} q_i^2 \right) + 6\alpha_6 q_N \left(\sum_{i=1}^{N-1} q_i^2 \right)^3 + 32\alpha_6 q_N^3 \left(\sum_{i=1}^{N-1} q_i^2 \right)^2 \\
&\quad + 32\alpha_6 q_N^5 \left(\sum_{i=1}^{N-1} q_i^2 \right). \tag{6.57}
\end{aligned}$$

As a conclusion, we can appreciate from the examples here shown that the $sl(2, \mathbb{R}) \oplus h_6$ symmetry allows for a systematic and straightforward construction of new integrable ND Hamiltonians starting from known 2D ones.

6.6 Adding centrifugal terms

The previous construction in which a given 2D integrable Hamiltonian is written as a function of the generators of the Poisson algebra $sl(2, \mathbb{R}) \oplus h_6$ has also an interesting byproduct: the possibility of adding centrifugal terms to the corresponding ND Hamiltonian without breaking the integrability.

This can be achieved by considering the following ND symplectic realization of $sl(2, \mathbb{R}) \oplus h_6$:

$$J_+ = \sum_{i=1}^{N-1} \left(p_i^2 + \frac{b_i}{q_i^2} \right) \quad J_- = \sum_{i=1}^{N-1} q_i^2 \quad J_3 = \sum_{i=1}^{N-1} q_i p_i \quad (6.58)$$

$$\begin{aligned} A_+ &= p_N & A_- &= q_N & K &= q_N p_N - \frac{1}{2} \\ B_+ &= p_N^2 & B_- &= q_N^2 & M &= 1. \end{aligned} \quad (6.59)$$

This would be the $(b_1, b_2, \dots, b_{N-1}) \oplus (1, 0)$ symplectic realization of $sl(2, \mathbb{R}) \oplus h_6$, and its only difference with respect to (6.6) and (6.7) is the addition of $(N-1)$ centrifugal terms in the realization of the J_+ generator.

We know that when these terms are included, the constants of the motion (6.9) coming from the $sl(2, \mathbb{R})$ -sector of the Hamiltonian are converted into

$$\mathcal{C}^{(m)} = \sum_{1 \leq i < j}^m (q_i p_j - q_j p_i)^2 + \sum_{1 \leq i < j}^m \left(b_i \frac{q_j^2}{q_i^2} + b_j \frac{q_i^2}{q_j^2} \right) \quad m = 2, \dots, N-1. \quad (6.60)$$

But it is essential to stress that any $sl(2, \mathbb{R}) \oplus h_6$ Hamiltonian, when written in terms of the symplectic realization (6.58)-(6.59) will Poisson-commute with all the integrals (6.60). And the same will happens to the additional integral \mathcal{I} , that will be modified only through the centrifugal terms coming from the J_+ generator.

This elementary discussion shows that all the ND Hamiltonians with potentials of the type $\mathcal{V}(J_-, A_-)$ so far introduced in this Chapter can be supplemented with $(N-1)$ centrifugal terms and giving rise to ‘generalized’ Hamiltonians in the form

$$\mathcal{H}_b^{(N)} = \frac{1}{2} \sum_{i=1}^{N-1} \left(p_i^2 + \frac{b_i}{q_i^2} \right) + \frac{1}{2} p_N^2 + \mathcal{V} \left(\sum_{i=1}^{N-1} q_i^2, q_N \right) \quad (6.61)$$

and whose full set of integrals of the motion for \mathcal{H}_b can be obtained as a consequence of their $sl(2, \mathbb{R}) \oplus h_6$ symmetry. Let us make this statement more explicit through two particular examples.

The generalized ND Sawada-Kotera Hamiltonian

If we consider the symplectic realization (6.58)-(6.59), the ND Sawada-Kotera Hamiltonian (6.16) is generalized as

$$\mathcal{H}_b^{(N)} = \frac{1}{2} \sum_{i=1}^{N-1} \left(p_i^2 + \frac{b_i}{q_i^2} \right) + \frac{1}{2} p_N^2 + \delta \sum_{i=1}^N q_i^2 + \alpha \left(q_N \sum_{i=1}^{N-1} q_i^2 + \frac{1}{3} q_N^3 \right) \quad (6.62)$$

and the integral of the motion coming from (6.19) now reads

$$\begin{aligned} \mathcal{I}^{(N)} = & -2\alpha q_N \left(\sum_{i=1}^{N-1} p_i q_i \right)^2 - p_N^2 q_N^2 (3\delta + \alpha q_N) - q_N^4 (3\delta^2 + 2\alpha\delta q_N) \\ & - \frac{\alpha^2}{3} q_N^6 - 3\delta p_N^2 \left(\sum_{i=1}^{N-1} q_i^2 \right) - 3\alpha p_N^2 q_N \left(\sum_{i=1}^{N-1} q_i^2 \right) \\ & - 2\alpha\delta q_N^3 \left(\sum_{i=1}^{N-1} q_i^2 \right) - \frac{\alpha^2}{2} q_N^4 \left(\sum_{i=1}^{N-1} q_i^2 \right) - 3\delta^2 \left(\sum_{i=1}^{N-1} q_i^2 \right)^2 \\ & - 4\alpha\delta q_N \left(\sum_{i=1}^{N-1} q_i^2 \right)^2 - 2\alpha^2 q_N^2 \left(\sum_{i=1}^{N-1} q_i^2 \right)^2 + \frac{\alpha^2}{6} \left(\sum_{i=1}^{N-1} q_i^2 \right)^3 \\ & - \left[\sum_{i=1}^{N-1} \left(p_i^2 + \frac{b_i}{q_i^2} \right) (3\delta + \alpha q_N) \right] \left[q_N^2 + \left(\sum_{i=1}^{N-1} q_i^2 \right) \right] \\ & - \frac{3}{4} \left[\sum_{i=1}^{N-1} \left(p_i^2 + \frac{b_i}{q_i^2} \right) (3\delta + \alpha q_N) \right]^2 - \frac{3}{4} p_N^4. \end{aligned} \quad (6.63)$$

The fact that both quantities are in involution can be straightforwardly checked.

The generalized ND Ramani Hamiltonian

Just in the same way, the Hamiltonian (6.46) is written in the $(b_1, b_2, \dots, b_{N-1}) \oplus (1, 0)$ symplectic realization as

$$\mathcal{H}_R^{(N)} = \frac{1}{2} \sum_{i=1}^{N-1} \left(p_i^2 + \frac{b_i}{q_i^2} \right) + \frac{1}{2} p_N^2 + \sum_{i=2}^R \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} \alpha_i 2^{i-2k} \binom{i-k}{k} \left(\sum_{j=1}^{N-1} q_j^2 \right)^k q_N^{i-2k} \quad (6.64)$$

and the integral of the motion that incorporates the contributions coming from the centrifugal terms arises by taking the same symplectic realization of (6.47), namely

$$\begin{aligned} \mathcal{I}_R^{(N)} = & -q_N \sum_{i=1}^{N-1} \left(p_i^2 + \frac{b_i}{q_i^2} \right) + p_N \left(\sum_{i=1}^{N-1} p_i q_i \right) \\ & + \left(\sum_{l=1}^{N-1} q_l^2 \right) \left(\sum_{i=2}^R \sum_{k=0}^{\lfloor \frac{i-1}{2} \rfloor} \alpha_i 2^{i-1-2k} \binom{i-1-k}{k} \left(\sum_{j=1}^{N-1} q_j^2 \right)^k q_N^{i-1-2k} \right). \end{aligned} \quad (6.65)$$

The same procedure can be systematically performed onto any Hamiltonian of the type (6.13) and its corresponding integral (6.14).

6.7 The $sl(2, \mathbb{R}) \oplus sl(2, \mathbb{R})$ construction

It seems also natural to wonder whether a 2D integrable Hamiltonian of the form

$$\mathcal{H}^{(2)} = \frac{1}{2}(p_x^2 + p_y^2) + \mathcal{V}(x^2, y^2) \quad (6.66)$$

could have any additional interesting feature from the point of view of its generalization to N dimensions. This is strongly motivated by the observation that, as described in in [106], the 2D integrable potentials that admit *two* centrifugal terms without altering the integrability are just functions of the type $\mathcal{V}(x^2, y^2)$.

The answer to this question is affirmative, and a similar coalgebra scheme based on the $sl(2, \mathbb{R}) \oplus sl(2, \mathbb{R})$ Poisson algebra can be introduced. This algebra is given by the following non-vanishing brackets

$$\begin{aligned} \{J_3, J_+\} = 2J_+ & \quad \{J_3, J_-\} = -2J_- & \quad \{J_-, J_+\} = 4J_3 \\ \{K_3, K_+\} = 2K_+ & \quad \{K_3, K_-\} = -2K_- & \quad \{K_-, K_+\} = 4K_3 \end{aligned} \quad (6.67)$$

and a possible ND symplectic realization for this Poisson algebra is

$$J_+ = \sum_{i=1}^M p_i^2 \quad J_- = \sum_{i=1}^M q_i^2 \quad J_3 = \sum_{i=1}^M q_i p_i \quad (6.68)$$

$$K_+ = \sum_{i=M+1}^N p_i^2 \quad K_- = \sum_{i=M+1}^N q_i^2 \quad K_3 = \sum_{i=M+1}^N q_i p_i. \quad (6.69)$$

Obviously, the Hamiltonian (6.66) is just the $M = 1, N = 2$ symplectic realization for the abstract $sl(2, \mathbb{R}) \oplus sl(2, \mathbb{R})$ object

$$\mathcal{H} = \frac{1}{2}(J_+ + B_+) + \mathcal{V}(J_-, K_-) \quad (6.70)$$

and if the integral of the motion $\mathcal{I}^{(2)}$ for (6.66) can be written as a function

$$\mathcal{I}^{(2)} = \mathcal{I}^{(2)}(p_x^2, x^2, x p_x, p_y^2, y^2, y p_y) \quad (6.71)$$

this integral can be also written in abstract algebraic terms as

$$\mathcal{I} = \mathcal{I}(J_+, J_-, J_3, K_+, K_-, K_3). \quad (6.72)$$

From this viewpoint, the ND generalization of the Hamiltonian (6.66) will be given by the ND symplectic realization of (6.68) and (6.69):

$$\mathcal{H}^{(N)} = \frac{1}{2} \left(\sum_{i=1}^M p_i^2 + \sum_{i=M+1}^N p_i^2 \right) + \mathcal{V} \left(\sum_{i=1}^M q_i^2, \sum_{i=M+1}^N q_i^2 \right). \quad (6.73)$$

The complete integrability of this Hamiltonian is guaranteed by the existence of the following functionally independent integrals: the ND symplectic realization of the ‘abstract’ integral \mathcal{I}

$$\mathcal{I}^{(N)} = \mathcal{I} \left(\sum_{i=1}^M p_i^2, \sum_{i=1}^M q_i^2, \sum_{i=1}^M q_i p_i, \sum_{i=M+1}^N p_i^2, \sum_{i=M+1}^N q_i^2, \sum_{i=M+1}^N q_i p_i \right), \quad (6.74)$$

the $(M - 1)$ integrals coming from the first $sl(2, \mathbb{R})$ sector

$$\mathcal{C}_b^{(m)} = \sum_{1 \leq i < j}^m (q_i p_j - q_j p_i)^2 \quad m = 2, \dots, M \quad (6.75)$$

and another set of $(N - M - 1)$ integrals coming from the second $sl(2, \mathbb{R})$ sector

$$\mathcal{D}_b^{(k)} = \sum_{M+1 \leq i < j}^{M+k} (q_i p_j - q_j p_i)^2 \quad k = 2, \dots, N - M. \quad (6.76)$$

Finally, it is immediate to realize that centrifugal terms can be also included in this case by making use of the same ‘abstract’ Hamiltonian (6.79), but now under the

$(b_1, b_2, \dots, b_M) \oplus (b_{M+1}, b_{M+2}, \dots, b_N)$ symplectic realization

$$J_+ = \sum_{i=1}^M \left(p_i^2 + \frac{b_i}{q_i^2} \right) \quad J_- = \sum_{i=1}^M q_i^2 \quad J_3 = \sum_{i=1}^M q_i p_i \quad (6.77)$$

$$K_+ = \sum_{i=M+1}^N \left(p_i^2 + \frac{b_i}{q_i^2} \right) \quad K_- = \sum_{i=M+1}^N q_i^2 \quad K_3 = \sum_{i=M+1}^N q_i p_i. \quad (6.78)$$

This gives rise to the ND Hamiltonian

$$\mathcal{H}_b^{(N)} = \frac{1}{2} \sum_{i=1}^M \left(p_i^2 + \frac{b_i}{q_i^2} \right) + \frac{1}{2} \sum_{i=M+1}^N \left(p_i^2 + \frac{b_i}{q_i^2} \right) + \mathcal{V} \left(\sum_{i=1}^M q_i^2, \sum_{i=M+1}^N q_i^2 \right) \quad (6.79)$$

that will Poisson-commute with the ND integral given by

$$\mathcal{I}_b^{(N)} = \mathcal{I} \left(\sum_{i=1}^M \left(p_i^2 + \frac{b_i}{q_i^2} \right), \sum_{i=1}^M q_i^2, \sum_{i=1}^M q_i p_i, \sum_{i=M+1}^N \left(p_i^2 + \frac{b_i}{q_i^2} \right), \sum_{i=M+1}^N q_i^2, \sum_{i=M+1}^N q_i p_i \right) \quad (6.80)$$

together with the integrals

$$\mathcal{C}_b^{(m)} = \sum_{1 \leq i < j}^m \left\{ (q_i p_j - q_j p_i)^2 + \left(b_i \frac{q_j^2}{q_i^2} + b_j \frac{q_i^2}{q_j^2} \right) \right\} \quad m = 2, \dots, M \quad (6.81)$$

$$\mathcal{D}_b^{(k)} = \sum_{M+1 \leq i < j}^{M+k} \left\{ (q_i p_j - q_j p_i)^2 + \left(b_i \frac{q_j^2}{q_i^2} + b_j \frac{q_i^2}{q_j^2} \right) \right\} \quad k = 2, \dots, N - M. \quad (6.82)$$

6.7.1 ND nonlinear oscillators with quartic coupling

As an example of this construction we consider the anharmonic coupled 2D oscillator considered as the system VI in [97]:

$$\mathcal{H}^{(2)} = \frac{1}{2} (p_1^2 + p_2^2) + (q_1^4 + 6q_1^2 q_2^2 + 8q_2^4) + a(q_1^2 + 4q_2^2) + \frac{\lambda}{q_1^2} + \frac{\mu}{q_2^2} \quad (6.83)$$

together with its 2D integral of the motion

$$\begin{aligned} \mathcal{I}^{(2)} &= p_1^4 + 4p_1^2 \left(q_1^4 + 6q_1^2 q_2^2 + aq_1^2 + \frac{\lambda}{q_1^2} \right) - 16q_1^3 p_1 q_2 p_2 + 4q_1^4 p_2^2 \\ &\quad + 4q_1^4 [q_1^4 + 4q_1^2 q_2^2 + 4q_2^4 + 2a(q_1^2 + 2q_2^2) + a^2] + \frac{8\mu q_1^4}{q_2^2} \\ &\quad + 8\lambda(q_1^2 + 2q_2^2) + \frac{4\lambda^2}{q_1^4}. \end{aligned} \quad (6.84)$$

Now, by considering the $(0) \oplus (0)$ symplectic realization of $sl(2, \mathbb{R}) \oplus sl(2, \mathbb{R})$, we can write the previous expressions in the algebraic form

$$\begin{aligned} \mathcal{H}_b &= \frac{1}{2} (J_+ + K_+) + (J_-^2 + 6J_-K_- + 8K_-^2) \\ &\quad + a(J_- + 4K_-) + \frac{\lambda}{J_-} + \frac{\mu}{K_-} \end{aligned} \quad (6.85)$$

and

$$\begin{aligned} \mathcal{I} &= J_+^2 + J_+ \left(\frac{4\lambda}{J_-} + 3aJ_- + 4J_-^2 + 8J_-K_- \right) + 4K_+J_-^2 \\ &\quad + \frac{4\lambda^2}{J_-^2} + 8\lambda J_- + 4a^2J_-^2 + 8aJ_-^3 + 4J_-^4 + aJ_-^5 \\ &\quad + \frac{8\mu J_-^2}{K_-} + 16\lambda K_- + 16J_-^2 K_- (a + J_- + K_-^2) \\ &\quad + 16J_3(J_3K_- - J_-K_3). \end{aligned} \quad (6.86)$$

A direct computation show that both functions are in involution with respect to the Poisson bracket (6.67).

The ND integrable generalization of (6.83), including all the possible centrifugal terms, is obtained by substituting in (6.85) and (6.86) the $(b_1, b_2, \dots, b_M) \oplus (b_{M+1}, b_{M+2}, \dots, b_N)$ symplectic realization. The resulting Hamiltonian is

$$\begin{aligned} \mathcal{H}_b^{(N)} &= \frac{1}{2} \sum_{i=1}^M \left(p_i^2 + \frac{b_i}{q_i^2} \right) + \frac{1}{2} \sum_{i=M+1}^N \left(p_i^2 + \frac{b_i}{q_i^2} \right) \\ &\quad + \left(\sum_{i=1}^M q_i^2 \right)^2 + 6 \left(\sum_{i=1}^M q_i^2 \right) \left(\sum_{i=M+1}^N q_i^2 \right) + 8 \left(\sum_{i=M+1}^N q_i^2 \right)^2 \\ &\quad + a \left(\sum_{i=1}^M q_i^2 + 4 \sum_{i=M+1}^N q_i^2 \right) + \frac{\lambda}{\left(\sum_{i=1}^M q_i^2 \right)} + \frac{\mu}{\left(\sum_{i=M+1}^N q_i^2 \right)}. \end{aligned} \quad (6.87)$$

The integrals of the motion for $\mathcal{H}_b^{(N)}$ are just the sets (6.81) and (6.82), together with the ND additional one in which the centrifugal terms are non-trivially intertwined within the integral (6.86):

$$\begin{aligned}
\mathcal{I}_b^{(N)} &= \left(\sum_{i=1}^M \left(p_i^2 + \frac{b_i}{q_i^2} \right) \right)^2 \\
&+ \left(\frac{4\lambda}{\sum_{i=1}^M q_i^2} + 3a \sum_{i=1}^M q_i^2 + 4 \left(\sum_{i=1}^M q_i^2 \right)^2 + 8 \left(\sum_{i=1}^M q_i^2 \right) \left(\sum_{i=M+1}^N q_i^2 \right) \right) \sum_{i=1}^M \left(p_i^2 + \frac{b_i}{q_i^2} \right) \\
&+ 4 \left(\sum_{i=M+1}^N p_i^2 + \frac{b_i}{q_i^2} \right) \left(\sum_{i=1}^M q_i^2 \right)^2 + \frac{4\lambda^2}{\left(\sum_{i=1}^M q_i^2 \right)^2} + 8\lambda \sum_{i=1}^M q_i^2 + 4a^2 \left(\sum_{i=1}^M q_i^2 \right)^2 \\
&+ 8a \left(\sum_{i=1}^M q_i^2 \right)^3 + 4 \left(\sum_{i=1}^M q_i^2 \right)^4 + a \left(\sum_{i=1}^M q_i p_i \right)^2 + \frac{8\mu \left(\sum_{i=1}^M q_i^2 \right)^2}{\sum_{i=M+1}^N q_i^2} + 16\lambda \sum_{i=M+1}^N q_i^2 \\
&+ 16 \left(\sum_{i=1}^M q_i^2 \right)^2 \left(\sum_{i=M+1}^N q_i^2 \right) \left(a + \left(\sum_{i=1}^M q_i^2 \right) + \left(\sum_{i=M+1}^N q_i^2 \right)^2 \right) \\
&+ 16 \left(\sum_{i=1}^M q_i p_i \right) \left(\left(\sum_{i=1}^M q_i p_i \right) \left(\sum_{i=M+1}^N q_i^2 \right) - \left(\sum_{i=1}^M q_i^2 \right) \left(\sum_{i=M+1}^N q_i p_i \right) \right). \quad (6.88)
\end{aligned}$$

Certainly, this expression would be really difficult to find without the aid of the underlying coalgebra symmetry. The very same procedure can be applied to obtain the ND integrable generalization (with and without centrifugal terms) of the System VII in [97].

Chapter 7

Conclusions and open problems

*“All truths are easy to understand once they are discovered
the point is to discover them.”*

G. Galilei

The main original results and conclusions of the work here presented can be summarized as follows:

- We have introduced a necessary condition for the generic symplectic realization of a given Poisson algebra in order to provide the integrability of the ND Hamiltonian systems that can be obtained from it through the coalgebra formalism.
- We have worked out explicitly the generic symplectic realizations for different real Lie-Poisson coalgebras with dimension 3, 4, 5 and 6, and we have introduced an infinite family of algebras for which the integrability criterion holds. In all these cases the corresponding integrable ND Hamiltonians have been presented, together with their explicit integrals of the motion.
- We have shown that the generic ND Hamiltonians defined through a one-particle symplectic realization of the two-photon coalgebra h_6 are quasi-integrable systems, since an extra integral that does not come from the coalgebra has to be found in order to achieve complete integrability.
- By using different algebraic techniques, we have identified several infinite families of two-photon Hamiltonians for which such remaining integral does exist, and we have found it explicitly. Among these systems we have found new families of ND natural systems, electromagnetic Hamiltonians and geodesic flows defining ND spaces with non-constant curvature.

- As particular examples of these integrable h_6 systems, we have found two new families of nonlinear integrable perturbations of the ND isotropic oscillator. The first one is a family of integrable perturbations depending on N parameters and two arbitrary functions including as particular cases several known quartic and sextic coupled nonlinear oscillators. The second family includes homogeneous functions with degree -2 in the coordinates and a radial arbitrary function. As a concrete example of this type we have proposed a new type of integrable ND generalized Calogero-Moser model.
- Finally, we have shown that the ND generalization of many known 2D integrable Hamiltonians can be achieved by making use of the direct sum of certain coalgebras. In particular, we have proven that any 2D integrable potential of the type $\mathcal{V}(x^2, y)$ can be generalized to ND by preserving its complete integrability, and a similar result is obtained for potentials of the type $\mathcal{V}(x^2, y^2)$. Moreover, centrifugal terms can be added onto the ND systems and as outstanding examples of this construction we have presented the ND generalized versions of the three integrable Hénon-Heiles systems, as well as of the Ramani series of potentials and we have obtained new ND systems of coupled nonlinear oscillators.

Open problems

Finally, in the sequel we will propose several open questions that, to our opinion, would deserve further investigation in the near future.

- In general, the integrability conditions presented in Chapter 2 can be generalized to the case in which symplectic realizations with different dimension s are used on each copy of the symmetry algebra, and this possibility would lead to different Hamiltonians.
- Another open possibility is the search for higher dimensional coalgebras that could give rise to new integrable Hamiltonian systems under appropriate symplectic realizations.
- Certainly, it would be also interesting to perform a detailed analysis of some of the new Hamiltonians here presented, working out explicitly the solutions for the equations of motion. We recall that the latter problem can be faced through the cluster variables technique [19] that makes use of the coalgebra symmetry in order to define the appropriate collective dynamical variables.
- In this direction, we would like to mention that the electromagnetic systems and the geodesic flows here presented will be the subject of further investigations. Also, the generalized Calogero-Moser model introduced in Chapter 5 seems to deserve some attention.

- For most of the Hamiltonians here presented there exists two different sets of integrals of the motion which are quadratic in the momenta. This fact raises the question concerning the separability of all these systems in (two) different coordinate systems. This separability problem would be also related to the explicit solvability of these Hamiltonians.
- It is clear that many other ND Hamiltonian systems can be written as a function of the generators of h_6 . In that case the $(N - 2)$ ‘universal’ integrals (5.8) do exist by construction, and the only task in order to solve the complete integrability problem is to find the additional constant \mathcal{I} , in case it exists. A systematic search for this type of systems is worth to be continued.
- It is also well known that the existence of a given coalgebra symmetry for \mathcal{H} allows for the direct construction of integrable deformations of \mathcal{H} through the use of q -deformations of the underlying Poisson coalgebra. In the case of h_6 , quantum two-photon/Schrödinger algebras have been constructed [10, 11] and its Poisson versions could be used to provide integrable deformations of some of the systems here presented.
- Another interesting possibility would be the construction of the analogues on ND spaces with constant curvature of the Euclidean Hamiltonians here presented. This could be feasible by using the $sl(2, \mathbb{R})$ approach [24, 26, 29] but now applying the corresponding h_6 -coalgebra symmetry.
- Finally, it is clear that the study of the Quantum Mechanical analogue of some of the new ND integrable systems here presented should be interesting.

Chapter 8

Supplements

*“Mathematics, rightly viewed, possesses not only truth,
but supreme beauty - a beauty cold and austere,
like that of sculpture.”*

B. Russell

8.1 Lie and Poisson algebras

In this supplement, we collect the basic definitions concerning Lie and Poisson algebras. We refer the reader to [8, 59, 74, 88, 94, 149, 163, 172] for a much deeper presentation.

• **Lie algebra.** A Lie algebra A is a vector space \mathcal{V} over \mathbb{C} endowed with a bilinear application $[\cdot, \cdot]$ named Lie bracket.

$$[\cdot, \cdot] : A \times A \rightarrow A \quad (8.1)$$

The Lie bracket must satisfy:

1. *Antisimmetry*

$$[X, Y] = -[Y, X] \quad \forall X, Y \in A \quad (8.2)$$

2. *Bilinearity*

$$\begin{aligned} [\alpha X + \beta Y, Z] &= \alpha[X, Z] + \beta[Y, Z] \\ [Z, \alpha X + \beta Y] &= \alpha[Z, X] + \beta[Z, Y] \\ \forall \alpha, \beta, \gamma \in \mathbb{C} \quad \forall X, Y, Z \in A & \end{aligned} \quad (8.3)$$

3. *Jacobi Identity*

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \quad \forall X, Y, Z \in A. \quad (8.4)$$

• **Lie subalgebra.** Let A' be a subspace of A as a vector space. Then, if

$$[P, R] \in A' \quad \forall P, R \in A' \quad (8.5)$$

then A' is said to be a Lie subalgebra of A .

• **Lie-algebra isomorphism.** Let A_1 and A_2 be two Lie algebras defined over the same field \mathcal{K} and let ϕ be a linear invertible mapping between A_1 and A_2 .

Then if,

$$\phi([X, Y]_{A_1}) = [\phi(X), \phi(Y)]_{A_2} \quad X, Y \in A_1 \quad (8.6)$$

we say A_1 and A_2 are isomorphic.

• **Ideal.** Let A be a Lie algebra and B a subalgebra of A , if

$$[B, A] \subseteq B \quad (8.7)$$

then B is said to be an ideal of A .

• **Structure constants.** Let A be a Lie algebra over \mathbb{C} , and let $\{e_i\}$ be a basis of A . Then any element of A can be written as

$$X = \sum_i x_i e_i \quad (8.8)$$

where $x_i \in \mathbb{C}$. The bracket between two elements of the basis can be written as

$$[e_i, e_j] = \sum_k C_{ij}^k e_k. \quad (8.9)$$

Where the C_{ij}^k quantities are known as structure constants for A . From them, the Lie bracket between two arbitrary elements of A can be obtained by applying the bilinearity property. Antisymmetry and the Jacobi relations imply the following relations for the C_{ij}^k constants:

$$C_{ij}^k = -C_{ji}^k$$

$$\sum_l \left(C_{jk}^l C_{il}^m + C_{ki}^l C_{jl}^m + C_{ij}^l C_{kl}^m \right) = 0. \quad (8.10)$$

• **Derived algebra.** The derived algebra $A^{(1)}$ of a Lie algebra A is defined as the subset

$$A^{(1)} = \{Z \in A / \exists X, Y \in A / Z = [X, Y]\}. \quad (8.11)$$

- **Simple Lie algebra.** A Lie algebra A is said to be ‘simple’ if it is not abelian and its only ideals are $\{0\}$ and itself.
- **Semisimple Lie algebra.** A Lie algebra A is said to be ‘semisimple’ if it does not contain any non-zero abelian ideal.
- **Solvable algebra.** Let us consider the sequence of derivations

$$A^{(1)} = [A, A] \quad A^{(2)} = [A^{(1)}, A^{(1)}] \quad , \dots , \quad A^{(i)} = [A^{(i-1)}, A^{(i-1)}] \quad (8.12)$$

if then there exists some $n \in \mathbb{N}$ such that $A^{(n)} = \{0\}$ then A is said to be solvable.

- **Radical of A .** The maximal solvable ideal of A is called *radical*.
- **Nilpotent algebra.** Let us now consider the sequence

$$A^{[1]} = [A, A] \quad A^{[2]} = [A, A^{[1]}] \quad , \dots , \quad A^{[i]} = [A, A^{[i-1]}]. \quad (8.13)$$

Then, if there exists some $n \in \mathbb{N}$ such that $A^{[n]} = \{0\}$, then A is said to be nilpotent.

The following properties can be easily proven:

- a) A is solvable if and only if the derived algebra of A is nilpotent.
- b) If A is nilpotent, then A is solvable.

- **Nilradical of a solvable Lie algebra.** Let A be a solvable Lie algebra. If it admits the following decomposition as a vector space

$$A = \mathfrak{n} \oplus \mathfrak{t} \quad (8.14)$$

and such that

$$[\mathfrak{t}, \mathfrak{t}] \subset \mathfrak{n}, \quad [\mathfrak{t}, \mathfrak{n}] \subset \mathfrak{n} \quad (8.15)$$

then \mathfrak{n} is called the nilradical of A , and the following relation holds:

$$\dim(\mathfrak{n}) \geq \frac{1}{2} \dim(\mathfrak{t}) \quad (8.16)$$

Casimir operators

Casimir Operators play a crucial role in many modern physical theories, and the literature concerning its explicit forms and properties for many Lie algebras can be traced back from, for instance, [1, 33, 34, 43, 44, 58, 59, 60, 61, 62, 103, 118, 139, 140, 141, 145, 159, 170].

• **Universal Enveloping Algebra.** Let A be a Lie Algebra defined over the field \mathcal{K} where $[\ , \]$ is the Lie bracket. Let A^\otimes be the tensor algebra defined from A , and let \mathcal{I} be the ideal of A^\otimes generated by

$$[X, Y] - (X \otimes Y - Y \otimes X) \quad X, Y \in A, \quad (8.17)$$

then the Universal Enveloping Algebra $\mathcal{U}(A)$ is defined as the quotient

$$\mathcal{U}(A) = A^\otimes / \mathcal{I}. \quad (8.18)$$

• **Center of an algebra.** Let A be a Lie algebra, the center of A^\otimes is defined as the ideal \mathcal{I} of A^\otimes commuting with all the elements of A^\otimes .

• **Casimir Operator.** Let A be an l -dimensional Lie Algebra and let $U(A)$ be the universal enveloping algebra of A . The Casimir elements for A are the elements of the center of $U(A)$ verifying

$$[\mathcal{C}_i, J_j] = 0 \quad \forall i = 1, \dots, r, \quad j = 1, \dots, l. \quad (8.19)$$

• **Number of independent invariants.** Let A be a real Lie algebra of dimension l , the number of independent invariants (Casimir operators) r of A is

$$r = l - \mathfrak{R} \quad (8.20)$$

where \mathfrak{R} is the rank of the commutator table of A when considered as a matrix. For the purpose of computing this rank, the $(J_i, i = 1, \dots, l)$ algebra generators are regarded as independent numbers [145].

Poisson algebras

• **Poisson algebra.** A Poisson algebra P is an associative algebra over a field \mathcal{K} endowed with a Poisson bracket:

$$\{ \ , \ } : A \otimes A \rightarrow A \quad (8.21)$$

satisfying the following relations $\forall f, g, h \in P$

$$\begin{aligned} \{f, g\} &= -\{g, f\} && \text{Antisymmetry} \\ \{f, gh\} &= g\{f, h\} + \{f, g\}h && \text{Leibnitz rule} \\ \{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} &= 0 && \text{Jacobi identity.} \end{aligned} \quad (8.22)$$

• **Poisson bivector.** Let M be a N -dimensional manifold and let us consider functions belonging to $\mathcal{C}^\infty(M)$. If we want to define a Poisson algebra over $\mathcal{C}^\infty(M)$, we have to

define the Poisson bracket by means of a skew-symmetric bivector Λ over the local coordinates of the manifold (x_1, \dots, x_N) :

$$\Lambda = \lambda^{ij}(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} \quad (i < j) \quad \lambda^{ij}(x) \in \mathcal{C}^\infty(M). \quad (8.23)$$

By using the former expression in terms of the bivector, the Poisson bracket becomes

$$\{f, g\} = \lambda^{ij}(x) \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}, \quad (8.24)$$

where,

$$\{x^i, x^j\} = \lambda^{ij}(x) \quad (8.25)$$

is called ‘the fundamental Poisson bracket’.

• **Casimir functions.** Let $f(x)$ be a function in $\mathcal{C}^\infty(M)$. The function $f(x)$ is said to be a Casimir function of the Poisson algebra $\mathcal{C}^\infty(M)$ if

$$\{h(x), f(x)\} = 0 \quad \forall h(x) \in \mathcal{C}^\infty(M). \quad (8.26)$$

8.2 The Yoshida-Ziglin test

We are going to revisit a useful tool as far as the non-integrability of homogeneous potentials is concerned. Starting from the Ziglin’s approach, Yoshida in 1983 was able to provide a sufficient condition for the non integrability of 2D homogeneous Hamiltonian systems, where the conditions coming from Ziglin’s results [186] were given in an explicit form. Several years latter, Yoshida established [183] the same conditions for N -dimensional homogeneous potentials. This so-called Yoshida-Ziglin approach to the integrability problem also called the monodromy group approach, has focused many efforts during the last two decades, see for instance [3, 125, 126, 135, 151, 182, 184]. On the other hand, the lack of a generic algorithm or method for proving the non integrability of a general dynamical system remains as one of the most difficult problems in this area.

• **Theorem.** [3, 183] *Let $\mathcal{V}(q_1, q_2, \dots, q_N)$ be an k -homogeneous potential, where $(k \neq 0, \pm 2)$ for the following N -dimensional Hamiltonian on the N -dimensional Euclidean space:*

$$\mathcal{H}^{(N)} = \sum_{i=1}^N \frac{p_i^2}{2} + \mathcal{V}(q_1, \dots, q_N) \quad (8.27)$$

we can define the following quantities:

$$\Delta\rho_i = \sqrt{1 + \frac{8k\xi_i}{(k-2)^2}} \quad (8.28)$$

where ξ_i , are the eigenvalues evaluated in c_i , of the following Hessian Matrix

$$\mathbb{H} = \left(\frac{\partial^2 \mathcal{V}}{\partial q_i \partial q_j} \Big|_{[q_i=c_i]} \right) \quad i, j = 1, \dots, N \quad (8.29)$$

and where c_i , are the solutions of the algebraic system coming from

$$\frac{\partial V}{\partial q_i} \Big|_{q_i=c_i} = c_i. \quad (8.30)$$

Then, if there exists some solution c_i to (8.30) such that the quantities $(\Delta\rho_1, \dots, \Delta\rho_N)$ are rationally independent, the ND Hamiltonian \mathcal{H} has no additional global analytical integral $(\mathcal{I}(q_i, p_i) = \text{constant})$ other than the Hamiltonian itself.

Some comments seem to be pertinent:

- a) The quantities $\Delta\rho_i$ are known as the Kowaleskaia's exponents, and are crucial in the Painlevé's analysis of integrable systems.
- b) $\Delta\rho_i, \Delta\rho_j$ are said to be rationally independent if $\nexists \mathbf{q} \in \mathbb{Q} / \frac{\Delta\rho_i}{\Delta\rho_j} = \mathbf{q}$.
- c) The analysis of the Kowaleskaia's exponents must be carried out for all the possible solutions c_i of the system (8.30).

Hereafter we are showing several examples of the application of this test.

• **Example 1.** Let us consider the Hamiltonian:

$$\mathcal{H}^{(3)} = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + q_1^3 + q_1^2 q_2 + q_1 q_2^2 + q_2^3 + q_1^2 q_3 + q_2^2 q_3 + q_1 q_3^2 + q_2 q_3^2 + q_3^3. \quad (8.31)$$

Therefore, we have to analyze the homogeneous potential with degree $k = 3$ given by

$$\mathcal{V} = q_1^3 + q_1^2 q_2 + q_1 q_2^2 + q_2^3 + q_1^2 q_3 + q_2^2 q_3 + q_1 q_3^2 + q_2 q_3^2 + q_3^3 \quad (8.32)$$

solving the system,

$$\frac{\partial V}{\partial q_i} \Big|_{q_i=c_i} = c_i \quad i = 1, 2, 3 \quad (8.33)$$

$$\begin{cases} 3c_1^2 + 2c_1 c_2 + c_2^2 + 2c_1 c_3 + c_3^2 = c_1 \\ c_1^2 + 2c_1 c_2 + 3c_2^2 + 2c_2 c_3 + c_3^2 = c_2 \\ c_1^2 + c_2^2 + 2c_1 c_3 + 2c_2 c_3 + 3c_3^2 = c_3 \end{cases} \quad (8.34)$$

we choose this particular solution

$$c_1 = c_2 = c_3 = \frac{1}{9}. \quad (8.35)$$

The Hessian matrix is

$$\mathbb{H} = \begin{pmatrix} \frac{10}{9} & \frac{4}{9} & \frac{4}{9} \\ \frac{4}{9} & \frac{10}{9} & \frac{4}{9} \\ \frac{4}{9} & \frac{4}{9} & \frac{10}{9} \end{pmatrix} \quad (8.36)$$

and its eigenvalues are

$$\xi_1 = 2 \quad \xi_2 = \frac{2}{3} \text{ (two times)}. \quad (8.37)$$

The Kowaleskaia's exponents are

$$\Delta\rho_1 = \sqrt{1 + \frac{8 \cdot 3 \cdot 2}{(3-2)^2}} = 7 \quad \Delta\rho_2 = \Delta\rho_3 = \sqrt{1 + \frac{8 \cdot 3 \cdot \frac{2}{3}}{(3-2)^2}} = \sqrt{17}. \quad (8.38)$$

As the Kowaleskaia's exponents for this solution are rationally independent

$$\nexists \mathbf{q} \in \mathbb{Q}/ \rightarrow \sqrt{17} = \mathbf{q} \cdot 7 \quad (8.39)$$

so the previous Theorem guarantees that our Hamiltonian is not integrable.

If we formulate the system within the h_6 language, we would have

$$\mathcal{H} = \frac{B_+}{2} + A_- B_- \quad \lambda_1 = \lambda_2 = \lambda_3 = 1. \quad (8.40)$$

The Yoshida-Ziglin criterion tell us that this Hamiltonian is not integrable. In fact, if we look for possible integrals \mathcal{I} commuting with (8.40), this search does not lead to any result.

• **Example 2.** Let us now consider

$$\mathcal{H}^{(2)} = \frac{1}{2} (p_1^2 + p_2^2) + 2q_1^4 + 12q_1^2 q_2^2 + 2q_2^4. \quad (8.41)$$

We have to solve this system

$$\left. \frac{\partial V}{\partial q_i} \right|_{q_i=c_i} = c_i \quad i = 1, 2 \quad (8.42)$$

$$\begin{cases} 8c_1^3 + 24c_1 c_2^3 = c_1 \\ 24c_1^2 c_2 + 8c_2^3 = c_2. \end{cases} \quad (8.43)$$

We choose this particular solution

$$c_1 = -\frac{1}{4\sqrt{2}} \quad c_2 = -\frac{1}{4\sqrt{2}}. \quad (8.44)$$

The Hessian matrix becomes

$$\mathbb{H} = \begin{pmatrix} 24(q_1^2 + q_2^2) & 48q_1q_2 \\ 48q_1q_2 & 24(q_1^2 + q_2^2) \end{pmatrix} \Big|_{[q_1=c_1, q_2=c_2]} = \begin{pmatrix} \frac{3}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{3}{2} \end{pmatrix} \quad (8.45)$$

and the eigenvalues are

$$\xi_1 = 3 \quad \xi_2 = 0. \quad (8.46)$$

The Kowaleskaia's exponents become

$$\Delta\rho_1 = \sqrt{1 + \frac{8 \cdot 4 \cdot 3}{(4-2)^2}} = 5 \quad \Delta\rho_2 = \sqrt{1 + \frac{8 \cdot 4 \cdot 0}{(4-2)^2}} = 0 \quad (8.47)$$

and they are rationally dependent. Since we obtain the same result for any other solution c_i , we conclude that the system would be integrable.

But unfortunately, the Theorem does not provide the expression for the integral. Matching the former potential within the h_6 framework we can re-write the Hamiltonian in the following form

$$\mathcal{H} = \frac{B_+}{2} + \delta_1 B_- + \delta_2 \mathcal{F}(\mathcal{C}_{\bar{\mathcal{G}}_+}) + \delta_3 \mathcal{G}(A_-) \quad (8.48)$$

where the arbitrary functions and parameters become $\mathcal{F}, \mathcal{G}, \delta, \lambda$ are

$$\delta_1 = 0, \quad \delta_2 = \delta_3 = 1, \quad \mathcal{F} = \left(\mathcal{C}_{\bar{\mathcal{G}}_+}\right)^2, \quad \mathcal{G} = A_-^4, \quad \lambda_1 = \lambda_2 = 1. \quad (8.49)$$

Actually, we are dealing with one our family of integrable systems (5.1), and the integral in the 2D case is as follows

$$\mathcal{I} = A_+^2 + \delta_1 A_-^2 + 2\delta_3 M A_-^4 \quad (8.50)$$

$$\mathcal{I}^{(2)} = (p_1 + p_2)^2 + 4(q_1 + q_2)^4. \quad (8.51)$$

8.3 Symplectic realizations for 5D algebras with $R = 3$

For the sake of completeness, in this supplement we summarize the symplectic realizations of the 5D Poisson algebras with $R = 3$. The construction of completely integrable systems coming from them would be obtained by following the procedure described in Chapters 2 and 3. For each algebra we give the Poisson brackets, the Casimir operators and the expression for the generic one-particle symplectic realization.

- $A_{5,13}^{\alpha,\beta,\gamma}$ algebra ($\alpha\gamma \neq 0, |\alpha| \leq 1$)

Poisson brackets

$$\begin{aligned} \{J_1, J_5\} &= J_1 & \{J_2, J_5\} &= \alpha J_2 \\ \{J_4, J_5\} &= \gamma J_3 + \beta J_4 & \{J_3, J_5\} &= \beta J_3 - \gamma J_4. \end{aligned} \quad (8.52)$$

Casimir functions

$$\mathcal{C}_1 = \frac{J_1^\alpha}{J_2} \quad \mathcal{C}_2 = \frac{J_1^{2\beta}}{J_3^2 + J_4^2} \quad \mathcal{C}_3 = J_3^{2\gamma} e^{-2\arctan\left(\frac{J_4}{J_3}\right)}. \quad (8.53)$$

Symplectic realization

$$\begin{aligned} J_1 &= e^{-p} \\ J_2 &= \frac{1}{k_1} e^{\alpha p} \\ J_3 &= e^{-\beta p} (A \cos \gamma p + B \sin \gamma p) \\ J_4 &= e^{\beta p} (B \cos \gamma p - A \sin \gamma p) \\ J_5 &= q \end{aligned} \quad (8.54)$$

$$k_1 \neq 0 \quad A = A(k_2, k_3) \quad B = B(k_2, k_3). \quad (8.55)$$

- $A_{5,16}^{\alpha,\beta}$ algebra ($\beta \neq 0$)

$$\begin{aligned} \{J_1, J_5\} &= J_1 & \{J_2, J_5\} &= J_1 + J_2 \\ \{J_3, J_5\} &= \alpha J_3 - \beta J_4 & \{J_4, J_5\} &= \beta J_3 + \alpha J_4. \end{aligned} \quad (8.56)$$

Casimir functions

$$\mathcal{C}_1 = \frac{J_1^{2\alpha}}{J_3^2 + J_4^2} \quad \mathcal{C}_2 = J_1^{2\beta} e^{-2\arctan\left(\frac{J_4}{J_3}\right)} \quad \mathcal{C}_3 = J_1 e^{-\frac{J_2}{J_1}}. \quad (8.57)$$

Symplectic realization

$$\begin{aligned} J_1 &= e^{-\alpha p} \\ J_2 &= -e^{\alpha p} (p + \log k_3) \\ J_3 &= e^{\alpha p} (A \cos \beta p + B \sin \beta p) \\ J_4 &= e^{\beta p} (B \cos \beta p - A \sin \beta p) \\ J_5 &= q \end{aligned} \quad (8.58)$$

$$k_3 > 0 \quad A = A(k_1, k_2, k_3) \quad B = B(k_1, k_2, k_3). \quad (8.59)$$

• $A_{5,17}^{s,\alpha,\beta}$ algebra ($s \neq 0$)

$$\begin{aligned} \{J_1, J_5\} &= \alpha J_1 - J_2 & \{J_2, J_5\} &= J_1 + \alpha J_2 \\ \{J_4, J_5\} &= sJ_3 + \beta J_4 & \{J_3, J_5\} &= \beta J_3 - sJ_4. \end{aligned} \quad (8.60)$$

Casimir functions

$$\mathcal{C}_1 = \frac{(J_1^2 + J_2)^{\beta}}{(J_3^2 + J_4)^{\alpha}} \quad \mathcal{C}_2 = (J_1^2 + J_2) e^{-2\alpha \arctan\left(\frac{J_2}{J_1}\right)} \quad \mathcal{C}_3 = (J_3^2 + J_4) e^{-\frac{2\beta}{s} \arctan\left(\frac{J_4}{J_3}\right)}. \quad (8.61)$$

Symplectic realizations

$$\begin{aligned} J_1 &= e^{-\alpha p} (A \cos p + A \sin p) \\ J_2 &= e^{-\alpha p} (B \cos p - A \sin p) \\ J_3 &= \sqrt{k_3} e^{-\beta p} \cos sp \\ J_4 &= -\sqrt{k_3} e^{-\beta p} \sin sp \\ J_5 &= q \end{aligned} \quad (8.62)$$

$$k_1, k_2, k_3 > 0 \quad (8.63)$$

$$A = (k_1 k_3^{\alpha})^{\frac{1}{4\beta}} \cos\left(\frac{\log[k_2(k_1 k_3^{\alpha})^{-\frac{1}{2\beta}}]}{2\alpha}\right) \quad B = -(k_1 k_3^{\alpha})^{\frac{1}{4\beta}} \sin\left(\frac{\log[k_2(k_1 k_3^{\alpha})^{-\frac{1}{2\beta}}]}{2\alpha}\right). \quad (8.64)$$

• $A_{5,18}^{\alpha}$ algebra ($\alpha \geq 0$)

$$\begin{aligned} \{J_1, J_5\} &= \alpha J_1 - J_2 & \{J_2, J_5\} &= J_1 + \alpha J_2 \\ \{J_4, J_5\} &= J_2 + J_3 + \alpha J_4 & \{J_3, J_5\} &= J_1 + \alpha J_3 - J_4. \end{aligned} \quad (8.65)$$

Casimir functions

$$\mathcal{C}_1 = \frac{J_1 J_4 - J_2 J_3}{J_1^2 + J_2^2} \quad \mathcal{C}_2 = (J_1^2 + J_2^2) e^{-2\alpha \arctan\left(\frac{J_2}{J_1}\right)} \quad \mathcal{C}_3 = (J_1^2 + J_2^2) e^{-2\alpha \left(\frac{J_1 J_3 - J_2 J_4}{J_1^2 + J_2^2}\right)} \quad (8.66)$$

$$\begin{aligned} J_1 &= e^{-\alpha p} A (\cos p + \sin p) \\ J_2 &= e^{-\alpha p} A (\cos p - \sin p) \\ J_3 &= e^{-\alpha p} \mathcal{R}(p) \\ J_4 &= e^{-\alpha p} \mathcal{S}(p) \\ J_5 &= q \end{aligned} \quad (8.67)$$

$$k_2, k_3 > 0 \quad (8.68)$$

$$\begin{aligned}\mathcal{R}(p) &= ((F \cos p + G \sin p) - Ap(\cos p + \sin p)) \\ \mathcal{S}(p) &= ((F \cos p - G \sin p) - Ap(\cos p - \sin p))\end{aligned}\tag{8.69}$$

$$\begin{aligned}A &= e^{\frac{\pi\alpha}{4}} \sqrt{\frac{k_2}{2}} \\ F &= \frac{1}{2\alpha} \sqrt{\frac{k_2}{2}} e^{\frac{\pi\alpha}{4}} \left(\log \left(\frac{e^{\frac{\pi\alpha}{2}} k_2}{k_3} \right) - 2\alpha k_1 \right) \\ G &= \frac{1}{2\alpha} \sqrt{\frac{k_2}{2}} e^{\frac{\pi\alpha}{4}} \left(\log \left(\frac{e^{\frac{\pi\alpha}{2}} k_2}{k_3} \right) + 2\alpha k_1 \right).\end{aligned}\tag{8.70}$$

8.4 The h_9 Poisson algebra

It is worth mentioning that if we consider the six h_6 generators and the Casimir operators of $(\mathcal{C}_{\bar{g}_+}, \mathcal{C}_{\bar{g}_-}, \mathcal{C}_{h_4})$, this nine objects close a Poisson algebra that we can call h_9 . On one hand, the Casimir operators close an algebra which is isomorphic to $sl(2)$. On the other hand, the brackets between the $sl(2)$ -part and h_6 are not vanishing, and we have the semidirect product structure.

$$\{h_6, h_6\} \subset h_6 \quad \{h_6, sl(2)\} \subset sl(2) \quad \{sl(2), sl(2)\} \subset sl(2) \quad (8.71)$$

$$h_9 = h_6 \odot sl(2). \quad (8.72)$$

Therefore the h_9 algebra is generated by the following objects

$$\{K, A_+, A_-, B_+, B_-, M, \mathcal{C}_{\bar{g}_+}, \mathcal{C}_{\bar{g}_-}, \mathcal{C}_{h_4}\} \quad (8.73)$$

and its commutation rules can be split into the following different subsets:

Between h_6 generators and $\mathcal{C}_{\bar{g}_+}$:

$$\begin{aligned} \{K, \mathcal{C}_{\bar{g}_+}\} &= 2\mathcal{C}_{\bar{g}_+} & \{A_+, \mathcal{C}_{\bar{g}_+}\} &= 0 & \{A_-, \mathcal{C}_{\bar{g}_+}\} &= 0 \\ \{B_+, \mathcal{C}_{\bar{g}_+}\} &= 0 & \{B_-, \mathcal{C}_{\bar{g}_+}\} &= 4\mathcal{C}_{h_4} & \{M, \mathcal{C}_{\bar{g}_+}\} &= 0. \end{aligned} \quad (8.74)$$

Between h_6 generators and $\mathcal{C}_{\bar{g}_-}$:

$$\begin{aligned} \{K, \mathcal{C}_{\bar{g}_-}\} &= -2\mathcal{C}_{\bar{g}_-} & \{A_+, \mathcal{C}_{\bar{g}_-}\} &= 0 & \{A_-, \mathcal{C}_{\bar{g}_-}\} &= 0 \\ \{B_+, \mathcal{C}_{\bar{g}_-}\} &= -4\mathcal{C}_{h_4} & \{B_-, \mathcal{C}_{\bar{g}_-}\} &= 0 & \{M, \mathcal{C}_{\bar{g}_-}\} &= 0. \end{aligned} \quad (8.75)$$

Between h_6 generators and \mathcal{C}_{h_4} :

$$\begin{aligned} \{K, \mathcal{C}_{h_4}\} &= 0 & \{A_+, \mathcal{C}_{h_4}\} &= 0 & \{A_-, \mathcal{C}_{h_4}\} &= 0 \\ \{B_+, \mathcal{C}_{h_4}\} &= -2\mathcal{C}_{\bar{g}_+} & \{B_-, \mathcal{C}_{h_4}\} &= 2\mathcal{C}_{\bar{g}_-} & \{M, \mathcal{C}_{h_4}\} &= 0. \end{aligned} \quad (8.76)$$

Between the $\{\mathcal{C}_{\bar{g}_+}, \mathcal{C}_{\bar{g}_-}, \mathcal{C}_{h_4}\} \equiv sl(2)$ Casimirs:

$$\{\mathcal{C}_{h_4}, \mathcal{C}_{\bar{g}_+}\} = 2M\mathcal{C}_{\bar{g}_+} \quad \{\mathcal{C}_{h_4}, \mathcal{C}_{\bar{g}_-}\} = -2M\mathcal{C}_{\bar{g}_-} \quad \{\mathcal{C}_{\bar{g}_+}, \mathcal{C}_{\bar{g}_-}\} = -4M\mathcal{C}_{h_4}. \quad (8.77)$$

As a consequence, if we take M as a constant h_9 is a Lie-Poisson algebra.

Finally the Casimir operators for h_9 algebra are M and

$$\begin{aligned} \mathcal{C} &= \frac{1}{M} (\mathcal{C}_{\bar{g}_+} \mathcal{C}_{\bar{g}_-} - \mathcal{C}_{h_4}^2) \\ \mathcal{C}_{h_6} &= (MB_+ - A_+^2) (MB_- - A_-^2) - (MK - A_- A_+ + M^2/2)^2 \end{aligned} \quad (8.78)$$

where we recall that the Casimirs of the subalgebras are taken as independent objects with respect to the h_6 generators.

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