LIE-POISSON REDUCTION

Reducción de Lie-Poisson

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"It is not the mountain we conquer
but ourselves".

E. Hillary

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Contents

1 Rigid body **8**
- 1.1 Lagrangian and Hamiltonian Mechanics . . . 8
- 1.2 Equations of motion for the rigid body . . . 10
  - 1.2.1 Hamiltonian form . . . 12

2 Lagrangian and Hamiltonian formalisms **16**
- 2.1 Symplectic manifolds . . . 16
  - 2.1.1 Canonical symplectic structure of the cotangent bundle . . . 18
- 2.2 Mechanics on manifolds . . . 19
  - 2.2.1 Configuration manifold, phase space of the velocities and phase space of momenta . . . 19
  - 2.2.2 Lagrangian mechanics . . . 19
  - 2.2.3 Legendre transformation and Hamilton Equations . . . 21
  - 2.2.4 Hamiltonian mechanics and the symplectic structure of the cotangent bundle . . . 23

3 Lie groups **25**
- 3.1 Lie groups . . . 25
  - 3.1.1 Lie groups and Lie algebras . . . 25
  - 3.1.2 Homomhorpisms of Lie groups and Lie algebras . . . 27
  - 3.1.3 Exponential map . . . 29
  - 3.1.4 Lie subgroups . . . 33
- 3.2 Actions of Lie groups . . . 36
  - 3.2.1 Infinitesimal generators . . . 41
  - 3.2.2 Adjoint and coadjoint action . . . 42

4 Momentum map **46**
- 4.1 Definitions and examples . . . 46
- 4.2 Properties of the momentum map . . . 48
- 4.3 Momentum maps on the cotangent bundle . . . 49

5 Poisson manifolds **51**
- 5.1 Generalities on Poisson structures . . . 51
  - 5.1.1 Poisson brackets . . . 51
  - 5.1.2 Poisson 2-vector . . . 52
  - 5.1.3 Symplectic manifolds and Poisson structures . . . 54
  - 5.1.4 Linear Poisson structures . . . 56
- 5.2 Poisson morphisms . . . 57
  - 5.2.1 Infinitesimal automorphims and Hamiltonian vector fields . . . 58
- 5.3 Symplectic foliation of a Poisson manifold . . . 60
  - 5.3.1 Symplectic foliation of linear Poisson structures . . . 62

6 Lie-Poisson reduction theorem **64**
6.1 Lie-Poisson reduction theorem 64
6.2 Rigid body 67

A Schouten-Nijenhuis bracket **69**
A.1 k-vectors and k-forms 69
A.1.1 2-vectors 70
A.2 Lie derivative 71
A.2.1 Schouten-Nijenhuis bracket 72

B Distributions **75**
B.1 Regular distributions 75
B.2 Generalized distributions 75

Introducción

Contexto histórico

El objetivo principal de este trabajo es probar el Teorema de Reducción de Lie-Poisson. Uno de los elementos más importantes para la demostración del Teorema serán precisamente las estructuras de Lie-Poisson. Estas estructuras fueron introducidas por primera vez por Lie en 1880 en su tratado de grupos de transformaciones. De hecho, en esta obra introdujo el concepto de estructura de Poisson y luego trató el caso particular de las estructuras de Poisson lineales en el dual de una álgebra de Lie. Posteriormente, estas estructuras han sido bautizadas por Marsden y Weinstein con el nombre de estructuras de Lie-Poisson debido a su historia.

Cabe destacar que el trabajo de Lie sobre estructuras de Poisson pasó inadvertido mucho tiempo e incluso Elie Cartan no era consciente de este aspecto del trabajo de Lie que le habría ayudado con la descripción Hamiltoniana de algunos de los sistemas que estudió. No fue hasta la década de 1950 cuando se hace patente la importancia de trabajar en los duales de álgebras de Lie para la descripción de la mecánica Hamiltoniana.

Es en 1977 cuando se da la primera definición formal de estructura de Poisson en un artículo de Lichnerowicz (véase [Lic]). Finalmente, en 1983 Weinstein [Wei] en su trabajo fundacional da un importante impulso a la geometría de Poisson describiendo la estructura local de las variedades de Poisson.

Teorema de reducción de Lie-Poisson

Es bien sabido que el espacio fase de momentos de un sistema Hamiltoniano es el fibrado cotangente T^*Q del espacio de configuración Q. Por tanto, usando la estructura simpléctica canónica de T^*Q y la función Hamiltoniana H del sistema mecánico se puede obtener el campo Hamiltoniano de H . La curvas integrales de dicho campo vienen dadas por las soluciones de las ecuaciones de Hamilton para H (véase, por ejemplo, [AbMa],[LeRo]).

En el caso particular en que el espacio de configuración es un grupo de Lie G, la estructura simpléctica canónica del espacio fase de momentos T^*G es invariante bajo la acción usual de G en T^*G. Además, si la función Hamiltoniana es también G-invariante, se puede definir una función Hamiltoniana reducida en el espacio dual $\mathfrak{g}^*$ del álgebra de Lie de G.

El espacio $\mathfrak{g}^*$ no es simpléctico pero si es una variedad de Poisson. Así, usando la estructura de Lie-Poisson de $\mathfrak{g}^*$ y la función Hamiltoniana reducida, obtenemos un nuevo sistema dinámico en $\mathfrak{g}^*$: el sistema Hamiltoniano reducido. Las soluciones del sistema Hamiltoniano original en T^*G se proyectan a través de la aplicación momento en las soluciones del sistema Hamiltoniano reducido en g^*. Esto es lo que establece el teorema de reducción de Lie-Poisson (véase, por ejemplo [MaRa]).

El proceso de reducción de Lie-Poisson puede aplicarse a numerosos sistemas mecánicos. Uno de estos sistemas es el sólido rígido (véase [MaRa]).

El objetivo principal de este proyecto es probar el teorema de reducción de Lie-Poisson usando la teoría básica de grupos de Lie, la aplicación momento y la geometría de Poisson.

El proyecto está estructurado como sigue. Los capítulos 1 y 2 están dedicados a la revisión de algunas definiciones y resultados básicos de la dinámica del sólido rígido (como ejemplo motivador) y a la formulación simpléctica de la mecánica Hamiltoniana, respectivamente. En el capítulo 3 se dará la teoría de grupos de

Lie que se usará en el resto del proyecto. En la primera parte del capítulo se introducirá el álgebra de Lie de un grupo de Lie y se tratarán algunos aspectos de la aplicación exponencial y de subgrupos de Lie. La segunda parte está dedicada a acciones de grupos de Lie en variedades diferenciables. En el capítulo 4 introducimos la noción de aplicación momento, damos ejemplos y probamos algunas de sus propiedades. La geometría de Poisson es tratada en el capítulo 5. Se prestará especial atención a las estructuras de Lie-Poisson y a la foliación simpléctica generalizada de una variedad de Poisson. En el capítulo 6 probamos el teorema de reducción de Lie-Poisson. El proyecto se cierra con dos apéndices; el primero sobre el corchete de Schouten-Nijenhuis de multi-vectores en una variedad diferenciable y el segundo sobre la integrabilidad de las distribuciones generalizadas.

Introduction

Historical context

The main objective of the project is to proof the Lie-Poisson reduction theorem. One of the most important elements needed to the proof are the Lie-Poisson structures. Such structures were first introduced by Lie about 1880 in his treatise on transformation groups. In fact, in this work Lie introduced the concept of Poisson structure and then considered the particular case of linear Poisson structures on the dual of a Lie algebra. Subsequently, these structures have been called by Marsden and Weinstein Lie-Poisson structures due to its history.

It is noteworthy that the Lie work on Poisson structures passed unnoticed and even Elie Cartan was unaware of this aspect of the Lie work that would have helped him with the Hamiltonian description of some of the systems that he studied. It was in the 1950s when it arose the importance of working in the dual Lie algebras for the description of Hamiltonian mechanics.

It was 1977 when Lichnerowicz gives for the first time the formal definition of a Poisson structure (see [Lic]). Finally, in 1983 Weinstein [Wei] in their seminal work gave a boost to the Poisson Geometry describing the local structure of Poisson manifolds.

Lie-Poisson reduction Theorem

It is well-known that the phase space of momenta of a Hamiltonian system is the cotangent bundle T^*Q of the configuration space Q. So, using the canonical symplectic structure of T^*Q and the Hamiltonian function H of the mechanical system one may construct the Hamiltonian vector field of H. The integral curves of this vector field are just the solutions of the Hamilton equations for H (see, for instance, [AbMa],[LeRo]).

In the particular case when the configuration space is a Lie group G then the canonical symplectic structure of the phase space of momenta T^*G is invariant under the standard action of G on T^*G. In addition, if the Hamiltonian function H also is G-invariant one may define a reduced Hamiltonian function on the dual space $\mathfrak{g}^*$ of the Lie algebra of G.

The space $\mathfrak{g}^*$ is not a symplectic manifold but a Poisson manifold. Thus, using the Lie-Poisson structure of $\mathfrak{g}^*$ and the reduced Hamiltonian function, we obtain a new dynamical system on $\mathfrak{g}^*$: the reduced Hamiltonian system. The solutions of the original Hamiltonian system on T^*G project, via the momentum map, over the solutions of the reduced Hamiltonian system on $\mathfrak{g}^*$. This is the statement of the Lie-Poisson reduction theorem (see, for instance, [MaRa]).

The Lie-Poisson reduction process may be applied to several interesting mechanical systems. One of these systems is the rigid body (see [MaRa]).

The aim of this project is to prove the Lie-Poisson reduction theorem using the basic theory of Lie groups, momentum map and Poisson geometry.

The project is structured as follows. Chapters 1 and 2 are devoted to review some definitions, results and basic constructions on the dynamics of the rigid body (as a motivating example) and the symplectic formulation of the Hamiltonian Mechanics, respectively. In Chapter 3, we will present the theory of Lie groups which will be used in the rest of the project. In the first part of the chapter, we will introduce the

Lie algebra of a Lie group and we will discuss some aspects on the exponential map and Lie subgroups. The second part is devoted to actions of Lie groups on smooth manifolds. In Chapter 4, the notion of a momentum map for a symplectic action is introduced. Examples of such maps are presented and some of its properties are proved. Poisson geometry is discussed in Chapter 5. Special attention is paid to the Lie-Poisson structures and the generalized symplectic foliation of a Poisson manifold. In Chapter 6, we prove the Lie-Poisson reduction theorem. The project closes with two appendices on the integrability of generalized distributions and the Schouten-Nijenhuis bracket of multi-vector fields on a smooth manifold.

Chapter 1

Rigid body

We will start studying, from a physical point of view and using only the fundamental and basic mathematical tools, the equations of motion of the rigid body. We will use this example as a reference throughout the project that will provide the necessary mathematical tools to prove that such a case is part of a much more general theory.

Before going further, it is important to underline that this chapter does not intend to study the mathematical concepts in great depth. Our goal right now is to present a first approach and, over next few chapters, develope the idea further.

1.1 Lagrangian and Hamiltonian Mechanics

Let us begin recalling what the equations of motion are. Given a physical system, our aim is to describe the behaviour of that system over time and we achieve it by means of the equations of motion. Formally, they are differential equations which give the existing relationship between the time derivative of the variables describing the system and the magnitudes which cause its evolution on time. As an example and starting point of our next discussions we have the well known *Euler-Lagrange equations* for a mechanical system

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0 \qquad \text{for } i = 1, \dots, n \tag{1.1}$$

where $q^i = q^i(t)$ are the coordinates describing the position of the system, $\dot{q}^i = \frac{dq^i}{dt}$ are the velocities and L is the *Lagrangian* function $L(q^i, \dot{q}^i) = L(q^1, \dots, q^n, \dot{q}^1, \dots, \dot{q}^n)$. The space of the variables (q^i) is the *configuration space* of the mechanical system and the space of the variables $(q^i, \dot{q}^i)$ is the *phase space of velocities.*

As we will show in the next chapter, the Euler-Lagrange equations arise from the *principle of minimum action*[1] which states that every mechanical system is characterized by the Lagrangian function and satisfies the following condition:
Let t_0 and t_1 be two instants of time where the system ocuppies two diferent and fixed positions, then, in the time interval $[t_0, t_1]$, the system will evolve in such a way that the following integral is minimized

$$S = \int_{t_0}^{t_1} L(q^i(t), \dot{q}^i(t))dt \tag{1.2}$$

Assuming this statement, some calculations lead to (1.1).

Before going further, let us study a very significant case of Euler-Lagrange equations. Suppose L is the

[1] Also known as Hamilton variational principle.

kinetic minus the potential energy for a N-particle system, that is

$$L(q^i, \dot{q}^i) = \frac{1}{2}\sum_{i=1}^{n} m_i \left\|\dot{q}^i\right\|^2 - V \qquad \text{for } i = 1, \dots, N \tag{1.3}$$

where q^i are the position vectors of the particles whose mass is m_i and $V = V(q^i)$ is the potential energy. In this case, Euler-Lagrange equations are just Newton's second law for this potential system, i.e.,

$$\frac{d}{dt}(m_i \dot{q}^i) = -\frac{\partial V}{\partial q^i} \qquad \text{for } i = 1, \dots, N$$

Even though we have introduced the Euler-Lagrange equations of motion, our main interest will be in the Hamiltonian ones. Nevertheless, the Hamiltonian formulation can be obtained from the Lagrangian one, and that is the reason why we first introduce Lagrangian. From now on, we will focus on obtaining Hamilton equations.

Let $L = L(q^i, \dot{q}^i)$ be a Lagrangian, to pass to the Hamiltonian formalism define the *conjugate momenta* as

$$p_i = \frac{\partial L}{\partial \dot{q}^i} \qquad \text{for } i = 1, \dots, n \tag{1.4}$$

and following that, introduce a change of variables

$$(q^i, \dot{q}^i) \to (q^i, p_i) \qquad \text{for } i = 1, \dots, n \tag{1.5}$$

called the *Legendre transformation*. The space of the variables (q^i, p_i) is the *phase space of momenta*.

We remark that the matrix of the differential of the Legendre transformation is

$$\begin{pmatrix} I & 0 \\ \frac{\partial^2 L}{\partial q^i \partial \dot{q}^j} & \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \end{pmatrix} \tag{1.6}$$

which is invertible if and only if $\left|\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right| \neq 0$. In this case, we will say that the Lagrangian is *regular*. If the Legendre transformation is a global diffeomorphism, we will say that the Lagrangian is *hyperregular*. From (1.6) we conclude that if a Lagrangian is hyperregular then it is regular. For instance, the Legendre transformation asociated with (1.3) is regular since we are assuming non-zero masses,

$$\left|\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right| = \begin{vmatrix} m_1 & & 0 \\ & \ddots & \\ 0 & & m_N \end{vmatrix} \neq 0$$

and hyperregular due to the transformation is defined by $p_i = m_i \dot{q}^i$ which has inverse $\dot{q}^i = \frac{1}{m_i} p_i$. Moreover, in this particular case, the conjugate momenta $p_i = m_i \dot{q}^i$, are exactly the linear momentum.

Now, given any hyperregular Lagrangian L, define the *Hamiltonian* function as

$$H(q^i, p_i) = \sum_{j=1}^{n} p_j \dot{q}^j - L(q^i, \dot{q}^i) \tag{1.7}$$

Note that the Legendre transformation is a diffeomorphism and, thus, $\dot{q}^i$ may be seen as a real function on the phase space of momenta. Therefore, H also is a real function on the phase space of momenta.

Keeping Legendre transformation in mind and making some rearrangements using (1.1), we obtain the *Hamilton equations*

$$\dot{q}^i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i} \qquad \text{for } i = 1, \dots, n \tag{1.8}$$

In the following chapter, we will proof that, in the regular case, they are equivalent to Euler-Lagrange equations (1.1). Specifically, we remark that when the hyperregular Lagrangian is given by (1.3), the Hamiltonian function coincides with the total energy of the system. Indeed, using the Legendre transformation calculated above for this particular case and keeping in mind (1.3) we get

$$H(q^i, p_i) = \frac{1}{2} \sum_{i=1}^{n} \frac{\|p_i\|^2}{m_i} + V$$

which is the kinetic plus the potential energy, i.e., the total energy of the system.

Up to this point, we have worked with two equivalent but different ways of obtaining equations of motion. The important fact is that the latter one allows us to reduce the order of the differential equations describing the system. While Euler-Lagrange equations are second order ODE's, in the Hamiltonian formulation the equations of motion are given by first order ODE's.

Hamiltonian mechanics is very suitable for studying conserved quantities and we achieve it by introducing a new operation between functions called the Poisson bracket. Given two functions $F(q^i, p_i)$ and $G(q^i, p_i)$ on the phase space of momenta, the *canonical Poisson bracket* of F and G is a new function in such a space given by

$$\{F, G\}_{can} := \sum_{i=1}^{n} \left(\frac{\partial F}{\partial q^i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q^i} \right) \tag{1.9}$$

Then, for any function F and any Hamiltonian H, we deduce from (1.8) that $\dot{F} = \{F, H\}_{can}$ along the solutions of the system. As well, it is not hard to check that Hamilton equations (1.8) for a Hamiltonian function H, are equivalent to

$$\dot{F} = \{F, H\}_{can} \qquad \text{for all real function } F = F(q^i, p_i) \tag{1.10}$$

Finally, a *conserved quantity* for a given Hamiltonian is a function $F(q^i, p_i)$ such that it is constant along any solution of the system. Particularly, F is a constant of motion, if and only if, $\{F, H\}_{can} = 0$.

For more details on the above topics we remit to [AbMa], [LeRo].

Regarding our current concern, we do not expect the reader to comprehend these concepts in depth, but to remember that the time evolution of a mechanical system might be described from both Euler-Lagrange and Hamilton equations (see (1.1) and (1.8)). It would be also suitable to retain the Legendre transformation and the canonical Poisson bracket in order to follow the next example. The next chapter will do a more through revision of these concepts and will study them from a geometric point of view recalling the Lagrangian and Hamiltonian formalisms.

1.2 Equations of motion for the rigid body

Once we have reviewed Lagrangian and Hamiltonian mechanics, we are ready to start dealing with the particular case of the rigid body. Let us start defining it. A *rigid body* is any solid body in which the distance between any of two of its points remains constant over time. Furthermore, we will assume no external forces acting on the body and there is a fixed point we will call it the *center of mass.*

We set an inertial frame, called *spatial coordinate system*, in which the origin and the center of mass coincides. We also fix the reference configuration and note with X the position of a given particle in this configuration. Normally, X is called the *label* of the particle. Denote by $x(X, t)$ and $\dot{x}(X, t)$ (also denoted by $x(t)$ and $\dot{x}(t)$) the position and velocity of the particle X of the body at time t in the inertial frame. Since the distance between any two particles is constant over time, the map which relates position $x(t)$ with X must preserve the distance, so it is an isometry in $\mathbb{R}^3$. As it is known, any isometry in the three dimensional space is obtained by composing a rotation with a translation. According to this, the resulting map might be written as

$$X \to RX + b$$

where R is a rotation matrix and $b \in \mathbb{R}^3$. Imposing the extra condition that the center of mass is fixed, we conclude that $b = 0$. Then, the relationship between $x(t)$ and X is given by

$$x(t) = R(t)X \tag{1.11}$$

Since $R(t)$ is a rotation matrix, it must be orthogonal and the following condition is satisfied

$$R(t)R^T(t) = Id \quad \forall t \;\Rightarrow\; |R(t)| = \pm 1$$

On the other hand, the motion is assumed to be continuous and $R(0) = Id$. Thus,

$$|R(t)| = 1 \quad \forall t \tag{1.12}$$

that is, $R(t)$ is a rotation matrix which preserves the orientation. Moreover, using again that $R(t)$ is a rotation matrix, we deduce that

$$\dot{R}R^T = -R\dot{R}^T \tag{1.13}$$

which means that $\dot{R}R^T$ is skew-symmetric. In other words, there exist ω_1, ω_2, ω_3 such that

$$\dot{R}R^T = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

On the other hand, taking derivatives with respect to time in (1.11) we get

$$\dot{x} = \dot{R}X = \dot{R}R^{-1}x = \dot{R}R^T x$$

If we define $\omega = (\omega_1, \omega_2, \omega_3)$, it is easy to check that,

$$\dot{x}(t) = \omega(t) \times x(t) \tag{1.14}$$

which gives the relationship between *linear velocity* and *angular velocity*. As well, if we define the *hat map* as

$$\hat{\omega} := \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

then,

$$\dot{R}R^{-1} = \hat{\omega} \quad \text{and} \quad \dot{x}(t) = \hat{\omega}(t)x(t) \tag{1.15}$$

It is interesting to point out that, later, in the next chapters, we will come back over the hat map but in a much more general way.

Due to equation (1.14), the *inertia tensor*, I, which will depend only on the geometry of the body and its mass distribution, can be introduced. Moreover, it can be shown that I is a symmetric and positive definite matrix, a fact which will simplify our work in the following lines.

Now, we will set a non-inertial frame whose origin coincides again with the center of mass and which is fixed in the body and moves with it. We will call it *body coordinate system*. Usually, our previous inertial frame is chosen in order to match with the body frame at some instant of time t . Notice that the position of a particle in body coordinates will remain constant and will be equal to its label X. It is in this sense that our new frame will be very useful to obtain the equations of motion of the body. Indeed, inertia tensor in body coordinates remains constant unlike in spatial coordinates where its components depend on time. Using all previous considerations it can be shown that the equations of motion of the rigid body in body coordinates are given by

$$I\dot{\omega} + \omega \times I\omega = 0 \tag{1.16}$$

Where I does not depend on time. Since I is symmetric and positive definite we can find a coordinate system in which I is diagonal. Let $I_1,\ I_2,\ I_3$ be the eigenvalues of I (which are positive), also called *moments of inertia*, then, equation (1.16) is rewritten as

$$\begin{aligned} I_1\dot{\omega}_1 &= (I_2 - I_3)\omega_2\omega_3 \\ I_2\dot{\omega}_2 &= (I_3 - I_1)\omega_3\omega_1 \\ I_3\dot{\omega}_3 &= (I_1 - I_2)\omega_1\omega_2 \end{aligned} \tag{1.17}$$

These are called the *Euler-Poincaré equations* for the motion of a rigid body and they are written in the body coordinate system. They are equations of motion since the integration of them determines the angular velocity ω which enables us to obtain the position by integration of equation (1.14).

Finally, if instead of working with angular velocity we do with *angular momentum* $\Pi = I\omega$ in body coordinates, we still get another form for the equations of motion

$$\dot{\Pi} = I \times \dot{\omega} \tag{1.18}$$

which leads us to

$$\begin{aligned} \dot{\Pi}_1 &= \frac{I_2 - I_3}{I_2 I_3}\Pi_2\Pi_3 \\ \dot{\Pi}_2 &= \frac{I_3 - I_1}{I_3 I_1}\Pi_3\Pi_1 \\ \dot{\Pi}_3 &= \frac{I_1 - I_2}{I_1 I_2}\Pi_1\Pi_2 \end{aligned} \tag{1.19}$$

These equations are known as the *Lie-Poisson equations* for the motion of a rigid body and they are also given in body coordinates. Finally, let us consider another form of writing (1.19) through the Lie-Poisson bracket of functions and the Hamiltonian defined below.

Let $f = f(\Pi_1, \Pi_2, \Pi_3)$ and $g = g(\Pi_1, \Pi_2, \Pi_3)$ be two real functions. We define the *Lie-Poisson bracket* for the rigid body as a new function $\{f, g\}$ in the variables (Π_1, Π_2, Π_3) given by

$$\{f,g\}(\Pi_1,\Pi_2,\Pi_3) = -(\Pi_1,\Pi_2,\Pi_3)\left(\left(\frac{\partial f}{\partial \Pi_1},\frac{\partial f}{\partial \Pi_2},\frac{\partial f}{\partial \Pi_3}\right)\times\left(\frac{\partial g}{\partial \Pi_1},\frac{\partial g}{\partial \Pi_2},\frac{\partial g}{\partial \Pi_3}\right)\right) \tag{1.20}$$

Next, we will introduce a real function h in the variables (Π_1, Π_2, Π_3) as follows,

$$h(\Pi_1,\Pi_2,\Pi_3) = \frac{1}{2}\left(\frac{\Pi_1^2}{I_1} + \frac{\Pi_2^2}{I_2} + \frac{\Pi_3^2}{I_3}\right) \tag{1.21}$$

and call it the *Hamiltonian* of the rigid body. By now, there is no reason for this name since, apparently, it has nothing to do with the Hamiltonian H defined in section 1.1 (note that H is defined on a space with an even number of variables and, moreover, h is a real function on $\mathbb{R}^3$). In any case, this new Hamiltonian is needed in order to write (1.19) in a much more suitable way. In fact, now (1.19) can be written as

$$\dot{\Pi}_i = \{\Pi_i, h\}\ \ i = 1,2,3 \tag{1.22}$$

Last theorem in this chapter will help us, among many other things, to understand why this function is called Hamiltonian.

1.2.1 Hamiltonian form

So far we have found the equations that describe the state of the system we would like to learn whether or not they are Hamiltonian, i.e., if their canonical form can be written as (1.8). Obviously, since (1.8) has an even number of equations and (1.19) has an odd number, they can not be equivalent. However, we are going to see that if we describe the rigid body using Euler angles, then the dynamical equations may be written in a Hamiltonian form. Once we build the Hamiltonian form of the equations of motion, the only thing that will be left is to study how the solutions of both systems are related. It will be our goal during the section and we will reach it in the last theorem.

Denote by

$$R_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad R_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \quad R_\psi = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.23}$$

the rotation through an angle φ, (θ, ψ respectively) about the axis OZ, (OX, OZ respectively). Denote also by $R_{\varphi\theta\psi}$ their composition

$$R_{\varphi\theta\psi} = \begin{pmatrix} \cos\varphi\cos\psi - \sin\varphi\cos\theta\sin\psi & -\cos\varphi\sin\psi - \sin\varphi\cos\theta\cos\psi & \sin\varphi\sin\theta \\ \sin\varphi\cos\psi + \cos\varphi\cos\theta\sin\psi & -\sin\varphi\sin\psi + \cos\varphi\cos\theta\cos\psi & -\cos\varphi\sin\theta \\ \sin\varphi\sin\psi & \sin\varphi\cos\psi & \cos\theta \end{pmatrix} \tag{1.24}$$

Then, it can be shown that any rotation R which preserves the orientation can be expressed in terms of $\varphi,\ \theta,\ \psi$, that is $R = R_{\varphi\theta\psi}$. The angles $\varphi,\ \theta,\ \psi$ are called *Euler angles* and they form a set of generalized coordinates.

Back to our example and reinterpreting the equation (1.11) from a geometric point of view, we can think it as a rotation from the body frame to the inertial frame . As in general this relationship depends on the time, the matrix R which gives this rotation also does. Using the results we have just proved, we can write $R(t) = R_{\varphi(t)\theta(t)\psi(t)}$. Therefore, a long computation using (1.15) proves that

$$\omega = \begin{pmatrix} \dot\theta\cos\psi + \dot\varphi\sin\psi\sin\theta \\ -\dot\theta\sin\psi + \dot\varphi\cos\psi\sin\theta \\ \dot\varphi\cos\theta + \dot\psi \end{pmatrix} \tag{1.25}$$

Or, in terms of angular the momentum Π,

$$\Pi = \begin{pmatrix} I_1(\dot\theta\cos\psi + \dot\varphi\sin\psi\sin\theta) \\ I_2(-\dot\theta\sin\psi + \dot\varphi\cos\psi\sin\theta) \\ I_3(\dot\varphi\cos\theta + \dot\psi) \end{pmatrix} \tag{1.26}$$

Now, we may introduce a new real function $T = T(\varphi,\ \theta,\ \psi,\ \dot\varphi,\ \dot\theta,\ \dot\psi)$. T is the Lagrangian function L on the phase space of velocities $(\varphi,\ \theta,\ \psi,\ \dot\varphi,\ \dot\theta,\ \dot\psi)$. Indeed, using (1.21) and (1.26) we define T to be

$$T(\varphi,\ \theta,\ \psi,\ \dot\varphi,\ \dot\theta,\ \dot\psi) = h\left(I_1(\dot\theta\cos\psi + \dot\varphi\sin\psi\sin\theta),\ I_2(-\dot\theta\sin\psi + \dot\varphi\cos\psi\sin\theta),\ I_3(\dot\varphi\cos\theta + \dot\psi)\right) \tag{1.27}$$

However, what we are looking for is a Hamiltonian function in the phase space of momenta. To do this we can calculate the Legendre transformation according to (1.5) and (1.4) and verify that it is a diffeomorphism. This allows us to ensure that we can solve for $(\dot\varphi,\ \dot\theta,\ \dot\psi)$ in terms of $\varphi,\ \theta,\ \psi$ and the conjugate momenta p_φ, p_θ and p_ψ.

If all indicated calculations and verifications are made, then we can rewrite (1.26) in terms of $\varphi,\ \theta,\ \psi,\ p_\varphi,\ p_\theta,\ p_\psi$ as follows

$$\Pi = \begin{pmatrix} \frac{1}{\sin\theta}\left((p_\varphi - p_\psi\cos\theta)\sin\psi + p_\theta\sin\theta\cos\psi\right) \\ \frac{1}{\sin\theta}\left((p_\varphi - p_\psi\cos\theta)\cos\psi - p_\theta\sin\theta\sin\psi\right) \\ p_\psi \end{pmatrix} \tag{1.28}$$

Now, using (1.21) and (1.28), the real function T may be considered as a Hamiltonian function in the phase space of momenta, that is, a real function in the variables $(\varphi,\ \theta,\ \psi,\ p_\varphi,\ p_\theta,\ p_\psi)$. H is given by

$$H(\varphi,\ \theta,\ \psi,\ p_\varphi,\ p_\theta,\ p_\psi) =$$
$$\frac{1}{2}\left(\frac{((p_\varphi - p_\psi\cos\theta)\sin\psi + p_\theta\sin\theta\cos\psi)^2}{I_1\sin^2\theta} + \frac{((p_\varphi - p_\psi\cos\theta)\cos\psi + p_\theta\sin\theta\sin\psi)^2}{I_1\sin^2\theta} + \frac{p_\psi^2}{I_3}\right) \tag{1.29}$$

Now that we have defined the Hamiltonian function, do not lose sight of our goal. We want to express in a Hamiltonian form the equations of motion (1.19) for the rigid body. First of all, we will need to relate the variables $\varphi,\ \theta,\ \psi,\ p_\varphi,\ p_\theta,\ p_\psi$ with the variables Π_1, Π_2, Π_3. But we have already done it in (1.28). Just define J, called the *momentum map*, as the map which gives this relation

$$(\varphi,\ \theta,\ \psi,\ p_\varphi,\ p_\theta,\ p_\psi) \xrightarrow{J} \begin{pmatrix} \frac{1}{\sin\theta}\left((p_\varphi - p_\psi\cos\theta)\sin\psi + p_\theta\sin\theta\cos\psi\right) \\ \frac{1}{\sin\theta}\left((p_\varphi - p_\psi\cos\theta)\cos\psi - p_\theta\sin\theta\sin\psi\right) \\ p_\psi \end{pmatrix} \tag{1.30}$$

From the definition above, it is clear that $H = h \circ J$. Moreover, recalling (1.9) it is easy to check after some calculations that $\{f, g\} \circ J = \{f \circ J, g \circ J\}_{can}$, where f and g are two real functions in the variables Π_1, Π_2, Π_3. Indeed, if $\tilde{\Pi}_i = J_i(\varphi, \theta, \psi, p_\varphi, p_\theta, p_\psi)$ $i = 1, 2, 3$,

$$\{f, g\} \circ J(\varphi, \theta, \psi, p_\varphi, p_\theta, p_\psi) = \{f, g\} = (\tilde{\Pi}_1, \tilde{\Pi}_2, \tilde{\Pi}_3)$$

$$= -\tilde{\Pi}_1\left(\frac{\partial f}{\partial \Pi_2}\frac{\partial g}{\partial \Pi_3} - \frac{\partial f}{\partial \Pi_3}\frac{\partial g}{\partial \Pi_2}\right) - \tilde{\Pi}_2\left(\frac{\partial f}{\partial \Pi_3}\frac{\partial g}{\partial \Pi_1} - \frac{\partial f}{\partial \Pi_1}\frac{\partial g}{\partial \Pi_3}\right) - \tilde{\Pi}_3\left(\frac{\partial f}{\partial \Pi_1}\frac{\partial g}{\partial \Pi_2} - \frac{\partial f}{\partial \Pi_2}\frac{\partial g}{\partial \Pi_1}\right)$$

On the other hand, using that $\frac{\partial\tilde{\Pi}_1}{\partial\varphi} = \frac{\partial\tilde{\Pi}_2}{\partial\varphi} = \frac{\partial\tilde{\Pi}_3}{\partial\varphi} = \frac{\partial\tilde{\Pi}_3}{\partial\theta} = \frac{\partial\tilde{\Pi}_3}{\partial p_\theta} = \frac{\partial\tilde{\Pi}_3}{\partial\psi} = 0$, we deduce that

$$\{f \circ J, g \circ J\}_{can}(\varphi, \theta, \psi, p_\varphi, p_\theta, p_\psi) =$$

$$= \left(\frac{\partial f}{\partial \Pi_2}\frac{\partial g}{\partial \Pi_3} - \frac{\partial f}{\partial \Pi_3}\frac{\partial g}{\partial \Pi_2}\right)\left(\frac{\partial\tilde{\Pi}_2}{\partial\psi}\right) + \left(\frac{\partial f}{\partial \Pi_3}\frac{\partial g}{\partial \Pi_1} - \frac{\partial f}{\partial \Pi_1}\frac{\partial g}{\partial \Pi_3}\right)\left(-\frac{\partial\tilde{\Pi}_1}{\partial\psi}\right) +$$
$$+ \left(\frac{\partial f}{\partial \Pi_1}\frac{\partial g}{\partial \Pi_2} - \frac{\partial f}{\partial \Pi_2}\frac{\partial g}{\partial \Pi_1}\right)\left(\frac{\partial\tilde{\Pi}_1}{\partial\psi}\frac{\partial\tilde{\Pi}_2}{\partial p_\psi} + \frac{\partial\tilde{\Pi}_1}{\partial\theta}\frac{\partial\tilde{\Pi}_2}{\partial p_\theta} - \frac{\partial\tilde{\Pi}_1}{\partial p_\psi}\frac{\partial\tilde{\Pi}_2}{\partial\psi} - \frac{\partial\tilde{\Pi}_1}{\partial p_\theta}\frac{\partial\tilde{\Pi}_2}{\partial\theta}\right)$$

Then, the equality arises from

$$\tilde{\Pi}_1 = -\frac{\partial\tilde{\Pi}_2}{\partial\psi}$$
$$\tilde{\Pi}_2 = \frac{\partial\tilde{\Pi}_1}{\partial\psi}$$
$$-\tilde{\Pi}_3 = \frac{\partial\tilde{\Pi}_1}{\partial\psi}\frac{\partial\tilde{\Pi}_2}{\partial p_\psi} + \frac{\partial\tilde{\Pi}_1}{\partial\theta}\frac{\partial\tilde{\Pi}_2}{\partial p_\theta} - \frac{\partial\tilde{\Pi}_1}{\partial p_\psi}\frac{\partial\tilde{\Pi}_2}{\partial\psi} - \frac{\partial\tilde{\Pi}_1}{\partial p_\theta}\frac{\partial\tilde{\Pi}_2}{\partial\theta}$$

So far, we have proved two of the three statements of the following theorem.

Theorem 1.1. *Let h and J be the Hamiltonian and the momentum map for the rigid body and H be the Hamiltonian function on the phase space of momenta. Let $\{\ ,\ \}_{can}$ be the canonical Poisson bracket and $\{\ ,\ \}$ the Lie-Poisson bracket. Then,*

i. $H = h \circ J$

ii. If f and g are two real functions in the variables Π_1, Π_2, Π_3*,*

$$\{f,g\} \circ J = \{f \circ J, g \circ J\}_{can}$$

iii. If $\sigma : t \to (\varphi(t), \theta(t), \psi(t), p_\varphi(t), p_\theta(t), p_\psi(t))$ *is a solution of the Hamilton equations for the Hamiltonian* H*,* $J \circ \sigma : t \to (\Pi_1(t), \Pi_2(t), \Pi_3(t))$ *is a solution of the Lie-Poisson equations for h, i.e, a solution of (1.22).*

Proof. It is enough to check that,

$$\dot{\Pi}_i(t) = \{\Pi_i, h\}$$

is true under the assumption that Hamilton equations holds true for H, i.e,

$$\dot{\varphi} = \frac{\partial H}{\partial p_\varphi}, \qquad \dot{p_\varphi} = -\frac{\partial H}{\partial \varphi},$$
$$\dot{\theta} = \frac{\partial H}{\partial p_\theta}, \qquad \dot{p_\theta} = -\frac{\partial H}{\partial \theta},$$
$$\dot{\psi} = \frac{\partial H}{\partial p_\psi}, \qquad \dot{p_\psi} = -\frac{\partial H}{\partial \psi}.$$

But using *ii.* and *i.* we obtain,

$$\begin{aligned}\{\Pi_i, h\} (\tilde{\Pi}_1, \tilde{\Pi}_2, \tilde{\Pi}_3) = \{\Pi_i, h\} \circ J(\varphi, \theta, \psi, p_\varphi, p_\theta, p_\psi) &= \{\Pi_i \circ J, h \circ J\}_{can} (\varphi, \theta, \psi, p_\varphi, p_\theta, p_\psi) \\ &= \{\Pi_i, H\}_{can} (\varphi, \theta, \psi, p_\varphi, p_\theta, p_\psi)\end{aligned}$$

Then, using the Hamilton equations we get the result

$$\begin{aligned}\{\Pi_i, H\}_{can} &= \frac{\partial \Pi_i}{\partial \varphi}\frac{\partial H}{\partial p_\varphi} + \frac{\partial \Pi_i}{\partial \theta}\frac{\partial H}{\partial p_\theta} + \frac{\partial \Pi_i}{\partial \psi}\frac{\partial H}{\partial p_\psi} - \frac{\partial \Pi_i}{\partial p_\varphi}\frac{\partial H}{\partial \varphi} - \frac{\partial \Pi_i}{\partial p_\theta}\frac{\partial H}{\partial \theta} - \frac{\partial \Pi_i}{\partial p_\psi}\frac{\partial H}{\partial \psi} \\ &= \frac{\partial \Pi_i}{\partial \varphi}\dot{\varphi} + \frac{\partial \Pi_i}{\partial \theta}\dot{\theta} + \frac{\partial \Pi_i}{\partial \psi}\dot{\psi} + \frac{\partial \Pi_i}{\partial p_\varphi}\dot{p}_\varphi + \frac{\partial \Pi_i}{\partial p_\theta}\dot{p}_\theta + \frac{\partial \Pi_i}{\partial p_\psi}\dot{p}_\psi \\ &= \dot{\Pi}_i.\end{aligned}$$

□

The most important idea that we have to keep in mind after viewing this theorem is that the inital system of ODE's had three equations while the latter has six. In practice, usually, the fewer ODE's the system has the easier it is to solve. This is precisely the main motivation of the project, which aims to study under what conditions the number of equations describing a system can be reduced, and obviously, the rigid body is one particular case.

Chapter 2

Lagrangian and Hamiltonian formalisms

After a first part which was, essentially physics, here we will start working with mathematics. Nevertheless, this chapter will be, basiquely, a review of concepts which are already known. It means that we will look over it quickly in order to set the mathematical basis that we will use later. Anyway, for more information on these topics we remit to [AbMa],[LeRo], [Hol].

First of all, we associate to any mechanical system a structure of smooth manifold and we call it the configuration space. It allows us to introduce the phase space of the velocities and the phase space of momenta as the tangent and cotangent bundle of that manifold, respectively. Then, we will use the geometric structure of the tangent and the cotangent bundle in order to develop the Lagrangian and the Hamiltonian formalisms and their equivalence in the hyperregular case. But, before that, we will go over symplectic manifolds since they will be essential in order to present some important propierties in the Hamiltonian formalism.

2.1 Symplectic manifolds

Symplectic manifolds are one of the main ingredients of theory that we are working up, and that is why we give a brief overview. At first, we have that the cotangent bundle of any manifold has symplectic structure. This will allow us to see the Hamiltonian formalism from a more interesting perspective. As well, when we define Poisson manifolds, we will see that they have a symplectic foliation associated with them. These two natural symplectic structures will be crucial in order to prove the most important results of the project.

Let us start with the definition of an almost symplectic manifold and its immediate properties.

Definition 2.1. *An* almost symplectic form *over a manifold Q is a non-degenerate 2-form ω. The couple (Q, ω) is said to be an* almost symplectic manifold.

Proposition 2.2. *i. If ω is an almost symplectic form over Q, then Q has even dimension.*

ii. If ω is an almost symplectic form over Q and $dimQ = 2n$, then $\omega^n = \omega \wedge \overset{(n}{\ldots} \wedge \omega \neq 0$, that is,

$$\omega(x) = \omega(x) \wedge \overset{(n}{\ldots} \wedge \omega(x) \neq 0 \quad \forall\, x \in Q$$

iii. Any almost symplectic manifold (Q, ω) is orientable.

Let ω be any 2-form (not necessarily non-degenerate) defined over a manifold Q. It induces a vector bundle morphism between the tangent bundle and the cotangent bundle $b_\omega : TQ \to T^*Q$ as follows

$$b_{\omega|T_xQ} = b_{\omega(x)} : \begin{array}{ccc} T_xQ & \to & T_x^*Q \\ v & \to & \omega(x)(v, \cdot) \end{array} \tag{2.1}$$

Moreover, we have the following result.

Proposition 2.3. *Let ω be a 2-form over a manifold Q, then the following statements are equivalent:*

i. *ω is an almost symplectic form over Q.*

ii. *b_ω is an isomorphism of vector bundles.*

Let (Q, ω) be an almost symplectic manifold and b_ω the induced isomorphism. With the definition above, b_ω induces a map between the sapce $\mathcal{X}(Q)$ of the vector fields on Q and the space $\Omega^1(Q)$ of the 1-forms on Q,

$$b_\omega : \mathcal{X}(Q) \to \Omega^1(Q) \tag{2.2}$$

denoted again by b_ω by abuse of notation. Indeed, for $x \in Q$ and $X \in \mathcal{X}(Q)$ we define $b_\omega(X)(x) = b_{\omega(x)}(X(x))$. With this definition, it can be shown that b_ω is an isomorphism of $\mathcal{F}(Q)$-modules. All those results that we have just seen will be extensively used in order to characterize symplectic manifolds, so it is important to bear them in mind.

After being working on almost symplectic manifolds, we will go on to symplectic manifolds.

Definition 2.4. *An almost symplectic form ω over a manifold Q is a* symplectic form *if it is closed , i.e, $d\omega = 0$. The couple (Q, ω) is said to be a* symplectic manifold

Now, we will introduce the definition of a symplectic map.

Definition 2.5. *Let (Q, ω) and (S, α) be symplectic manifolds such that $dimQ = dimS = 2n$. A map $h : Q \to S$ is called* symplectic map *if $h^*\alpha = \omega$, i.e.,*

$$\alpha_{h(x)} = (T_x hX, T_x hY) = \omega_x(X, Y) \quad \forall x \in Q \text{ and } X, Y \in T_x Q$$

This last definition is telling us that if h is a symplectic map, then it is a local diffeomorphism.

Definition 2.6. *Let (Q, ω) and (S, α) be symplectic manifolds and $h : Q \to S$ be a symplectic map.*

i. *If h is bijective then h is said to be a* symplectomorphism.

ii. *If h is a symplectomorphism, $\omega = \alpha$ and $Q = S$, we will call h a* canonical transformation.

Due to these definitions we can define a symplectic vector field as follows.

Definition 2.7. *Let (Q, ω) be a symplectic manifold. A vector field X over Q is called a* symplectic vector field *if its flow consists on symplectic maps.*

Since it is not at all easy to check that the flow of a vector field is made up of symplectic maps, we give a last result which characterizes the symplectic vector fields in terms of the Lie derivative and the isomorphism b_ω. This result is proved by using the Poincaré lemma which is the reason motivating us to enunciate it. As well, although it is not exactly part of the symplectic theory, it is very helpful to prove many results.

Lemma 2.8 (Poincaré Lemma)**.** *Let ω be a closed p-form over a manifold Q. Then, for any $x \in Q$, there exists an open set U such that $x \in U$ and a (p-1)-form α defined over U, such that $\omega = d\alpha$ in U.*

Proposition 2.9. *Let (Q, ω) be a symplectic manifold and X be a vector field over Q. Then, the following statements are equivalent:*

i. *X is a symplectic vector field.*

ii. *The Lie derivative of ω with respect to X is zero, that is, $\mathcal{L}_X \omega = 0$.*

iii. *For any $x \in Q$, there exists an open set U in Q such that $x \in U$ and a smooth function $f : U \to \mathbb{R}$ such that $b_\omega(X) = df$ in U.*

Our review on symplectic geometry ends here (for more details on symplectic geometry see, for instance, [AbMa], [LeRo]). Nontheless, before moving on, we will see an example of what we have been doing. Specifically, we will show that given any manifold Q, its cotangent bundle T^*Q can be provided with a symplectic structure. Therefore, this will yield to plenty of examples and, at the same time, it will leave a very rich theory from which we can keep working on.

2.1.1 Canonical symplectic structure of the cotangent bundle

From now until the end of this section assume that Q is a manifold, T^*Q its cotangent bundle and $\Pi_Q : T^*Q \to Q$ the canonical projection. Let us begin defining the Liouville 1-form which is key in order to obtain a symplectic structure in T^*Q.

Definition 2.10. *The* Liouville 1-form*, is the 1-form $\lambda_Q : T^*Q \to T^*(T^*Q)$ over the cotangent bundle given by*

$$\lambda_Q(\alpha)(X) = \alpha(T_\alpha \Pi_Q(X)) \quad \textit{for} \quad \alpha \in T^*Q \quad \textit{and} \quad X \in T_\alpha(T^*Q)$$

It can be easly shown that if $(U, \varphi \equiv (q^1, \ldots, q^n))$ is a local chart in Q and $(\pi_Q^{-1}(U), \underline{\varphi} \equiv (q^1, \ldots, q^n, p_1, \ldots, p_n))$ is the corresponding induced chart, then λ_Q can be expressed locally as

$$\lambda_Q = \sum_{i=1}^{n} p_i dq^i \tag{2.3}$$

And not as easily, we have the following proposition which gives a characterization of the Liouville 1-form.

Proposition 2.11. *The Liouville 1-form λ_Q is the unique 1-form over T^*Q satisfying $\beta^*\lambda_Q = \beta$, for any 1-form β on Q.*

Now, let us define a closed 2-form ω_Q over T^*Q as $\omega_Q = -d\lambda_Q$. Taking the same charts as before we conclude

$$\omega_Q = \sum_{i=1}^{n} dq^i \wedge dp_i \tag{2.4}$$

which implies that the 2-form is non-degenerate.

Definition 2.12. *The 2-form ω_Q over T^*Q defined as $\omega_Q = -d\lambda_Q$ is called the* canonical symplectic structure *of the cotangent bundle.*

This symplectic 2-form is fundamental in order to prove the last result of this section. It will show a property that satisfies any diffeomorphism defined over a manifold. But first, remember the definition of the cotangent lift of a map.

Let $F : Q \to Q$ be a diffeomorphism. Define the *cotangent lift* of F, $T^*F : T^*Q \to T^*Q$ as

$$\begin{array}{rccc} T_x^*F = T^*F_{|T_x^*Q}: & T_x^*Q & \to & T_{F^{-1}(x)}^*Q \\ & \alpha & \to & T^*F(\alpha) \end{array} \tag{2.5}$$

where $T^*F(\alpha)(X) = \alpha((T_{F^{-1}(x)}F)(X))$ for $X \in T_{F^{-1}(x)}Q$.

Proposition 2.13. *If $F : Q \to Q$ is a diffeomorphism then the cotangent lift of F, $T^*F : T^*Q \to T^*Q$, is a symplectomorphism.*

With this proposition ends the part of the project dedicated explicitly to symplectic geometry. From now on, we will use all these results in order to reach our goal.

2.2 Mechanics on manifolds

We began the first chapter giving some basic background about Lagrangian and Hamiltonian mechanics. These notions were enough to deal with the example of the rigid body from a physical perspective. Nevertheless, since our current concern is to go back to the example but from a mathematical perspective, what we have introduced will not be enough. Thus, we will go over the same, but now, in a deeper way. The only matter that we will see later is the Poisson bracket that will be discussed in the chapter of Poisson manifolds.

Just like before, our main interest is the Hamiltonian formulation but we will obtain it from the Lagrangian formulation via the Legendre transformation. But first, we will focus on a fact that we have left out in the first chapter: identify in what kind of spaces we are formulating our theories.

2.2.1 Configuration manifold, phase space of the velocities and phase space of momenta

In the last chapter when we introduced the equations of motion, we referred to the variables describing the system, but we did not specify what they are. Here we will go back into it in depth.

First of all, let us define the *configuration space* of a given mechanical system as the set of all possible positions of the system. In general, it is not an Euclidean space but a manifold, and that is why it is also known as the *configuration manifold*. Under this definition, the variables describing the system will be the local coordinates of the manifold, noted as q^i. They are known as *generalized coordinates* and their number will determine the dimension of the configuration manifold which coincides with the degrees of freedom of the system.

As an example, recall the case of a N-particle system that we have already seen. In this case, the configuration manifold was $\mathbb{R}^{3N}$ since no restrictions were imposed over the positions of the system. A more general case will be one in which the system is subjected to certain constraints, to adjust onto a given shape or to verify some properties, for instance. In general, these constraints can be expressed as

$$f_j(x_i) = 0 \qquad \text{for} \quad i = 1, \dots N \quad \text{and} \quad j = 1, \dots k \leq 3N$$

Here $(x_1, \dots, x_{3N})$ are the standard coordinates on $\mathbb{R}^3$. Then, the configuration manifold is $Q = \{x_i \in \mathbb{R}^k | \ f_j(x_i) = 0 \ \}$.

However, it is not enough to determine the positions in order to describe the physical state of the system. For this reason, we introduce the phase space of velocities and the phase space of momenta. The *phase space of velocities* will be the set of all possible velocities of the system. Given a position $q \in Q$, mathematically, the set of all possible velocities in that position will be the tangent space at the point $q \in Q$, that is, T_qQ. So, the phase space of the velocities is the tangent bundle TQ with coordinates $(q^i, \dot{q}^i)$, since we are not imposing any constraint to the velocities. In an analogous way, define the *phase space of momenta* as the cotangent bundle T^*Q of the configuration manifold with coordinates (q^i, p_i).

We will spend the rest of the chapter studying two equivalent ways of obtaining the equations of motion of an autonomous and conservative mechanical system. First, we will obtain them in the phase space of velocities and after that in the phase space of momenta where we will use the symplectic structure that we have seen before.

2.2.2 Lagrangian mechanics

As one can deduce from the first chapter approach, Lagrangian mechanics is formulated over the phase space of velocities. We can define the Lagrangian function on it and deduce the Euler-Lagrange equations from a variational perspective. Our purpose now is to accurately show that Hamilton variational principle leads to the Euler-Lagrange equations for a given Lagrangian L.

From now on, suppose Q is the configuration manifold of some autonomous and conservative mechanical system, TQ its phase space of velocities and T^*Q its phase space of momenta. Let us define the Lagrangian function of that mechanical system as follows.

Definition 2.14. *A smooth function* $L : TQ \to \mathbb{R}$ *is called a* Lagrangian*.*

Suppose that $L = L(q^i, \dot{q}^i)$ is a Lagrangian function and $q^i(t_0)$ and $q^i(t_1)$ for $i = 1, \ldots, n$ are the positions of the system in two given instants of time t_0 and t_1. Our interest is to determine which is the physical trajectory that the system takes between those instants.

To do this, consider the family of trajectories $q^i(t, s)$ parameterized by $s \in [-\epsilon, \epsilon]$ such that $q^i(t_0, s) = q^i(t_0)$ and $q^i(t_1, s) = q^i(t_1)$ $\forall s$ for $i = 1, \ldots, n$. As well, we require that s parameterizes the family in a smooth way which guarantees that $\frac{\partial q^i(t,s)}{\partial s}$ exists in the time interval $[t_0, t_1]$. Let us consider also the following functional action

$$S(s) = \int_{t_0}^{t_1} L(q^i(t, s), \dot{q}^i(t, s))dt \tag{2.6}$$

which depends on s since it depends on the trajectory and the trajectory depends on s. What the *Hamilton variational principle* states is that among all possible trajectories, the system will take the one that minimizes that functional. Let us try to put it in mathematical terms.

We can assume without loss of generality, that the minimum is reached when $s = 0$. Denote by $\delta = \frac{d}{ds}_{|s=0}$ and call it *variation*. Then, the Hamilton variational principle states that the physical trajectories of the system verify

$$\delta S = \delta \int_{t_0}^{t_1} L(q^i, \dot{q}^i)dt = 0 \tag{2.7}$$

Due to the fact that the s-dependence of the second term comes form the s-dependence of q^i and $\dot{q}^i$ involved in the Lagrangian, (2.7) is equivalent to

$$\delta S = \int_{t_0}^{t_1} \delta L(q^i, \dot{q}^i)dt = 0 \tag{2.8}$$

Now, using

$$\delta L = \sum_{i=1}^{n} \left(\frac{\partial L}{\partial q^i}\delta q^i + \frac{\partial L}{\partial \dot{q}^i}\delta \dot{q}^i \right) \tag{2.9}$$

and

$$\frac{\partial L}{\partial \dot{q}^i}\delta \dot{q}^i = \frac{\partial L}{\partial \dot{q}^i}\frac{d}{dt}\delta q^i = \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}^i}\delta q^i\right] - \left[\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}\right]\delta q^i \qquad \text{for } i = 1, \ldots, n \tag{2.10}$$

where the equality of the mixed partials has been used, we have

$$\delta L = \sum_{i=1}^{n} \left(\left[\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}\right]\delta q^i + \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}^i}\delta q^i\right] \right). \tag{2.11}$$

Thus, inserting (2.11) to (2.8),

$$0 = \delta S = \sum_{i=1}^{n} \int_{t_0}^{t_1} \left[\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}\right]\delta q^i dt + \sum_{i=1}^{n} \int_{t_0}^{t_1} \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}^i}\delta q^i\right] dt \tag{2.12}$$

The second term of the integral is 0 since $q^i(s, t)$ has fixed endpoints in t_0 and t_1, which implies $\delta q^i(t_0) = \delta q^i(t_1) = 0$ for $i = 1, \ldots, n$. Finally, because δq^i are arbitrary, we conclude

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0 \qquad \text{for } i = 1, \ldots, n \tag{2.13}$$

which are the *Euler-Lagrange* equations. Then, we have just proved that a trajectory $q^i(t)$ verifies the Euler-Lagrange equations if and only if it verifies Hamilton variational principle.

Remark The proof above is not entirely general. In fact, the result is proved only in the case that the two endpoints and the whole family of trajectories lie in the same chart (for instance, the result is proved for Euclidean spaces). If that hypothesis was not true, we could break the trajectory into subtrajectories, each one of them remaining in one chart. Also, we could do the same with the paths of the family $q^i(t,s)$. These subpaths do not verify, necessarily, the propriety of leaving the endpoints fixed, so we could not use the result that we have jut proved. Nontheless, from (2.12) and assuming that the endpoints are not fixed, we get

$$\sum_{i=1}^{n}\int_{t_0}^{t_1}\left[\frac{\partial L}{\partial q^i}-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}\right]\delta q^i dt=\sum_{i=1}^{n}\frac{\partial L}{\partial \dot{q}^i}\delta q^i\Bigg|_{t_0}^{t_1} \tag{2.14}$$

Taking advantage of this result which holds for any path, it can be shown, without much work, that a trajectory $q^i(t)$ verifies the Hamilton variational principle, if and only if, it verifies Euler-Lagrange equations in every local coordinate system.

Finally, remember from the first chapter the definition of a regular Lagrangian.

Definition 2.15. *A Lagrangian L is* regular *if the Hessian matrix $\left|\frac{\partial L}{\partial \dot{q}^i \partial \dot{q}^j}\right|$ is invertible for every system of local coordinates $(q^i, \dot{q}^i)$ on TQ.*

In the next section we will also remember the definition of a hyperregular Lagrangian, the one we need. We are not giving it now because more definitions are necessary to do it.

2.2.3 Legendre transformation and Hamilton Equations

First of all, we will introduce the Legendre transformation associated with a Lagrangian function.

Definition 2.16. *Let $L : TQ \to \mathbb{R}$ be a Lagrangian, we define the corresponding* Legendre transformation *$Leg_L : TQ \to T^*Q$ as*

$$\begin{array}{rccc} Leg_{L|T_qQ}: & T_qQ & \to & T_q^*Q \\ & u_q & \to & Leg_L(u_q) \end{array}$$

where

$$\begin{array}{rcl} Leg_L(u_q) & : T_qQ & \longrightarrow \mathbb{R} \\ & v_q & \to \dfrac{d}{ds}_{|s=0} L(u_q+tv_q) \end{array}$$

Now, we are going to see that in local coordinates it has the same expression as the Legendre transformation defined in Chapter 1.

Let $(U, \varphi \equiv (q^1,\dots,q^n))$ be a local chart in Q, $(\tau_Q^{-1}(U), \overline{\varphi} \equiv (q^1,\dots,q^n,\dot{q}^1,\dots,\dot{q}^n))$ be the corresponding chart in TQ and $(\pi_Q^{-1}(U), \underline{\varphi} \equiv (q^1,\dots,q^n,p_1,\dots,p_n))$ be the corresponding chart in T^*Q. Then, $(\underline{\varphi}\circ Leg_L \circ \overline{\varphi}^{-1})$ is locally given by

$$\begin{array}{ccccccc} \overline{\varphi}(\tau_Q^{-1}(U)) & \xrightarrow{\overline{\varphi}^{-1}} & \tau_Q^{-1}(U) & \xrightarrow{Leg_L} & \pi_Q^{-1}(U) & \xrightarrow{\underline{\varphi}} & \underline{\varphi}(\pi_Q^{-1}(U)) \\ (q^i,\dot{q}^i) & \longrightarrow & \sum_{i=1}^n \dot{q}^i \frac{\partial}{\partial q^i}_{|\varphi^{-1}(q^j)} & \longrightarrow & \sum_{i=1}^n \frac{\partial L(\overline{\varphi}^{-1}(q^j,\dot{q}^j))}{\partial \dot{q}^i} dq^i_{|\varphi^{-1}(q^j)} & \longrightarrow & (q^i, \frac{\partial L(\overline{\varphi}^{-1}(q^j,\dot{q}^j))}{\partial \dot{q}^i}) \end{array} \tag{2.15}$$

Therefore, given a vector $x \in \tau_Q^{-1}(U)$ the matrix associated to the linear map $T_xLeg_L : T_x(TQ) \to T_{Leg_L}(T^*Q)$with respect to the bases $\left\{\frac{\partial}{\partial q^i}_{|x}, \frac{\partial}{\partial \dot{q}^i}_{|x}\right\}$ and $\left\{\frac{\partial}{\partial q^i}_{|Leg_L(x)}, \frac{\partial}{\partial p_i}_{|Leg_L(x)}\right\}$ is

$$\begin{pmatrix} I & 0 \\ \frac{\partial^2 L}{\partial q^i \partial \dot{q}^j} & \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}_j} \end{pmatrix} \tag{2.16}$$

Thus, the Lagrangian will be regular if and only if Leg_L is a local diffeomorphism.

Definition 2.17. *A Lagrangian L is* hyperregular *if the associated Legendre transformation is a global diffeomorphism.*

The hyperregular Lagrangians will be very important since they allow us to introduce the Hamiltonian function of the system. First, we will introduce the Lagrangian energy associated with a Lagrangian function.

Definition 2.18. *The* energy function *for a Lagrangian $L : TQ \to \mathbb{R}$ is the function E_L, defined by*

$$\begin{array}{rccc} E_L: & TQ & \longrightarrow & \mathbb{R} \\ & v & \longrightarrow & Leg_L(v)(v) - L(v) \end{array}$$

In the previous charts the local expression will be

$$E_L = \sum_{i=1}^{n} \dot{q}^i \frac{\partial L}{\partial \dot{q}^i}_{|(\overline{\varphi}^{-1}(q^j,\dot{q}^j))} - L(\overline{\varphi}^{-1}(q^j,\dot{q}^j)) \tag{2.17}$$

Remark Notice that if the Lagrangian is the kinetic minus the potential energy as in (1.3), then the energy function is

$$E_L = \sum_{i=1}^{n} m_i(\dot{q}^i)^2 - \frac{1}{2}\sum_{i=1}^{n} m_i(\dot{q}^i)^2 + V(q^i) = \sum_{i=1}^{n} m_i(\dot{q}^i)^2 + V(q^i) \tag{2.18}$$

that is, the kinetic plus the potential energy, so the total energy of the system.

Finally, we are ready to introduce the Hamiltonian function.

Definition 2.19. *Let $L : TQ \to \mathbb{R}$ be an hyperregular Lagrangian and E_L the energy function associated to L. Then, the* Hamiltonian *function $H : T^*Q \to \mathbb{R}$ is given by*

$$H = E_L \circ {Leg_L}^{-1}$$

Again, assuming that $Leg_L(q^i, \dot{q}^i) = (q^i, p_i)$, the local expression of the Hamiltonian function in the given charts will be

$$H(q^i, p_i) = \sum_{i=1}^{n} p_i \dot{q}^i(q^j, p_j) - L(q^i, \dot{q}^i(q^j, p_j)) \tag{2.19}$$

which makes sense since the Lagrangian is hyperregular.

Taking partial derivatives in (2.19) we obtain

$$\frac{\partial H}{\partial q^i} = \sum_{j=1}^{n} p_j \frac{\partial \dot{q}^j}{\partial q^i} - \frac{\partial L}{\partial q^i} - \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}^j}\frac{\partial \dot{q}^j}{\partial q^i} = -\frac{\partial L}{\partial q^i} \tag{2.20}$$

and,

$$\frac{\partial H}{\partial p_i} = \dot{q}^i + \sum_{j=1}^{n} p_j \frac{\partial \dot{q}^j}{\partial p_i} - \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}^j}\frac{\partial \dot{q}^j}{\partial p_i} = \dot{q}^i \tag{2.21}$$

On the other hand, using Euler-Lagrange equations (2.13), we get,

$$\frac{\partial L}{\partial q^i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) = \frac{dp_i}{dt} \tag{2.22}$$

which together with $\frac{dq^i}{dt} = \dot{q}^i$, (2.20) and (2.21), leads us to the Hamilton equations

$$\dot{q}^i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i} \qquad \text{for } i = 1, \dots, n \tag{2.23}$$

Conversely, if $t \to (q^i(t), p_i(t))$ is a solution of the Hamilton equations for H, then $t \to (q^i(t))$ is a solution of Euler-Lagrange equations for L.

So, we have just proved the following theorem.

Theorem 2.20. *Let $L : TQ \to \mathbb{R}$ be any hyperregular Lagrangian, $H : T^*Q \to \mathbb{R}$ the corresponding Hamiltonian function and $\sigma : I \to Q$ be a curve on Q. Then, σ is a solution of the Euler-Lagrange equations for L if and only if $Leg_L \circ \dot{\sigma} : I \to T^*Q$ is a solution of the Hamilton equations for H.*

As a result of the Legendre transformation, we have introduced Hamilton equations and we have shown their equivalence to Lagrangian ones. Nevertheless, this way of viewing the Hamiltonian formalism has some disadvantages since it has left some important properties of Hamiltonian systems out. In the next section, we will see that one may associate a vector field to an arbitrary Hamiltonian function on T^*Q. For this purpose, we will use the canonical symplectic structure of the cotangent bundle.

2.2.4 Hamiltonian mechanics and the symplectic structure of the cotangent bundle

Let Q be a smooth manifold. Denote by ω_Q the canonical symplectic structure of T^*Q and by $b_{\omega_Q} : \mathcal{X}(T^*Q) \to \Omega^1(T^*Q)$ the corresponding isomorphism of $\mathcal{F}(Q)$-modules.

It induces the following definitions.

Definition 2.21. *If $H : T^*Q \to \mathbb{R}$ is a Hamiltonian function on T^*Q then the* Hamiltonian vector field *of H is just the vector field X_H on T^*Q given by*

$$X_H = b_{\omega_Q}^{-1}(dH)$$

In other words, $X_H \in \mathcal{X}(T^*Q)$ is characterized by the following condition

$$i_{X_H}\omega = dH \tag{2.24}$$

Definition 2.22. *The triple (T^*Q, ω_Q, X_H) is called* Hamiltonian system*.*

Notice that from Proposition 2.9 we conclude that any Hamiltonian vector field is a symplectic vector field. The reciprocal is not always true, since the third condition of the proposition has local character. Nevertheless, it leads us to introduce the definition of a locally Hamiltonian vector field on a symplectic manifold.

Definition 2.23. *A vector field on a symplectic manifold (S, ω) is said to be a* locally Hamiltonian *if for every point $x \in S$ there exists an open set U of S, $x \in U$, and a real smooth function $H : S \to \mathbb{R}$ such that*

$$b_{\omega_S}^{-1}(dH) = (X_H)_{|U}$$

where $b_{\omega_S} : \mathcal{X}(S) \to \Omega^1(S)$ is the isomorphism of $\mathcal{F}(S)$-modules induced by ω. If $U = S$ then the vector field is said to be Hamiltonian.

Now, we will see that the integral curves of the Hamiltonian vector field of an arbitrary Hamiltonian function $H : T^*Q \to \mathbb{R}$ are just the solutions of the Hamilton equations. In fact, form (2.4) and (2.24), we deduce that the local expression of X_H is

$$X_H = b_{\omega_Q}^{-1}(dH) = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i} \right) \tag{2.25}$$

Thus, if $\sigma : (-\epsilon, \epsilon) \to T^*Q$ is a curve on T^*Q, then, σ will be an integral curve of X_H if and only if $X_H(\sigma) = \dot{\sigma}$, which yields, almost immediately, to Hamilton equations (2.23).

On the other hand, it is easy to prove the following result.

Proposition 2.24. *Let $h : Q \to S$ be a symplectic map between two symplectic manifolds (Q, ω) and (S, α). Then,*

$$T_x h(X_{F\circ h}(x)) = X_F(h(x))$$

for $F \in \mathcal{F}(S)$ and $x \in Q$.

From Proposition 2.24, we deduce the following corollary

Corollary 2.25. *Let $h : T^*Q \to T^*Q'$ be a symplectic map and $F' : T^*Q' \to \mathbb{R}$ be a real smooth function on T^*Q'. If $\gamma : I \to T^*Q$ is a solution of the Hamilton equations for $F' \circ h : T^*Q \to \mathbb{R}$, then $h \circ \gamma : I \to T^*Q'$ is a solution of the Hamilton equations for F'.*

This last result may be very useful. In fact, if $h : T^*Q \to T^*Q$ is a canonical transformation, $H : T^*Q \to \mathbb{R}$ is a Hamiltonian function such that $H \circ h = H$ and $\sigma : I \to T^*Q$ is a solution of the Hamilton equations for H, then $\sigma \circ h : I \to T^*Q$ also is a solution of such equations. Thus, from a solution of the Hamilton equations for H we may obtain a new solution of such equations.

This corollary ends the review on the Lagrangian and Hamiltonian formulation that we give. Over the next chapters we will use many of the stated results, specially those related with the phase space of momenta and Hamiltonian mechanics.

Chapter 3

Lie groups

The starting point of the study of any mechanical system is the configuration manifold. The theory we want to develop in this project deals with mechanical systems which have as a configuration manifold a Lie group. The properties that verify such manifolds will be key in order to obtain the most important results of the project and that is why we dedicate this chapter to Lie groups.

The chapter is divided in two main parts. The first one deals with the notions of Lie group and subgroup, its associated Lie algebra and the left invariant vector fields, group homomorphisms and the exponential map. In the second one we will discuss the definition of an action of a Lie group on a manifold and we will consider two particular actions: the adjoint and the coadjoint actions associated with a Lie group. These two actions together with the Lie algebra and the left invariant vector fields will be fundamental issues over the next chapters.

3.1 Lie groups

3.1.1 Lie groups and Lie algebras

Let us start defining a special type of manifolds, Lie groups, which are the fundamental tool of the chapter.

Definition 3.1. *A Lie group G is a smooth manifold which is endowed with a group structure such that the maps*

$$\begin{array}{rccc} \cdot: & G\times G & \to & G \\ & (g,h) & \to & g\cdot h \end{array} \qquad \begin{array}{rccc} i: & G & \to & G \\ & g & \to & g^{-1} \end{array}$$

are smooth, where $\cdot$ is the group operation.

This last condition might be also reformulated in terms of the map μ defined below:

$$\begin{array}{rccc} \mu: & G\times G & \to & G \\ & (g,h) & \to & g\cdot h^{-1} \end{array} \tag{3.1}$$

Indeed, the inversion map i and the group operation are smooth, if and only if, μ is smooth. In fact, if e is the identity element and $C_e : G \to G\times G$ is the map given by $C_e(h) = (e, h)$, then, the following equalities prove the result:

$$\mu = \cdot \circ (Id_G \times i), \qquad i = \mu \circ C_e, \qquad \cdot = \mu \circ (Id_G \times i).$$

Throughout the whole chapter G will denote a Lie group and e the identity element.

Example 3.2. i. Any vector space with the addition is a Lie group. Particularly, $\mathbb{R}^n$ with the vector addition is a Lie group.

ii. Define the *General linear group* $GL(n,\mathbb{R})$ as the set of all $n \times n$ matrices with non-zero determinant, that is, $GL(n,\mathbb{R}) = det^{-1}(\mathbb{R} - \{0\})$. Note that, $GL(n,\mathbb{R})$ is an open subset of the set of the square matrices $gl(n,\mathbb{R})$, so it is a submanifold of $gl(n,\mathbb{R})$. Thus, it can be easily checked that $GL(n,\mathbb{R})$ is a Lie group with the standard multiplication of $n \times n$ matries.

iii. If G and L are Lie groups, $G \times L$ is a Lie group with the product manifold structure and the induced operation between them, that is $(g_1, l_1)(g_2, l_2) = (g_1 g_2, l_1 l_2)$ for $g_1, g_2 \in G$ and $l_1, l_2 \in L$.

Next we will recall the definition of a Lie algebra and subalgebra and we will present some important examples.

Definition 3.3. *A* Lie algebra *is a couple* $(A, [\ ,\ ])$ *where* A *is a vector space and* $[\ ,\ ] : A \times A \to A$ *is a* $\mathbb{R}$*-bilinear map such that*

i. $[u, v] = -[v, u]$, $\forall u, v \in A$ (skew-symmetry)

ii. $[u, [v, w]] + [w, [u, v]] + [v, [w, u]] = 0$, $\forall u, v, w \in A$ (Jacobi identity)

Definition 3.4. *A* Lie subalgebra *of a Lie algebra* $(A, [\ ,\ ])$, *is a subespace of* A *which is closed under the bracket* $[\ ,\ ]$.

Clearly, a Lie subalgebra is also a Lie algebra with the same bracket.

Example 3.5. i. $\mathbb{R}^3$ endowed with the cross product $[u, v] = u \times v$ is a Lie algebra.

ii. Any vector space V with the trivial bracket $[x, y] = 0$ for all $x, y \in V$, is a Lie algebra. In particular, $\mathbb{R}^n$ is a Lie algebra with such a bracket.

iii. If M is a manifold, then the standard Lie bracket of vector fields defines a Lie algebra structure on the vector space $\mathcal{X}(M)$.

iv. $gl(n,\mathbb{R})$ is a Lie algebra when endowed with the bracket

$$[A, B] = AB - BA \qquad \forall A, B \in gl(n,\mathbb{R}) \tag{3.2}$$

As we pointed out, our current goal is to associate to any Lie group G a Lie algebra of finite dimension. In order to do it we will show that given a Lie group G, it exists a distinguished Lie subalgebra of $\mathcal{X}(G)$. Moreover, we will show that it is isomorphic to T_eG and hence, the tangent space of a Lie group at the identity element will also have Lie algebra structure. But before that we give two definitions.

Definition 3.6. *Let* G *be a Lie group and* $g \in G$. *Define the* left translation *and the* right translation *by* g *as the maps given by*

$$\begin{array}{rccc} L_g: & G & \to & G \\ & h & \to & g \cdot h \end{array} \qquad \begin{array}{rccc} R_g: & G & \to & G \\ & h & \to & h \cdot g \end{array}$$

From the definition of Lie group we can assure that L_g and R_g are smooth. Besides, since $L_{g_1} \circ L_{g_2} = L_{g_1 g_2}$, $R_{g_1} \circ R_{g_2} = R_{g_1 g_2}$ and $L_e = Id = R_e$, it is clear that $(L_g)^{-1} = L_{g^{-1}}$ and $(R_g)^{-1} = R_{g^{-1}}$. Therefore, L_g and R_g are diffeomorphisms. It can be also easily checked that $L_g \circ R_h = R_h \circ L_g$.

Now, we will introduce the notion of a left invariant vector field.

Definition 3.7. *A vector field* $X \in \mathcal{X}(G)$ *is called* left invariant *if*

$$T_h L_g(X(h)) = X(g \cdot h) \qquad \forall g, h \in G$$

that is $TL_g X = X$.

Denote by $\mathcal{X}_L(G)$ the set of left invariant vector fields.

Proposition 3.8. *$\mathcal{X}_L(G)$ is a Lie subalgebra of the Lie algebra $(\mathcal{X}(G), [\ ,\ ])$.*

Proof. Let $X, Y \in \mathcal{X}_L(G)$, we have to check that $[X, Y] \in \mathcal{X}_L(G)$ or what is the same

$$T_h L_g([X, Y](h)) = [X, Y](g \cdot h) = [X, Y](L_g(h)) \quad \forall g, h \in G$$

But this equality holds true since X and Y are L_g-projectable to themselves by hypothesis. □

Before seeing the next results, let us define another type of vector fields on a Lie group.

Definition 3.9. *The* left extension *of any $\xi \in T_e G$ is a vector field X_ξ given by*

$$X_\xi(g) = T_e L_g(\xi) \quad \forall g \in G$$

It is easy to show that the left extension of ξ is a left invariant vector field. Indeed,

$$T_h L_g(X_\xi(h)) = T_h L_g \circ T_e L_h(\xi) = T_e(L_g \circ L_h)(\xi) = T_e L_{gh}(\xi) = X_\xi(gh)$$

Finally, we are going to provide $T_e G$ with a structure of Lie algebra. In order to do it, first we will show that $T_e G$ is isomorphic to $\mathcal{X}_L(G)$ and then we will use the Lie algebra structure of $\mathcal{X}_L(G)$ to endow $T_e G$ with such a structure.

Proposition 3.10. *$\mathcal{X}_L(G) \cong T_e G$. In particular, $\mathcal{X}_L(G)$ is a real vector space of finite dimension.*

Proof. It is enough to define

$$\begin{array}{rccc} \rho_1 : & \mathcal{X}_L(G) & \to & T_e G \\ & X & \to & X(e) \end{array} \qquad\qquad \begin{array}{rccc} \rho_2 : & T_e G & \to & \mathcal{X}_L(G) \\ & \xi & \to & X_\xi \end{array}$$

It is clear that ρ_1 and ρ_2 are linear maps and $\rho_1 \circ \rho_2 = Id_{T_e G}$ and $\rho_2 \circ \rho_1 = Id_{\mathcal{X}_L(G)}$. □

Remark 3.11. Let us recall the following property of real vector spaces: "If V is a real vector space, then $T_x V \cong V\ \forall x \in V$". It is important to keep it in mind because it will be largely used along the project.

Once we have that $\mathcal{X}_L(G) \cong T_e G$, we define a bracket on $T_e G$ as follows

$$[\xi, \eta] = [X_\xi, X_\eta](e) \quad \forall \xi, \eta \in T_e G. \tag{3.3}$$

It is immediate to prove that such a bracket defines a Lie algebra structure on $T_e G$. On the other hand, since

$$\rho_1(X_{[\xi,\eta]}) = [\xi, \eta] = [X_\xi, X_\eta](e) = \rho_1([X_\xi, X_\eta])$$

we have

$$X_{[\xi,\eta]} = [X_\xi, X_\eta]. \tag{3.4}$$

Definition 3.12. *The vector space $T_e G$ endowed with the Lie algebra structure defined above is called the* Lie algebra of G.

We will use $\mathfrak{g}$ to refer to $T_e G$ or equivalently to $\mathcal{X}_L(G)$ with their Lie algebra structure defined previously.

3.1.2 Homomhorpisms of Lie groups and Lie algebras

In this section we present some basic results about the homomorphisms between Lie groups and Lie algebras. They will be extensively used along this chapter because they allow us to relate different Lie groups (or Lie algebras) and to regard some of them in a more suitable way. As well, the definitions and the results that we will give are the basic background that we need in order to define and give the properties of the exponential map in the next section.

Definition 3.13. *Let* $(G, \cdot)$ *and* $(H, +)$ *be two Lie groups and* $f : G \to H$ *be a smooth map.* f *is said to be a* Lie group homomorphism *if it is a group homomorphism, that is,*

$$f(g \cdot h) = f(g) + f(h) \qquad \forall g, h \in G$$

Moreover, if f *is a bijective then* f *is said to be a* Lie group isomorphism*.*

Example 3.14. $\mathbb{R} - \{0\}$ with the standard multiplication of real numbers is a Lie group. Furthermore, since $|AB| = |A| \cdot |B|$ for $A, B \in GL(n, \mathbb{R})$ the map $det : GL(n, \mathbb{R}) \to \mathbb{R} - \{0\}$ is a Lie group homomorphism.

Definition 3.15. *Let* $(\mathfrak{g}, [\ ,\ ])$ *and* $(\mathfrak{h}, [\ ,\ ]')$ *be two Lie algebras and* $f : \mathfrak{g} \to \mathfrak{h}$ *be a linear map.* f *is said to be a* Lie algebra homomorphism *if*

$$f([\xi, \eta]) = [f(\xi), f(\eta)]' \qquad \forall \xi, \eta \in \mathfrak{g}$$

Moreover, if f *is bijective then* f *is said to be* Lie algebra isomorphism

Example 3.16. Denote by $\mathfrak{gl}(n, \mathbb{R})$ the Lie algebra of $GL(n, \mathbb{R})$. Then, $\mathfrak{gl}(n, \mathbb{R})$ might be identified with $gl(n, \mathbb{R}), [\ ,\ ])$, where $[\ ,\ ]$ is the bracket defined in Example 3.5. In fact, there exists a Lie algebra isomorphism between $\mathfrak{gl}(n, \mathbb{R})$ and $gl(n, \mathbb{R})$. Let us prove it.

Let x_{ij} be the generalized coordinates of $gl(n, \mathbb{R})$ which are given by $x_{ij}(a_{kl}) = a_{ij}$. Consider the map

$$\begin{array}{rccc} \alpha : & T_e(gl(n, \mathbb{R})) & \longrightarrow & gl(n, \mathbb{R}) \\ & v & \to & \alpha(v)_{ij} = (v(x_{ij})) \end{array}$$

and using that $\mathcal{X}_L(GL(n, \mathbb{R})) = \mathfrak{gl}(n, \mathbb{R}) \cong T_e(GL(n, \mathbb{R})) = T_e(gl(n, \mathbb{R}))$ define β as

$$\begin{array}{rccc} \beta : & \mathfrak{gl}(n, \mathbb{R}) & \longrightarrow & gl(n, \mathbb{R}) \\ & X & \to & \beta(X) = \alpha(X(e)) \end{array}$$

It is easy to prove that β is a vector space isomorphism. Therefore, the only fact that is left is to prove that $\beta([X, Y]) = [\beta(X), \beta(Y)]$ for any $X, Y \in \mathcal{X}_L(GL(n, \mathbb{R})) = \mathfrak{gl}(n, \mathbb{R})$.

Let $A, B \in GL(n, \mathbb{R})$ be matrices and $X, Y \in \mathfrak{gl}(n, \mathbb{R})$ be left invariant vector fields. First of all notice that,

$$(x_{ij} \circ L_A)(B) = x_{ij}(AB) = \sum_k x_{ik}(A) x_{kj}(B)$$

Now, taking into account that X is a left invariant vector field we have

$$\begin{aligned} (X(x_{ij}))(A) &= T_e L_A(X(e))(x_{ij}) = X(e)(x_{ij} \circ L_A) \\ &= X(e)\left(\sum_k x_{ik}(A) x_{kj}\right) = \sum_k x_{ik}(A) X(e)(x_{kj}) \\ &= \sum_k x_{ik}(A) \alpha(X(e))_{kj} = \sum_k x_{ik}(A) \beta(X)_{kj} \end{aligned}$$

which means that

$$X(x_{ij}) = \sum_k \beta(X)_{kj} x_{ik}.$$

Finally, we obtain

$$\begin{aligned}
\beta([X,Y])_{ij} &= \alpha([X,Y](e))_{ij} = [X,Y](e)(x_{ij}) \\
&= X(e)(Y(x_{ij})) - Y(e)(X(x_{ij})) \\
&= X(e)\left(\sum_k \beta(Y)_{kj}x_{ik}\right) - Y(e)\left(\sum_k \beta(X)_{kj}x_{ik}\right) \\
&= \sum_k \left(X(e)(x_{ik})\beta(Y)_{kj} - Y(e)(x_{ik})\beta(X)_{kj}\right) \\
&= \sum_k \left(\beta(X)_{ik}\beta(Y)_{kj} - \beta(Y)_{ik}\beta(X)_{kj}\right) \\
&= [\beta(X),\beta(Y)]_{ij}
\end{aligned}$$

At last, we give an interesting result that allows us to define a Lie algebra homomorphism between the associated Lie algebras of two Lie groups provided that we have a Lie group homomorphism between them. It is important to bear in mind its proof because it will be used further on.

Proposition 3.17. *Let $(G,\cdot)$ and $(H,+)$ be two Lie groups with identity element e and e' respectively. If $f : G \to H$ is a Lie group homomorphism, then*

$$T_e f : T_e G \to T_{e'} H$$

is a Lie algebra homomorphism.

Proof. If $T_g f(X_\xi(g)) = X_{T_e f(\xi)}(f(g))$ for all $\xi \in T_e G$ and for all $g \in G$, then

$$T_e f[\xi,\eta] = T_e f\left([X_\xi, X_\eta](e)\right) = [X_{T_e f(\xi)}, X_{T_e f(\eta)}](e') = [T_e f(\xi), T_e f(\eta)]' \quad \forall \xi,\eta \in T_e G$$

Consequently, it is enough to prove that the previous equality holds. In fact, given any $g,h \in G$, we have that $(f \circ L_h)(g) = f(hg) = f(h)f(g) = (L_{f(h)} \circ f)(g)$. Thus,

$$T_g f(X_\xi(g)) = T_g f(T_e L_g(\xi)) = T_{e'} L_{f(g)}(T_e f(\xi)) = X_{T_e f(\xi)}(f(g)).$$

□

3.1.3 Exponential map

We will introduce the exponential map, give some of its properties and study some important particular examples. It is important to point out that those concepts will be widely used throughout the rest of the chapter.

Let us give a first definition which will help us to characterize the integral curves of left invariant vector fields starting at e.

Definition 3.18. *A* one-parameter subgroup *of a Lie group $(G,\cdot)$ is a Lie group homomorphism $\gamma : (\mathbb{R},+) \to (G,\cdot)$.*

Notice that since γ is a Lie group homomorphism we have $\gamma(0) = e$. Following this definition, the next result gives the existing relationship between such homomorphisms and the mentioned integral curves.

Proposition 3.19. *There exists a bijective correspondence between the one-parameter subgroups of G and the integral curves of the left invariant vector fields of G starting at e.*

Proof. Assume that γ_ξ is a one-parameter subgroup of G such that $\xi = T_0\gamma_\xi(\frac{d}{dt}_{|s=0}) \in T_eG$ and X_ξ is the corresponding left extension. We have that $\gamma_\xi \circ L_t = L_{\gamma_\xi(t)} \circ \gamma_\xi$, where $L_t : \mathbb{R} \to \mathbb{R}$ is the left translation by t in the additive group $(\mathbb{R}, +)$ (see the proof of Proposition 3.17). Thus, it follows that

$$\begin{aligned} T_t\gamma_\xi\left(\frac{d}{dt}_{|t}\right) &= T_t\gamma_\xi\left(T_0L_t\left(\frac{d}{ds}_{|s=0}\right)\right) = T_0(\gamma_\xi \circ L_t)\left(\frac{d}{ds}_{|s=0}\right) \\ &= T_eL_{\gamma_\xi(t)}\left(T_0\gamma_\xi\left(\frac{d}{ds}_{|s=0}\right)\right) = T_eL_{\gamma_\xi(t)}(\xi) \\ &= X_\xi(\gamma_\xi(t)) \end{aligned}$$

Conversely, let X_ξ be a left invariant vector field and γ_ξ be its integral curve starting at e, i.e, $\gamma_\xi(0) = e$. We have to prove that γ_ξ is a one-parameter subgroup. Let U be an open set of G and $I = (-\epsilon, \epsilon)$ be an open interval in $\mathbb{R}$ and consider the local one-parameter group $\phi_{X_\xi} : U \times I \to G$ associated with X_ξ. Clearly, we have that $\gamma_\xi(t) = \phi_{X_\xi}(e, t)$.
Given any $g \in G$, define the curve $\sigma_g : I \to G$ as $\sigma_g(t) = g \cdot \gamma_\xi(t)$. We claim that σ_g is the integral curve of X_ξ starting at g. Indeed,

$$\begin{aligned} T_t\sigma_g\left(\frac{d}{dt}_{|t}\right) &= T_t(L_g \circ \gamma_\xi)\left(\frac{d}{dt}_{|t}\right) = T_{\gamma_\xi(t)}L_g\left(T_t\gamma_\xi\left(\frac{d}{dt}_{|t}\right)\right) \\ &= T_{\gamma_\xi(t)}L_g(X_\xi(\gamma_\xi(t))) = X_\xi(g \cdot \gamma_\xi(t)) \\ &= X_\xi(\sigma_g(t)). \end{aligned}$$

Therefore, $\sigma_g(t) = \phi_{X_\xi}(g, t)$ for $(g, t) \in U \times I$. Thus, if t_1 and t_2 are small enough,

$$\begin{aligned} \gamma_\xi(t_1 + t_2) &= \phi_{X_\xi}(e, t_1 + t_2) \\ &= ((\phi_{X_\xi})_{t_2} \circ (\phi_{X_\xi})_{t_1})(e) = (\phi_{X_\xi})_{t_2}(\gamma_\xi(t_1)) \\ &= (\phi_{X_\xi})(\gamma_\xi(t_1), t_2) = \sigma_{\gamma_\xi(t_1)}(t_2) \\ &= \gamma_\xi(t_1) \cdot \gamma_\xi(t_2). \end{aligned}$$

It proves that γ_ξ is a local Lie homomorphism. However, using the same fact, it can be extended to the real line defining

$$\gamma_\xi(t) = \left(\gamma_\xi\left(\frac{t}{n}\right)\right)^n$$

for n such that $\frac{t}{n} \in I$. ☐

Remark 3.20. Let ξ be an element of $\mathfrak{g}$ and X_ξ be the corresponding left invariant vector field on G. Then X_ξ is complete and its global flow is given by

$$\phi_{X_\xi}(g, t) = g \cdot \gamma_\xi(t)$$

Once we have studied the integral curves of the left invariant vector fields let us move on to the exponential map.

Definition 3.21. *The* exponential map $exp_G : \mathfrak{g} \to G$ *is defined by*

$$exp_G(\xi) = \gamma_\xi(1)$$

Proposition 3.22. *i. exp_G is smooth.*

ii. $exp_G(t\xi) = \gamma_\xi(t)$.

iii. *There exists an open neighbourhood U of 0 in $\mathfrak{g}$ and an open neighbourhood V of e in G such that $exp_G : U \to V$ is a diffeomorphism.*

Proof. i. Define a vector field Z on $G \times \mathfrak{g}$ as follows:

$$\begin{array}{rccc} Z: & G \times \mathfrak{g} & \to & T(G \times \mathfrak{g}) \\ & (g, \xi) & \to & (X_\xi(g), 0) \end{array}$$

Its integral curve through (g, ξ) is given by $t \to (g \cdot \gamma_\xi(t), \xi)$. Thus its flow is

$$\begin{array}{rccc} F: & \mathbb{R} \times (G \times \mathfrak{g}) & \to & G \times \mathfrak{g} \\ & (t, (g, \xi)) & \to & (g \cdot \gamma_\xi(t), \xi) \end{array}$$

Finally, if $\pi : G \times \mathfrak{g} \to G$ denotes the projection onto G and $(C_1, C_e, Id) : \mathfrak{g} \to \mathbb{R} \times (G \times \mathfrak{g})$ is the map given by $\xi \to (1, (e, \xi))$, we have that

$$exp_G = \pi \circ F \circ (C_1, C_e, Id).$$

Since π, (C_1, C_e, Id) are smooth and F is also smooth due to the fact that it is a flow, we conclude that exp_G is smooth.

ii. By *i.* it is enough to prove that $\gamma_\xi(t) = \gamma_{t\xi}(1) = exp_G(t\xi)$. We will show that $\gamma_{t\xi}(s) = \gamma_\xi(st)$ for all $s \in \mathbb{R}$ and choosing $s = 1$ we will get the result.

First of all, using the fact that $\gamma_{t\xi}$ and γ_ξ are one-parameter subgroups, it is clear that $\gamma_{t\xi}(0) = \gamma_\xi(0) = e$. Then, the curves $s \to \gamma_{t\xi}(s)$ and $s \to \gamma_\xi(st)$ satisfy the same initial condition. Moreover, given $g \in G$,

$$X_{t\xi}(g) = T_e L_g(t\xi) = t T_e L_g(\xi) = t X_\xi(g)$$

Therefore,

$$\frac{d\gamma_{t\xi}(s)}{ds} = X_{t\xi}(\gamma_{t\xi}(s)) = t X_\xi(\gamma_{t\xi}(s)).$$

On the other hand,

$$\frac{d\gamma_\xi(st)}{ds} = t \frac{d\gamma_\xi}{ds}(st) = t X_\xi(\gamma_\xi(st)).$$

Therefore, $s \to \gamma_{t\xi}(s)$ and $s \to \gamma_\xi(st)$ are integral curves of the same left invariant vector field $tX_\xi = X_{t\xi}$. This implies that $\gamma_{t\xi}(s) = \gamma_\xi(st)$ for all $s \in \mathbb{R}$.

iii. It is sufficient to prove that $T_0 exp_G : T_0\mathfrak{g} \to T_e G$ is a linear isomorphism. Indeed, if it is true, using the inverse function theorem we obtain the result.

Using that $T_0\mathfrak{g} \cong \mathfrak{g}$ and $T_e G \cong \mathfrak{g}$ we might regard $T_0 exp_G$ as a map $\mathfrak{g} \to \mathfrak{g}$. Now, if $\xi \in \mathfrak{g}$ then the corresponding vector $X \in T_0\mathfrak{g}$ is given by

$$X = \rho'_\xi(0)$$

where $\rho_\xi : \mathbb{R} \to \mathfrak{g}$ is the curve defined by $\rho_\xi(t) = t\xi$. Thus,

$$T_0 exp_G(X) = T_0 exp_G(\rho'_\xi(0)) = (exp_G \circ \rho_\xi)'(0).$$

By *ii*, $exp_G \circ \rho_\xi = \gamma_\xi$, so

$$T_0 exp_G(X) = \gamma'_\xi(0) = \xi$$

□

From Proposition 3.22 and the previous proposition it follows that all one-parameter subgroups of G are of the form $t \to exp_G(t\xi)$, with $\xi \in \mathfrak{g}$.

Example 3.23. Let $(V, +)$ be a vector space with the addition as the group operation. From Remark 3.11 its Lie algebra can be identified with V and, easily one has that $exp : V \to V$ is the identity.

Example 3.24. If G is the general linear group $GL(n, \mathbb{R})$, the exponential map is the usual exponential map for matrices

$$exp(A) = \gamma_A(1) = \sum_{i=0}^{\infty} \frac{A^i}{i!} \quad \text{for } A \in \mathfrak{gl}(n, \mathbb{R}) \tag{3.5}$$

Let us see it. Consider the map

$$\begin{array}{rccc} \gamma_A & : \mathbb{R} & \to & GL(n, \mathbb{R}) \\ & t & \to & \sum_{i=0}^{\infty} \frac{t^i}{i!} A^i \end{array}$$

Clearly $\gamma_A(0) = Id$. Define the curve

$$\begin{array}{rccc} \sigma_{IA} & : \mathbb{R} & \to & \mathfrak{gl}(n, \mathbb{R}) \\ & s & \to & I + sA \end{array}$$

and remark that

$$\begin{array}{rccc} \gamma_A(t) \cdot \sigma_{IA} & : (-\epsilon, \epsilon) & \to & GL(n, \mathbb{R}) \\ & s & \to & \gamma_A(t) + s\gamma_A(t)A \end{array}$$

Then,

$$\begin{aligned} T_I L_{\gamma_A(t)}(A) &= T_I L_{\gamma_A(t)}(\sigma'_{IA}(0)) = (L_{\gamma_A(t)} \circ \sigma_{IA})'(0) \\ &= (\gamma_A(t) \cdot \sigma_{IA})'(0) = \gamma_A(t) \cdot A \end{aligned}$$

Finally we have that

$$\gamma'_A(t) = \sum_{i=0}^{n} \frac{t^{i-1}}{(i-1)!} A^i = \gamma_A(t) \cdot A = T_I L_{\gamma_A(t)}(A) = X_A(\gamma_A(t))$$

which proves that γ_A is a one-parameter subgroup. Thus the exponential map is given by (3.5)

At last, we prove a last result which gives an useful formula for the exponential map.

Proposition 3.25. *Let $f : G \to H$ be a Lie group homomorphism. If $e \in G$ denotes the identity element, then*

$$f(exp_G(\xi)) = exp_H(T_e f(\xi)) \qquad \forall \xi \in \mathfrak{g}$$

Proof. As we know (see the proof of the Proposition 3.17)

$$T_g f(X_\xi(g)) = X_{T_e f(\xi)}(f(g)) \qquad \forall g \in G.$$

Using this fact it is easy to check that if γ_ξ is the one-parameter subgroup associated with ξ, then $f \circ \gamma_\xi$ is the one parameter subgroup associated with $T_e f(\xi)$. Hence,

$$exp_H(T_e f(\xi)) = \gamma_{T_e f(\xi)}(1) = (f \circ \gamma_\xi)(1) = f(exp_G(\xi)).$$

□

3.1.4 Lie subgroups

Most of the Lie groups that one can find are, in fact, Lie subgroups of $GL(n,\mathbb{R})$. That is why in this section we give the formal definition of a Lie subgroup and enunciate the Cartan theorem which allows us to identify such subgroups. Nevertheless, the theory that we are working up is focused on the Lie algebras of Lie groups, so we need to study the relationship between the Lie algebra of a Lie group and the Lie algebra of a Lie subgroup.

Definition 3.26. *A* Lie subgroup *H of a Lie group G is subgroup of G such that the inclusion map $i : G \to H$ is an immersion.*

Theorem 3.27 (Cartan)**.** *Any closed subgroup of a Lie group G is a Lie subgroup.*

Proof. See, for instance, [AbMa], [War]. □

Now, our interest is to relate the Lie algebra of G with the Lie algebra of a Lie subgroup H. The next proposition gives this result, but we need a previous lemma in order to prove it.

Lemma 3.28. *Let $\varphi : N \to M$ be a smooth map and $\phi : P \to M$ be an integral submanifold of an involutive distribution $\mathcal{D}$ in M such that $\varphi(N) \subset \phi(P)$. Let $\varphi_0 : N \to P$ be the unique map such that $\phi \circ \varphi_0 = \varphi$. Then φ_0 is smooth.*

Proof. See for instance [War]. □

Proposition 3.29. *Let $i : H \to G$ be a Lie subgroup of G. Then $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$ and*

$$\mathfrak{h} = \{\xi \in \mathfrak{g} \mid exp_G(t\xi) \in i(H)\ \forall t \in \mathbb{R}\}$$

Proof. Let e be the identity element of G. Since i is an immersion, from Proposition 3.17 we have that $T_e i : T_e H \cong \mathfrak{h} \to T_e G \cong \mathfrak{g}$ is a Lie algebra monomorphism. Therefore, $T_e i(\mathfrak{h})$ is a Lie subalgebra of $\mathfrak{g}$ isomorphic to the Lie algebra $\mathfrak{h}$.
Let us prove the double inclusion:
$\subseteq\rfloor$ Fix $\xi = T_e i(\eta) \in T_e i(\mathfrak{h}) \cong \mathfrak{h}$ with $\eta \in \mathfrak{h}$. We have

$$exp_G(t\xi) = exp_G(tT_e i(\eta)) = exp_G(T_e i(t\eta)) = i(exp_H(t\eta)) \in i(H) \quad \forall t \in \mathbb{R}$$

$\supseteq\rfloor$ Suppose that $exp_G(t\xi) \in i(H)\ \forall t \in \mathbb{R}$. Then, $exp_G(t\xi) = (i \circ \phi)(t)$ with $\phi(t) \in H$.
Now, consider the distribution $\mathcal{D}$ on G whose characteristic space at the point $g \in G$ is,

$$\mathcal{D}(g) = \{X_\xi(g) | \xi \in \mathfrak{h}\}$$

From (3.4) we conlcude that $\mathcal{D}$ is involutive. Besides, taking into account that (H, i) is an integral submanifold of $\mathcal{D}$ we can use the previous lemma and we conclude that ϕ is smooth. Moreover, since i is a group homomorphism and $exp_G(t\xi) = (i \circ \phi)(t)$, we have that ϕ is a one-parameter subgroup. It means that there exists $\eta \in \mathfrak{h}$ such that

$$\phi(t) = \gamma_\eta(t) = exp_H(t\eta) \quad \forall t \in \mathbb{R}$$

Thus,

$$exp_G(t\xi) = i(exp_H(t\eta)) = exp_G(T_e i(t\eta))$$

Consequently, $\xi = T_e i(\eta)$, i.e, $\xi \in T_e i(\mathfrak{h}) \cong \mathfrak{h}$. □

Let us use all those results to study the particular Lie groups we are interested in.

Lie subgroups of $GL(n,\mathbb{R})$

As we know, the map $det : GL(n,\mathbb{R}) \to \mathbb{R} - \{0\}$ is a Lie group homomorphism (see Example 3.14). Moreover we have the following result.

Proposition 3.30. *The tangent map to* $det : GL(n,\mathbb{R}) \to \mathbb{R} - \{0\}$ *at* $A \in GL(n,\mathbb{R})$ *is given by*

$$\begin{array}{rccc} T_A det: & T_A(GL(n,\mathbb{R})) \cong \mathfrak{gl}(n,\mathbb{R}) & \to & T_{|A|}\mathbb{R} - \{0\} \cong \mathbb{R} \\ & B & \to & |A| tr(A^{-1}B) \end{array}$$

where $det(A) = |A|$ *and* $tr(A)$ *denotes the trace of* A.

Proof. Given $B \in \mathfrak{gl}(n,\mathbb{R}) \cong T_A(GL(n,\mathbb{R}))$ define the curve

$$\begin{array}{rccc} \sigma_{AB}: & \mathbb{R} & \to & gl(n,\mathbb{R}) \\ & t & \to & A + tB \end{array}$$

Remark that since the determinant map is continuous, there exists an interval $I \subset \mathbb{R}$ such that $\sigma_{AB}(I) \subset GL(n,\mathbb{R})$. Then, using the following equalities

$$|A + tB| = |A||I + tA^{-1}B| \qquad\qquad |I + tB| = 1 + tr(C)t + \cdots + t^n|C|$$

we conclude that,

$$\begin{aligned} T_A det(B) &= T_A det(\sigma'_{AB}(0)) = (det \circ \sigma_{AB})'(0) \\ &= \frac{d}{dt}_{|t=0}(|A||I + tA^{-1}B|) \\ &= |A|\frac{d}{dt}_{|t=0}(1 + tr(A^{-1}B)t + \cdots + t^n|A^{-1}B|) \\ &= |A| tr(A^{-1}B) \end{aligned}$$

□

Special linear group $SL(n,\mathbb{R})$ $SL(n,\mathbb{R})$ is defined as

$$SL(n,\mathbb{R}) = det^{-1}(1) = \{B \in gl(n,\mathbb{R}) | \ |B| = 1\} \tag{3.6}$$

Thus, $SL(n,\mathbb{R})$ is a closed subgroup of $GL(n,\mathbb{R})$ and by the Cartan theorem we can conclude that it is a Lie subgroup of $GL(n,\mathbb{R})$.

From Proposition 3.30 it follows that the Lie algebra $\mathfrak{gl}(n,\mathbb{R})$ of $SL(n,\mathbb{R})$ is given by

$$\mathfrak{sl}(n,\mathbb{R}) = T_I(SL(n,\mathbb{R})) = \{B \in gl(n,\mathbb{R}) | \ tr(B) = 0\} \tag{3.7}$$

Orthogonal group $O(n)$ A matrix $A \in gl(n,\mathbb{R})$ is called orthogonal if $< Ax, Ay >=< x, y >$, for all $x, y \in \mathbb{R}^n$ where $< \ , \ >$ denotes the standard scalar product on $\mathbb{R}^n$. Equivalently, A is orthogonal if $< Ax, y >=< x, A^t y >$, being A^t the transposed matrix. It leads to $A^t A = I$, which implies that the determinant of an orthogonal matrix A is ± 1. Indeed,

$$1 = |I| = |A^t A| = |A|^2$$

Hence, $A \in GL(n,\mathbb{R})$.
Now, let us see that the set of the orthogonal matrices, $O(n)$, has group structure and therefore it is a subgroup of $GL(n,\mathbb{R})$.

i. Obviously $I \in O(n)$.

ii. $A, B \in O(n) \Rightarrow < ABx, ABy >=< Bx, By >=< x, y >$ for all $x, y \in \mathbb{R}^n \Rightarrow AB \in O(n)$

iii. If $A \in O(n)$ and $x, y \in \mathbb{R}^n$ we may define $x' = A^{-1}x$ and $y' = A^{-1}y$. Then, $< x, y >=< Ax', Ay' >=< x', y' >=< A^{-1}x, A^{-1}y >$ which implies that $A^{-1} \in O(n)$.

Next, we will prove that $O(n)$ is a Lie subgroup of $GL(n, \mathbb{R})$. Define ϕ as

$$\begin{array}{rccc} \phi : & GL(n,\mathbb{R}) & \to & S(n) \\ & A & \to & AA^t \end{array}$$

where $S(n)$ is the vector space of the symmetric matrices of order n. Clearly ϕ is smooth and the tangent map in $A \in GL(n, \mathbb{R})$ is given by

$$\begin{array}{rccc} T_A\phi : & T_A(GL(n,\mathbb{R})) \cong \mathfrak{gl}(n,\mathbb{R}) & \to & T_{AA^t}(S(n)) \cong S(n) \\ & B = \sigma'_{AB}(0) & \to & (\phi \circ \sigma_{AB})'(0) \end{array}$$

where σ_{AB} is the curve defined previously in the proof of Proposition 3.30. In fact,

$$(\phi \circ \sigma_{AB})(t) = AA^t + t(AB^t + BA^t) + t^2 BB^t$$

which yields to

$$T_A\phi(B) = \frac{d}{dt}_{|t=0}(AA^t + t(AB^t + BA^t) + t^2BB^t) = AB^t + BA^t$$

Now, for $A \in O(n)$ and $C \in S(n) \cong T_{AA^t}S(n)$, if we put $B = \frac{CA}{2}$ we have

$$T_A\phi(B) = A\ A^t\frac{C^t}{2} + \frac{C}{2}AA^t = I\frac{C^t}{2} + \frac{C}{2}I = \frac{C}{2} + \frac{C}{2} = C$$

which proves that $T_A\phi$ is a linear epimorphism. Moreover, $O(n) = \phi^{-1}(I)$ is a closed subgroup of $GL(n, \mathbb{R})$. Thus we conclude that $O(n)$ is a Lie subgroup of $GL(n, \mathbb{R})$ and, in addition, the Lie algebra of $O(n)$ is

$$\mathfrak{o}(n) = \left\{A \in gl(n,\mathbb{R}) | A = -A^t\ \right\} \tag{3.8}$$

Special orthogonal group $SO(n)$ Define $SO(n)$ as

$$SO(n) = \{A \in O(n) |\ |A| = 1\} \tag{3.9}$$

In fact, $SO(n) = det^{-1}(1)$ where $det : O(n) \to \{-1, 1\}$ is a Lie group homomorphism. It proves that $SO(n)$ is a closed subgroup of $O(n)$, i.e, a Lie subgroup. On the other hand we have

$$SO(n) = O(n) \cap SL(n, \mathbb{R})$$

which tells us that $SO(n)$ is an open subset of $O(n)$ and that their Lie algebras coincide, that is, $\mathfrak{so}(n) = \mathfrak{o}(n)$.

In particular, we are going to see that the Lie algebra $\mathfrak{so}(3)$ can be identified with the Lie algebra $(\mathbb{R}^3, \times)$ via the hat map[1] defined as

$$\begin{array}{cccc} \hat{} : & \mathbb{R}^3 & \longrightarrow & \mathfrak{so}(3) \\ & v = (v_1, v_2, v_3) & \to & \hat{v} = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix} \end{array} \tag{3.10}$$

Remark that

$$\hat{v} \cdot w = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = v \times w$$

[1] Recall the section 1.2 where we have introduced the hat map for the rigid body.

which, using the Jacobi identity for the cross product in $\mathbb{R}^3$ gives

$$\begin{aligned}[\hat{u},\hat{v}]\cdot w &= (\hat{u}\hat{v}-\hat{v}\hat{u})\cdot w = \hat{u}\cdot(v\times w)-\hat{v}\cdot(u\times w)\\ &= u\times(v\times w)-v\times(u\times w) = (u\times v)\times w\\ &= \widehat{u\times v}\cdot w\end{aligned}$$

Thus,

$$\widehat{u\times v} = [\hat{u},\hat{v}] \tag{3.11}$$

and the hat map is a Lie algebra isomorphism.

Remark 3.31. We recall that in section 1.2 (Chapter 1), we have shown that the position $x(t)$ at the time t of a particle with label X of a rigid body is given by

$$x(t) = R(t)X$$

where $R(t)$ is a rotation matrix which preserves the orientation, that is $R(t)\in SO(3)$. It means that $SO(3)$ is the configuration manifold of that mechanical system. On the other hand, if $(\varphi,\ \theta,\ \psi)$ are the Euler angles on $SO(3)$, we have that $(\varphi,\ \theta,\ \psi)$ are generalized coordinates for the rigid body. Consequentely, we deduce that the configuration manifold for the rigid body is a Lie group of dimension 3. This fact will allow us to work over its associated Lie algebra where we have properties helping us to reduce the number of ODE's that describe the system.

3.2 Actions of Lie groups

After studying the main properties of Lie groups and their Lie algebras we are ready to begin working with actions of Lie groups on manifolds. In this section we introduce the main notions and results that we will need further.

Let us start with the definition of action of a Lie group on a manifold and the first examples.

Definition 3.32. *Let M be a smooth manifold and G be a Lie group. A* (left) action *of G on M is a smooth map $\phi: G\times M\to M$ such that*

i. $\phi(e,q)=q$ for all $q\in M$.

ii. $\phi(g,\phi(h,q))=\phi(g\cdot h,q)$, for all $g,h\in G$ and $q\in M$.

Notice that if $\phi: G\times M\to M$ is an action and $g\in G$, then ϕ induces a smooth map $\phi_g: M\to M$ for all $g\in G$ given by $\phi_g(q)=\phi(g,q)$ for all $q\in M$. Moreover, this map verifies:

i. $\phi_e = Id_{|M}$

ii. $\phi_g\circ\phi_h=\phi_{gh}$

iii. $(\phi_g)^{-1}=\phi_{g^{-1}}$

Thus, ϕ_g is a diffeomorphism and the action can be also seen as an homomorphism of G into the group of diffeomorphisms of M. In particular, if M is a vector space and ϕ_g is linear for all $g\in G$, the action is called a *representation* of G on M.

Example 3.33. i. The flow $F:\mathbb{R}\times M\to M$ of a complete vector field $X\in\mathcal{X}(M)$ is an action of $(\mathbb{R},+)$ on M.

ii. Given a Lie subgroup H of a Lie group G, consider the map

$$\begin{array}{rccc}\varphi: & H\times G & \to & G\\ & (h,g) & \to & h\cdot g\end{array}$$

It is easy to prove that φ is an action of H on G.

iii. There is an standard action of the linear general group $GL(n,\mathbb{R})$ on $\mathbb{R}^n$ given by $(A,v) \to Av$ for $A \in GL(n,\mathbb{R})$ and $v \in \mathbb{R}^n$

Before going further, we introduce the tangent and cotangent lift of an action because we will need them in the incoming chapters.

Definition 3.34. *Let $\phi : G \times M \to M$ be an action.*

i. The tangent lift *of ϕ is the action ϕ^T given by*

$$\begin{array}{rccc} \phi^T: & G \times TM & \to & TM \\ & (g, v_q) & \to & T_q\phi_g(v_q) \end{array}$$

for $v_q \in T_qM$.

ii. The cotangent lift *of ϕ is the action ϕ^{T^*} given by*

$$\begin{array}{rccc} \phi^{T^*}: & G \times T^*M & \to & T^*M \\ & (g, \alpha_q) & \to & T_q^*\phi_{g^{-1}}(\alpha_q) \end{array}$$

for $\alpha_q \in T_q^*M$.

It is easy to check that they are actions. For instance, we give the prove for the cotangent lift[2].

i. Since ϕ is an action it is clear that $\phi^{T^*}(e,\alpha) = \alpha$.

ii. For any $X \in T_{\phi_h g(q)}M$ we have

$$\begin{aligned} \phi^{T^*}(h, \phi^{T^*}(g,\alpha_q))(X) &= \phi^{T^*}(h, T_q^*\phi_{g^{-1}}(\alpha_q))(X) = T^*_{\phi_g(q)}\phi_{h^{-1}}(T_q^*\phi_{g^{-1}}(\alpha_q))(X) \\ &= T_q^*\phi_{g^{-1}}(\alpha_q)\left(T_{\phi_h\circ\phi_g(q)}\phi_{h^{-1}}(X)\right) = \alpha_q\left(T_{\phi_g(q)}\phi_{g^{-1}}\left(T_{\phi_h\circ\phi_g(q)}\phi_{h^{-1}}(X)\right)\right) \\ &= \alpha_q\left(T_{\phi_h g(q)}(\phi_{g^{-1}}\circ\phi_{h^{-1}})(X)\right) = \alpha_q\left(T_{\phi_h g(q)}\phi_{(hg)^{-1}}(X)\right) \\ &= T_q^*\phi_{(hg)^{-1}}(\alpha_q)(X) = \phi^{T^*}(h\cdot g, \alpha_q)(X) \end{aligned}$$

Consequently, $\phi^{T^*}(h, \phi^{T^*}(g,\alpha_q)) = \phi^{T^*}(h\cdot g,\alpha_q)$

Example 3.35. Assume that V is a vector space of finite dimension and $\phi : G \times V \to V$ is an action of a Lie group G such that ϕ_g is a linear isomorphism for every $g \in G$. We will see that in such a case, the tangent lift of ϕ under the identification $TV \cong V \times V$ is given by

$$\begin{array}{rccc} \phi^T: & G \times (V \times V) & \to & V \times V \\ & (g, (u,v)) & \to & (\phi_g(v), \phi_g(v)) \end{array}$$

First, if $u, v \in V$ define a curve on V as follows

$$\begin{array}{rccc} \alpha_{uv}: & \mathbb{R} & \to & V \\ & t & \to & v + tu \end{array}$$

Then one has that the isomorphism $TV \cong V \times V$ is given by,

$$\begin{array}{rccc} \psi: & (V \times V) & \to & TV \\ & (u,v) & \to & \psi_v(u) = \alpha'_{uv}(0) \end{array}$$

[2]Recall the definition of the cotangent lift of a map from section 2.1.1

On the other hand, bearing in mind that ϕ_g is linear, one gets

$$\phi_g(\alpha_{uv}) = \phi_g(v) + t\phi_g(u) = \alpha_{\phi_g(u)\phi_g(v)}(t)$$

which yields to

$$T_v\phi_g(\psi_v(u)) = T_v\phi_g(\alpha'_{uv}(0)) = \alpha'_{\phi_g(u)\phi_g(v)}(0) = \psi_{\phi_g(v)}(\phi_g(u))$$

that is, the following diagram is commutative,

$$\begin{array}{ccc} V & \xrightarrow{\phi_g} & V \\ \psi_v \big\downarrow & & \big\downarrow \psi_{\phi_g(v)} \\ T_vV & \xrightarrow{T_v\phi_g} & T_{\phi_g(v)}V \end{array}$$

Thus, we have that the diagram

$$\begin{array}{ccc} V \times V & \xrightarrow{(\phi_g,\phi_g)} & V \times V \\ \psi \big\downarrow & & \big\downarrow \psi \\ TV & \xrightarrow{T\phi_g} & TV \end{array}$$

is commutative and the result is proved.

Now, we introduce the definition of the orbit of a point with respect to an action.

Definition 3.36. *Let $\phi : G \times M \to M$ be an action of a Lie group G on a manifold M, and $q \in M$ be a point. The* orbit *of q is defined by*

$$G \cdot q = \{\phi_g(q) \mid g \in G\}$$

Example 3.37. Remark that the usual action of $SO(3)$ on $\mathbb{R}^3$, $(A, v) \to Av$, verifies that $||Av|| = ||v||$. Therefore, the orbit of a vector $v \neq 0$ will be the sphere of radius $||v||$.

One can define an equivalence relation in terms of the orbits. Indeed, two elements will be in the same class if they belong to the same orbit, that is,

$$x, y \in M \quad x \sim y \;\Leftrightarrow\; \exists g \in G \text{ such that } \phi_g(x) = y$$

Thus $[q] = G \cdot q$ and the equivalence class of q is just the orbit of q. If we denote by M/G the quotient which is induced by this equivalence relation we have the following results.

Theorem 3.38. *Let $\phi : G \times M \to M$ be an action of a Lie group G on a manifold M and consider the subset $\mathcal{R}$ of $M \times M$ defined by*

$$\mathcal{R} = \{(q, \phi_g(q)) \in M \times M \mid (g, q) \in G \times M\}$$

Then, $\mathcal{R}$ is a closed submanifold of $M \times M$ if and only if the quotient space M/G is a smooth manifold such that the canonical projection $\pi : M \to M/G$ is a submersion.

Proof. See [AbMa]. □

Proposition 3.39. *Let M and N be two manifolds, G be a Lie group acting on M such that the space of orbits M/G is a quotient manifold and $\pi : M \to M/G$ be the canonical projection. A map $\phi : M/G \to N$ is smooth, if and only if, $\phi \circ \pi : M \to N$ is smooth.*

Proof. $\Rightarrow\rfloor$ It is obvious since ϕ and π are both smooth.
$\Leftarrow\rfloor$ Let q be a point on M. Due to $\pi : M \to M/G$ is a submersion, there exists an open set $U \subset M/G$, $\pi(q) = [q] \in U$ and a local smooth section $s : U \to M$ such that $s([q]) = q$. As $\phi_{|U} = (\phi \circ \pi) \circ s$ we conclude that $\phi_{|U}$ is smooth. □

Now, we come to the isotropy group.

Definition 3.40. *Let $\phi : G \times M \to M$ be an action of a Lie group G on a manifold M, and q be a point of M. The* isotropy group *of ϕ on q is given by*

$$G_q = \{g \in G | \phi_g(q) = q\}$$

Using Cartan theorem we can prove that the isotropy group is a Lie subgroup of G.

Proposition 3.41. *Let $\phi : G \times M \to M$ be an action of G on M and q be a point of M. The isotropy subgroup of ϕ on q, G_q, is a closed Lie subgroup of G.*

Proof. On the one hand we have that G_q is a subgroup of G. Indeed, $e \in G_q$ and

i. $g, h \in G_q \Rightarrow \phi_{gh}(q) = \phi_g \circ \phi_h(q) = q$

ii. $g \in G_q \Rightarrow \phi_{g^{-1}}(q) = \phi_{g-1} \circ \phi_g(q) = \phi_e(q) = q$

Let us see that it is closed. Consider the following map

$$\begin{array}{cccc} \phi_q : & G & \longrightarrow & M \\ & g & \to & \phi_g(q) = \phi(g, q) \end{array}$$

and remark that $G_q = (\phi_q)^{-1}(q)$. Thus G_q is a closed subgroup and by the Cartan theorem one has that the isotropy subgroup is a Lie subgroup of G. □

Using theorem 3.38, it follows the next result.

Lemma 3.42. *Let H be a closed subgroup of a Lie group G acting on H by left translations. Then, G/H is a manifold and $\pi : G \to G/H$ is a submersion.*

Remark 3.43. Note that $G/H = \{gH \mid g \in G\}$ and, however, the space of orbits of the action H on G by left translations is $\{Hg \mid g \in G\} = H/G$. Anyway, there exists a one-to-one correspondance between G/H and H/G. So G/H is a quotient manifold if and only if H/G is a quotient manifold.

Now, it makes sense to define the manifold $G/G_q = \{[g] \mid g \in G\}$ where $[g] = \{g \cdot h \mid h \in G_q\} = g \cdot G_q$. Besides, with the notation above, for $h \in G_q$ one has that $\phi_q(g \cdot h) = \phi_{g \cdot h}(q) = \phi_g \circ \phi_h(q) = \phi_g(q) = \phi_q(g)$. Thus, it allows us to introduce the well-defined map

$$\begin{array}{cccc} \tilde{\phi_q} : & G/G_q & \to & M \\ & [g] & \to & \phi(g, q) \end{array} \tag{3.12}$$

and we have following theorem.

Theorem 3.44. *If $\phi : G \times M \to M$ is an action of a Lie group G on a manifold M, and q is a point of M, then, $\tilde{\phi_q} : G/G_q \to M$ is an injective immersion and $\tilde{\phi_q}(G/G_q) = G \cdot q$.*

Proof. Let $\pi : G \to G/G_q$ be the canonical projection and consider the following commutative triangle

$$\begin{array}{ccc} G & \xrightarrow{\phi_q} & M \\ \downarrow\pi & \nearrow\tilde{\phi_q} & \\ G/G_q & & \end{array}$$

By the Proposition 3.39, we conclude that $\tilde{\phi_q}$ is smooth.
Now, let us see the injectivity. Suppose that $\tilde{\phi_q}([g]) = \tilde{\phi_q}([h])$ for $g, h \in G$. Then,

$$\phi_q(g) = \phi_q(h) \Rightarrow \phi_g(q) = \phi_h(q) \Rightarrow \phi_{g^{-1}h}(q) = q \Rightarrow hg^{-1} \in G_q \Rightarrow [g] = [h]$$

Finally, we are going to prove that $\tilde{\phi_q}$ is an immersion but we need two previous steps:

i. We show that $\mathfrak{g}_q = Ker(T_e\phi_q)$, where $\mathfrak{g}_q = T_eG_q$ is the Lie algebra of G_q.

$\subseteq\rfloor$ If $\xi \in \mathfrak{g}_q$ we have that $(T_e\phi_q)(\xi) = 0$ since $\phi_{q|G_q} : G_q \to M$ is the constant map. Therefore, $\xi \in Ker(T_e\phi_q)$.

$\supseteq\rfloor$ Given $\xi \in Ker(T_e\phi_q)$ consider the curve

$$\begin{array}{rccc} \alpha_{q\xi}: & \mathbb{R} & \longrightarrow & M \\ & t & \to & \phi_q(exp_G(t\xi)) = \phi_q(\gamma_\xi(t)) \end{array}$$

and note that if $g \in G$

$$\phi_q \circ L_{exp_G(t\xi)}(g) = \phi(exp_G(t\xi) \cdot g, q) = \phi_{exp_G(t\xi)} \circ \phi_q(g) \Rightarrow \phi_q \circ L_{exp_G(t\xi)} = \phi_{exp_G(t\xi)} \circ \phi_q$$

Using it, we deduce that

$$\begin{aligned} \alpha'_{q\xi}(t) &= (T_{\gamma_\xi(t)}\phi_q)(\gamma'_\xi(t)) = (T_{\gamma_\xi(t)}\phi_q)(X_\xi(\gamma_\xi(t))) \\ &= (T_{\gamma_\xi(t)}\phi_q)(T_eL_{\gamma_\xi(t)}(\xi)) \\ &= (T_{exp_G(t\xi)}\phi_q)(T_eL_{exp_G(t\xi)}(\xi)) \\ &= (T_q\phi_{exp_G(t\xi)})(T_e\phi_q(\xi)) \\ &= 0 \end{aligned}$$

Therefore, $\alpha_{q\xi}(t) = \alpha_{q\xi}(0) = \phi_q(e) = q$, which means that $exp_G(t\xi) \in G_q$ for all t. Consequently, by Proposition 3.29, we have that $\xi \in \mathfrak{g}_q$.

ii. Let us prove that $T_{[g]}(G/G_q) \cong \frac{T_gG}{(T_eL_g)(\mathfrak{g}_q)}$. From the previous lemma 3.42 we have that $\pi : G \to G/G_q$ is an exhaustive submersion. Thus, if $\pi(g) = [g]$ for $g \in G$,

$$T_{[g]}(G/G_q) \cong \frac{T_gG}{Ker(T_g\pi)}$$

Now remark that $\pi^{-1}(\pi(g)) = L_g(G_q)$ which using the regular value theorem leads to

$$Ker(T_g\pi) = T_g(\pi^{-1}(\pi(g))) = (T_eL_g)(T_eG_q) = (T_eL_g)(\mathfrak{g}_q)$$

Hence,

$$T_{[g]}(G/G_q) \cong \frac{T_gG}{(T_eL_g)(\mathfrak{g}_q)}$$

Using i. and ii. we prove that $\tilde{\phi_q}$ is an immersion. Suppose that $X_{[g]} \in T_{[g]}(G/G_q)$ and $(T_{[g]}\tilde{\phi_q})(X_{[g]}) = 0$. Since π is a submersion there exists $X_g \in T_gG$ such that $(T_g\pi)(X_g) = X_{[g]}$. Thus, it follows that,

$$0 = (T_{[g]}\tilde{\phi_q})((T_g\pi)(X_g)) = (T_g\phi_q)(X_g).$$

Now, define $\xi \in \mathfrak{g}$ to be such that $X_g = (T_eL_g)(\xi)$ and notice that

$$0 = (T_g\phi_q)((T_eL_g)(\xi)) = T_e(\phi_g \circ \phi_q)(\xi) = (T_q\phi_g)((T_e\phi_q)(\xi)).$$

Since ϕ_g is a diffeomorphism, $T_q\phi_g$ is an isomorphism and $(T_e\phi_q)(\xi) = 0$. Thus, $\xi \in \mathfrak{g}_q = Ker(T_e\phi_q)$ and $X_g \in (T_eL_g)(\mathfrak{g}_q) = Ker(T_g\pi)$, which yields to $X_{[g]} = (T_g\pi)(X_g) = 0$.

□

3.2.1 Infinitesimal generators

We are going to associate to any action $\phi : G \times M \to M$ and any $\xi \in \mathfrak{g}$ a vector field on M. These vector fields will have some important properties that will help us over the project. In particular, they are the last tool that we need in order to determine the tangent space to the orbits. Formally, they are defined as follows.

Let $\phi : G \times M \to M$ be an action of a Lie group G on a manifold M and ξ be an element of $\mathfrak{g}$. Define the map

$$\begin{array}{rccc} \phi^{\xi}: & \mathbb{R} \times M & \to & M \\ & (t,q) & \to & \phi(exp_G(t\xi), q) \end{array}$$

ϕ^{ξ} is an action of $\mathbb{R}$ on M.

Definition 3.45. *The vector field ξ_M on $\mathcal{X}(M)$ whose flow is given by ϕ^{ξ} is called* infinitesimal generator *of the action corresponding to ξ, that is,*

$$\xi_M(q) = \frac{d}{dt}_{|t=0} \phi^{\xi}_q(exp_G(t\xi))$$

Notice that

$$\xi_M(q) = T_0\phi^{\xi}_q \left(\frac{d}{dt}_{|t=0} \right) = T_0(\phi_q \circ \gamma_{\xi}) \left(\frac{d}{dt}_{|t=0} \right) = T_e\phi_q \left(T_0\gamma_{\xi} \left(\frac{d}{dt}_{|t=0} \right) \right) = T_e\phi_q(\xi) \tag{3.13}$$

Using it we get the result that we are looking for.

Proposition 3.46. *The tangent space to an orbit $G \cdot q$ at q is given by*

$$T_q(G \cdot q) = \{\xi_M(q) \mid \xi \in \mathfrak{g}\}$$

Proof. From theorem 3.44 we deduce that $T_{[g]}\tilde{\phi_q} : T_{[g]}G/G_q \to T_{\tilde{\phi}_g(q)}G \cdot q$ is an isomorphism for all $g \in G$. Taking $g = e$ and using the previous equality we get

$$T_q(G \cdot q) = T_e\tilde{\phi_q}(T_{[e]}G/G_q) = \{T_e\phi_q(\xi) \mid \xi \in \mathfrak{g}\} = \{\xi_M(q) \mid \xi \in \mathfrak{g}\}.$$

□

Next, we will present an interesting example.

Example 3.47. Let G be a Lie group, $\xi \in \mathfrak{g}$, and consider the action ϕ of G on itself by left translations. Then,

$$\begin{array}{rccc} \phi^{\xi}: & \mathbb{R} \times G & \to & G \\ & (t,h) & \to & exp_G(t\xi) \cdot h \end{array}$$

that is, $\phi^{\xi}_h = R_h \circ exp_G(t\xi)$. Hence, given $g \in G$

$$\xi_G(g) = T_e\phi_g(\xi) = T_eR_g(\xi)$$

which means that the infinitesimal generator of our action associated with ξ is the right invariant vector field whose value in the identity element is ξ.

Now, let us introduce a particular class of smooth maps and discuss how they transform the infinitesimal generators.

Definition 3.48. *Let M and N be two smooth manifolds and $\phi : G \times M \to M$ (respectively, $\varphi : G \times N \to N$) be an action of a Lie group G on M (respectively, N). A smooth map $f : M \to N$ is said to be* equivariant *with respect to these actions if*

$$f \circ \phi_g = \varphi_g \circ f \quad \forall g \in G$$

Proposition 3.49. *Let $f : M \to N$ be a smooth map and G be a Lie group. If $\phi : G \times M \to M$ (respectively, $\varphi : G \times N \to N$) is an action of G on M (respectively, N) and f is equivariant with respect to these actions then,*

$$T_q f(\xi_M(q)) = \xi_N(f(q)) \quad \xi \in \mathfrak{g} \text{ and } q \in M$$

Proof. Given $g \in G$ and taking into account the hypothesis of equivariance one has

$$\varphi_{f(q)}(g) = \varphi_g(f(q)) = (\varphi_g \circ f)(q) = (f \circ \phi_g)(q) = (f \circ \phi_q)(g)$$

Using this fact and (3.13) we get

$$\xi_N(f(q)) = T_e\varphi_{f(q)}(\xi) = T_e(f \circ \phi_q)(\xi) = T_q f(\xi_M(q)).$$

□

Finally, we enunciate an important result that relates the elements of $\mathfrak{g}$ with their associated infinitesimal generators. However, we still do not have the necessary tools in order to give a proof. We will come back to it in the next section.

Proposition 3.50. *Let ϕ be an action of a Lie group G on a manifold M. The map $\xi \in \mathfrak{g} \to \xi_M \in \mathcal{X}(M)$ is a Lie algebra anti-homomorphism, that is*

$$[\xi, \eta]_M = -[\xi_M, \eta_M] \qquad \forall \xi, \eta \in \mathfrak{g}$$

3.2.2 Adjoint and coadjoint action

As we have already pointed out, we are going to study two specific and important actions. In particular, they will help us to prove at the end of the section, the remaining Proposition 3.50. First, we will introduce the adjoint action of a Lie group on its Lie algebra $\mathfrak{g}$ and we will describe the corresponding infinitesimal generators. Then, the coadjoint action will be defined in terms of the adjoint action.

We start by introducing the notion of the inner automorphism associated with an element of G.

Definition 3.51. *Let G be a Lie group and $g \in G$. The* inner automorphism *associated with g is defined by*

$$\begin{array}{rccc} I_g : & G & \to & G \\ & h & \to & g \cdot h \cdot g^{-1} \end{array}$$

Remark that I_g is a Lie group homomorphism,

$$I_g(h \cdot h') = (g \cdot h \cdot g^{-1}) \cdot (g \cdot h' \cdot g^{-1}) = I_g(h) \cdot I_g(h').$$

Moreover, since $I_g = R_{g^{-1}} \circ L_g$, I_g is diffeomorphism and the following definition makes sense.

Definition 3.52. *Let G be a Lie group, the* adjoint action *of G on $\mathfrak{g}$ is defined by*

$$\begin{array}{rccc} Ad : & G \times \mathfrak{g} & \to & \mathfrak{g} \\ & (g, \eta) & \to & Ad_g(\eta) = T_e I_g(\eta) \end{array}$$

Remark that from the Proposition 3.25 one has that for any $g \in G$ and $\xi \in \mathfrak{g}$,

$$exp_G(Ad_g \xi) = g \cdot (exp_G \xi) \cdot g^{-1} \tag{3.14}$$

Now, let us see that given $\xi \in \mathfrak{g}$, the infinitesimal generator $\xi_{\mathfrak{g}}$ of the adjoint action corresponding to ξ is given by $\xi_{\mathfrak{g}}(\eta) = [\xi, \eta]$. We will use the two following facts:

i. Let $X, Y \in \mathcal{X}(M)$ be two vector fields, ϕ_t the one-parameter subgroup associated to X and q a point of M. Then the following equality holds:

$$[X,Y](q) = \lim_{t\to 0} \frac{T_{\phi_t(q)}\phi_{-t}(Y_{\phi_t(q)}) - Y(q)}{t}$$

ii. The flow of the vector field $X_\xi \in \mathcal{X}_L(G)$ is given by $\varphi(t,g) = g \cdot \gamma_\xi(t) = R_{\gamma_\xi(t)}(g)$.

Thus,

$$\begin{aligned}
[\xi,\eta] &= [X_\xi, X_\eta](e) \\
&= \lim_{t\to 0} \frac{T_{\gamma_\xi(t)}R_{\gamma_\xi(-t)}(X_\eta(\gamma_\xi(t))) - X_\eta(e)}{t} \\
&= \lim_{t\to 0} \frac{T_{\gamma_\xi(t)}R_{\gamma_\xi(-t)}(T_e L_{\gamma_\xi(t)}(\eta)) - \eta}{t} \\
&= \lim_{t\to 0} \frac{T_e(R_{\gamma_\xi(t)^{-1}} \circ L_{\gamma_\xi(t)})(\eta) - \eta}{t} \\
&= \lim_{t\to 0} \frac{T_e I_{\gamma_\xi(t)}(\eta) - \eta}{t}
\end{aligned}$$

On the other hand, the one-parameter subgroup associated to $\xi_{\mathfrak{g}}$ is given by

$$\begin{array}{rccc}
Ad^\xi: & \mathbb{R}\times\mathfrak{g} & \longrightarrow & \mathfrak{g} \\
& (t,\eta) & \to & Ad_{exp_G(t\xi)}(\eta) = Ad_{\gamma_\xi(t)}(\eta)
\end{array}$$

Therefore,

$$\xi_{\mathfrak{g}}(\eta) = (Ad^\xi_\eta)'(0) = \lim_{t\to 0} \frac{Ad^\xi_\eta(t) - Ad^\xi_\eta(0)}{t} = \lim_{t\to 0} \frac{T_e I_{\gamma_\xi(t)}(\eta) - \eta}{t}$$

which proves that $\xi_{\mathfrak{g}}(\eta) = [\xi,\eta]$.

Before moving on to the definition of the coadjoint action let us see an interesting example.

Example 3.53. We have seen in section 3.1.4 that we can identify the Lie algebra $\mathfrak{so}(3)$ of $SO(3)$ with $\mathbb{R}^3$. Now, we will prove that under this identification the adjoint action of $SO(3)$ is the usual action of $SO(3)$ on $\mathbb{R}^3$, that is

$$\begin{array}{rccc}
\phi: & SO(3)\times\mathbb{R}^3 & \to & \mathbb{R}^3 \\
& (A,x) & \to & Ax
\end{array}$$

The adjoint action for $SO(3)$ is given by

$$\begin{array}{rccc}
Ad: & SO(3)\times\mathfrak{so}(3) & \to & \mathfrak{so}(3) \\
& (A,\hat{\xi}) & \to & T_e I_A(\hat{\xi})
\end{array}$$

where $I_A(B) = ABA^{-1}$. Taking a curve $\sigma_{I\hat{\xi}}: \mathbb{R} \to SO(3)$ such that $\sigma_{I\hat{\xi}}(0) = I$ and $\sigma'_{I\hat{\xi}}(0) = \hat{\xi}$ we have that

$$Ad_A\hat{\xi} = T_e I_A(\hat{\xi}) = \frac{d}{dt}_{|t=0} I_A(\sigma_{I\hat{\xi}}(t)) = \frac{d}{dt}_{|t=0} A\sigma_{I\hat{\xi}}(t)A^{-1} = A\hat{\xi}A^{-1}$$

Therefore, if $w \in \mathbb{R}^3$

$$(Ad_A\hat{\xi})(w) = (A\hat{\xi}A^{-1})(w) = A(\xi \times A^{-1}w) = A\xi \times w = \widehat{A\xi}w$$

and we get $\widehat{A\xi} = Ad_A\hat{\xi}$. Finally, if we identify $\mathfrak{so}(3)$ with $\mathbb{R}^3$ we have the result

$$\begin{array}{rcll} Ad: & SO(3)\times\mathbb{R}^3 & \to & \mathbb{R}^3 \\ & (A,\xi) & \to & Ad_A\xi = A\xi \end{array}$$

Moreover, under the same identification, the infinitesimal generator $\hat{\xi}_{\mathbb{R}^3}$ of the adjoint action $Ad : SO(3)\times\mathfrak{so}(3)\to\mathfrak{so}(3)$ corresponding to $\hat{\xi}$ is given by

$$\hat{\xi}_{\mathbb{R}^3}(x) = \frac{d}{dt}_{|t=0}\gamma_{\hat{\xi}}(t)x = \hat{\xi}x = \xi\times x \tag{3.15}$$

for $x\in\mathbb{R}^3$.

Now, we have the necessary elements to introduce the coadjoint action and to determine its associated infinitesimal generator.

Definition 3.54. *Let G be a Lie group and $Ad : G\times\mathfrak{g}\to\mathfrak{g}$ be the adjoint action of G on its Lie algebra $\mathfrak{g}$. The* coadjoint action *of G on $\mathfrak{g}^*$ is defined by*

$$\begin{array}{rcll} Ad^*: & G\times\mathfrak{g}^* & \to & \mathfrak{g}^* \\ & (g,\alpha) & \to & Ad^*_{g^{-1}}(\alpha) \end{array}$$

where $(Ad^*_{g^{-1}}(\alpha))(\xi) = \alpha(Ad_{g^{-1}}\xi)$ for $\xi\in\mathfrak{g}$.

Let us show that the infinitesimal generator $\xi_{\mathfrak{g}^*}$ of the coadjoint action associated with $\xi\in\mathfrak{g}$ is given by $\xi_{\mathfrak{g}^*}(\alpha)(\eta) = -\alpha[\xi,\eta]$ for $\alpha\in\mathfrak{g}^*$ and $\eta\in\mathfrak{g}$.
Let $Ad^{*\xi} : \mathbb{R}\times\mathfrak{g}\to\mathfrak{g}$ be the flow of $\xi_{\mathfrak{g}^*}$. Then,

$$\begin{aligned} \xi_{\mathfrak{g}^*}(\alpha)(\eta) &= (Ad^{*\xi}_{\alpha})'(0)(\eta) \\ &= \left(\lim_{t\to 0}\frac{Ad^{*\xi}_{\alpha}(t) - Ad^{*\xi}_{\alpha}(0)}{t}\right)(\eta) \\ &= \lim_{t\to 0}\frac{(Ad^*_{\gamma_\xi(-t)}(\alpha))(\eta) - \alpha(\eta)}{t} \\ &= \alpha\left(\lim_{t\to 0}\frac{Ad_{\gamma_\xi(-t)}(\eta) - \eta}{t}\right) \end{aligned}$$

Taking $s=-t$ and using the proved results for the adjoint action we deduce that,

$$\xi_{\mathfrak{g}^*}(\alpha)(\eta) = -\alpha\left(\lim_{s\to 0}\frac{Ad_{\gamma_\xi(s)}(\eta) - \eta}{s}\right) = -\alpha[\xi,\eta]$$

Example 3.55. We proved that the adjoint action of $SO(3)$ is the standard action of $SO(3)$ on $\mathbb{R}^3$. Likewise, we are going to show that the same happens with the coadjoint action.
First, remark that the dual space $\mathfrak{so}^*(3)$ might be identified with $\mathbb{R}^3$ via the breve map as follows,

$$\begin{array}{rcl} \breve{}: \quad \mathbb{R}^3 & \to & \mathfrak{so}^*(3) \\ \Pi = (\Pi_1,\Pi_2,\Pi_2) & \to & \breve{\Pi} \end{array}$$

where if $\hat{\xi}\in\mathfrak{so}(3)$, $\breve{\Pi}(\hat{\xi}) = \Pi\cdot\xi = \Pi_1\xi_1 + \Pi_2\xi_2 + \Pi_3\xi_3$.

The coadjoint action of $SO(3)$ is given by

$$\begin{array}{rccc} Ad^*: & SO(3)\times\mathfrak{so}^*(3) & \to & \mathfrak{so}^*(3) \\ & (A,\breve{\Pi}) & \to & Ad^*_{A^{-1}}(\breve{\Pi}) \end{array}$$

Hence, if $\hat{\xi}\in\mathfrak{so}(3)$,

$$Ad^*(A,\breve{\Pi})(\hat{\xi}) = \breve{\Pi}(Ad_{A^{-1}}(\hat{\xi})) = \breve{\Pi}(A^{-1}\hat{\xi}) = \Pi\cdot A^{-1}\hat{\xi} = A\Pi\cdot\hat{\xi} = \breve{A\Pi}(\hat{\xi})$$

which means that $Ad^*_{A^{-1}}\breve{\Pi} = \breve{A\Pi}$. Thus, identifying $\mathfrak{so}^*(3)\cong\mathbb{R}^3$ we conclude that,

$$\begin{array}{rccc} Ad^*_{A^{-1}}: & SO(3)\times\mathbb{R}^3 & \to & \mathbb{R}^3 \\ & (A,\Pi) & \to & Ad^*_{A^{-1}}\Pi = A\Pi \end{array}$$

Finally, we will prove the propostion 3.50. For this purpose, we will use the following result.

Lemma 3.56. *Let $\phi: G\times M\to M$ be an action of a Lie group G on a manifold M and q be a point of M. If $\xi\in\mathfrak{g}$, then*

$$(Ad_g\xi)_M(q) = T_{\phi_{g^{-1}(q)}}\phi_g(\xi_M(\phi_{g^{-1}}(q)))$$

Proof. Denote by $\phi^{Ad_g\xi}:\mathbb{R}\times M\to M$ the one-parameter subgroup associated to $(Ad_g\xi)_M$. Thus, for $q\in M$ and $t\in R$,

$$\begin{aligned} \phi^{Ad_g\xi}(t,q) &= \phi_q(exp_G(tAd_g(\xi))) = \phi_q(exp_G(Ad_g(t\xi))) = \phi_q(g\cdot exp_G(t\xi)\cdot g^{-1}) \\ &= \phi(g\cdot exp_G(t\xi),\phi_{g^{-1}}(q)) = \phi_{\phi_{g^{-1}(q)}}(g\cdot exp_G(t\xi)) = (\phi_g\circ\phi_{\phi_{g^{-1}(q)}})(exp_G(t\xi)) \\ &= (\phi_g\circ\phi^{\xi}_{\phi_{g^{-1}(q)}})(t) \end{aligned}$$

where the equality (3.14) has been used. Now,

$$(Ad_g\xi)_M(q) = T_0\phi_q^{Ad_g\xi}\left(\frac{d}{dt}_{|t=0}\right) = T_0(\phi_g\circ\phi^{\xi}_{\phi_{g^{-1}(q)}})\left(\frac{d}{dt}_{|t=0}\right) = T_{\phi_{g^{-1}(q)}}\phi_g(\xi_M(\phi_{g^{-1}(q)})).$$

□

Finally, we come to the proof.

Proof of Proposition 3.50. Replacing $g = exp_G(t\eta)$ in the previous lemma we get

$$(Ad_{\exp_G(t\eta)}\xi)_M(q) = T_{\phi_{\exp_G(-t\eta)}}\phi_{\exp_G(t\eta)}(\xi_M(\phi_{\exp_G(-t\eta)}(q)))$$

Then,

$$\begin{aligned} -[\eta_M,\xi_M](q) &= [-\eta_M,\xi_M](q) \\ &= \lim_{t\to 0}\frac{T_{\phi_{\exp_G(-t\eta)}}\phi_{\exp_G(t\eta)}(\xi_M(\phi_{\exp_G(-t\eta)}(q))) - \xi_M(q)}{t} \\ &= \lim_{t\to 0}\frac{(Ad_{\exp_G(t\eta)}\xi)_M(q) - \xi_M(q)}{t} \\ &= \lim_{t\to 0}\left(\frac{(Ad_{\exp_G(t\eta)}\xi) - \xi}{t}\right)_M(q) \\ &= [\eta,\xi]_M(q) \end{aligned}$$

□

Chapter 4

Momentum map

Given an action $\phi : G \times M \to M$ of a Lie group G on a symplectic manifold M, under certain conditions one can associate to it a map $J : M \to \mathfrak{g}^*$ verifying some properties. Such maps are known as momentum maps. For Hamiltoninan functions which are invariant under the action, momentum maps allow us to obtain constants of the motion for the corresponding Hamiltoninan system. Furthermore, assuming some extra hypothesis it can be shown that momentum maps are equivariant with respect to the original action and the coadjoint action of the Lie group G. As regards the end of the project, we anticipate that such maps will be fundamental to prove the Lie-Poisson reduction theorem.

We begin the chapter with the formal definition of symplectic actions and momentum map and give some examples. Then, we focus on the momentum maps associated with symplectic actions and we study its properties. Finally, we introduce a particular case of such maps, but very useful for our current concern. At the end we show how one can get momentum maps from the cotangent lift of an action and we have a quick overview over some significant examples.

4.1 Definitions and examples

First of all, we introduce the notion of a momentum map.

Definition 4.1. *Let (M, ω) be a connected symplectic manifold, G be a Lie group and $\mathfrak{g}$ its Lie algebra. A* momentum map *for an action $\phi : G \times M \to M$ is a real function $J : M \to \mathfrak{g}^*$ such that for every $\xi \in \mathfrak{g}$, ξ_M is the Hamiltonian vector field of the map J_ξ defined as*

$$\begin{array}{rccc} J_\xi : & M & \to & \mathbb{R} \\ & q & \to & J(q)(\xi) \end{array}$$

that is,

$$X_{J_\xi} = \xi_M$$

In other words, we have that $J : M \to \mathfrak{g}^*$ is a momentum map if for all $\xi \in \mathfrak{g}$ the following equation holds,

$$dJ_\xi = i_{\xi_M}\omega$$

In such conditions we have the following definition.

Definition 4.2. *(M, ω, ϕ, J) is known as a* Hamiltonian G-space

Now, introduce the notion of a symplectic action because our interest is to obtain momentum maps associated with them.

Definition 4.3. *Let (M, ω) be a connected symplectic manifold and G be a Lie group. A* symplectic action *is an action $\phi : G \times M \to M$ of G on M such that $\phi_g : M \to M$ is a symplectic map, for all $g \in G$.*

Remark 4.4. Although the chapter will focus on the study of momentum maps associated with symplectic actions, it is not true that any symplectic action has a momentum map associated with it. In fact, if $\phi : G \times M \to M$ is a symplectic action and $\xi \in \mathfrak{g}$, we have that the flow of ξ_M preserves the symplectic form. Therefore, the infinitesimal generators of a symplectic action are always locally Hamiltonian. Nevertheless, we can not ensure that they are globally Hamiltonian and consequently the symplectic action may not admit a momentum map.

Example 4.5. Following the previous remark, let us find a momentum map for a symplectic action $\phi : G \times M \to M$ whose infinitesimal generators ξ_M are globally Hamiltonian.
Let $\{\xi_1, \ldots, \xi_n\}$ be a basis of $\mathfrak{g}$ and J_i the Hamiltonian functions of the vector fields ξ_{iM}. Given any $\xi = \lambda^1\xi_1 + \ldots + \lambda^n\xi_n$ in $\mathfrak{g}$, $\lambda^i \in \mathbb{R}$, define the function $J_\xi = \sum_{i=1}^n \lambda^i J_{\xi_i} : M \to \mathbb{R}$. Notice that $J_{\xi_i} = J_i$. Thus,

$$dJ_\xi = \sum_{i=1}^n \lambda^i d(J_{\xi_i}) = \sum_{i=1}^n \lambda^i i_{\xi_{iM}}\omega = i_{\left(\sum_{i=1}^n \lambda^i \xi_i\right)_M}\omega = i_{\xi_M}\omega$$

Example 4.6. Denote by ϕ^T the tangent lift of the usual action of $SO(3)$ on $\mathbb{R}^3$ under the identification $T\mathbb{R}^3 \cong \mathbb{R}^3 \times \mathbb{R}^3$. From Example 3.35 it is given by $\phi^T_A(q, v) = (Aq, Av)$. Given a vector $v \in \mathbb{R}^3$, let $< v, >$ denote the corresponding covector under the identification of $\mathbb{R}^3$ with $\mathbb{R}^{3*}$ via the standard scalar product $< , >$. Thus, if $(Aq, w) \in T_{Aq}\mathbb{R}^3$ is a vector, the cotangent lift ϕ^{T^*} verifies

$$\begin{aligned}\phi^{T^*}(A,(q,v))(Aq,w) &= T_q^*\phi_{A^{-1}}(q,v)(Aq,w) = (q,v)\left(T_{Aq}\phi_{A^{-1}}(Aq,w)\right)\\ &= (q,v)\left(\phi^T_{A^{-1}}(Aq,w)\right) = (q,v)(q,A^{-1}w)\\ &= < v, A^{-1}w > = < Av, w >\\ &= (Aq,Av)(Aq,w)\end{aligned}$$

Then, under the given identification between vectors and covectors, ϕ^{T^*} is given by

$$\phi^{T^*}(A,(q,v)) = (Aq,Av)$$

From the definition of the cotangent lift and Proposition 2.13 it is clear that ϕ^{T^*} is a symplectic action. Let us show that ϕ^{T^*} admits a momentum map $J : T^*\mathbb{R}^3 \to \mathfrak{so}(3)$ and that it coincides with the angular momentum. Given $\hat{\xi} \in \mathfrak{so}(3)$,

$$\hat{\xi}_{T^*\mathbb{R}^3}(q,v) = \frac{d}{dt}_{|t=0}\left(\phi^{T^*}_{(q,v)}\right)^{\hat{\xi}}(\gamma_{\hat{\xi}}(t)) = (\hat{\xi}q, \hat{\xi}v) = (\xi \times q, \xi \times v)$$

where the identification of $\mathfrak{so}(3)$ with $\mathbb{R}^3$ has been used[1].
The momentum map we are looking for has to verify $X_{J_\xi} = \hat{\xi}_{T^*\mathbb{R}^3}$. Hence, from the local expression of the Hamiltonian vector fields (2.25) it is equivalent to

$$\frac{\partial J_\xi}{\partial v_i} = (\xi \times q)_i \qquad \frac{\partial J_\xi}{\partial q_i} = -(\xi \times v)_i \quad \text{for } i = 1,2,3$$

Therefore,

$$J_\xi(q,v) = (\xi \times q)\cdot v = (q \times v)\cdot \xi$$

and $J = (q \times v)$ is the angular momentum.

It is worth mentioning that further on we will see that the cotangent lift is always a symplectic action and admits a momentum map. This example will be a particular case of such a result. We will come back to it later from another perspective.

Finally, we specialize the definition of an equivariant momentum map[2]. As we will see, this property is satisfied by the particular momentum maps in which we will focus on.

[1]See the hat map in section 3.1.4
[2]See section 3.2.1

Definition 4.7. *Let (M, ω, ϕ, J) be a Hamiltonian G-space. J is said to be Ad^*-*equivariant *if*

$$J \circ \phi_g = Ad^*_{g^{-1}} \circ J \quad \forall g \in G$$

Example 4.8. Remember that the coadjoint action of $SO(3)$ is given by $\alpha \to A\alpha$. As well, in last example we have proved that the momentum map for the usual action of $SO(3)$ on $\mathbb{R}^3$ is given by $J(q, v) = q \times v$. Then, since

$$(Aq) \times (Av) = A(q \times v)$$

the angular momentum J is Ad^*-equivariant.

4.2 Properties of the momentum map

We will center our interest on the momentum maps associated with symplectic actions. That is why we will devote this section to study their main properties.

Proposition 4.9. *If J and J' are two momentum maps for the same symplectic action $\phi : G \times M \to M$, then there exists $\mu \in \mathfrak{g}^*$ such that $J - J' = \mu$.*

Proof. If $\xi \in \mathfrak{g}$

$$dJ_\xi = i_{\xi_M}\omega = dJ'_\xi \quad \Rightarrow \quad d(J_\xi - J'_\xi) = 0$$

Since M is connected, $J_\xi - J'_\xi = C(\xi)$, where $C(\xi)$ is a constant. Thus the 1-form μ that we are looking for is given by

$$\begin{array}{rccc} \mu : & \mathfrak{g} & \to & \mathbb{R} \\ & \xi & \to & C(\xi) \end{array}$$

□

Next, we will see that for a G-invariant Hamiltonian system the presence of a momentum map allows us to obtain constants of motion for the system.

Lemma 4.10. *Let M be a smooth manifold, X be a vector field on M whose flow is φ_t and $F : M \to \mathbb{R}$ be a smooth map. Then,*

$$F \circ \varphi_t = F \quad \Leftrightarrow \quad X(F) = 0$$

Theorem 4.11. *Let (M, ω, ϕ, J) be a Hamiltonian G-space. Suppose that the Hamiltonian function of the system $H : M \to \mathbb{R}$ is invariant under the action ϕ, i.e, $H = H \circ \phi_g$ for any $g \in G$. Then, J is a constant of motion for H, that is,*

$$J \circ \varphi_t = J$$

where φ_t is the flow of the Hamiltonian vector field X_H.

Proof. By the previous lemma it is enough to verify that $X_H(J_\xi) = 0$. Indeed, if this equality is satisfied the lemma states that given $\xi \in \mathfrak{g}$

$$J_\xi \circ \varphi_t(q) = J_\xi(q) \quad \forall q \in M$$

which implies that,

$$J(\varphi_t(q))(\xi) = J(q)(\xi) \quad \forall q \in M \quad \Rightarrow J \circ \varphi_t = J$$

Let us check that $X_H(J_\xi) = 0$. One the one hand we have,

$$X_H(J_\xi) = dJ_\xi(X_H) = i_{\xi_M}\omega(X_H) = -i_{X_H}\omega(\xi_M) = -dH(\xi_M) = -\xi_M(H).$$

On the other hand we have seen in the previous chapter that the flow of ξ_M is given by $\phi^\xi_x = \phi_{\gamma_\xi(t)}(x)$. Then, since the Hamiltonian is invariant under the action, using again the lemma, we conclude that

$$0 = -\xi_M(H) = X_H(J_\xi)$$

□

4.3 Momentum maps on the cotangent bundle

Let us end this chapter by giving a rule to obtain symplectic actions that admit a momentum map. It will allow us to introduce several examples, within which we find the previous Example 4.6. We will achieve it by means of the cotangent lift of a given action.

Theorem 4.12. *Let (M, ω) be a symplectic manifold such that the symplectic form $\omega = -d\theta$ is exact. Suppose that ϕ is an action of Lie group G on M such that $\phi_g^*\theta = \theta$ for any $g \in G$. Then the map $J : M \to \mathfrak{g}^*$ given by*

$$J(q)(\xi) = i_{\xi_M}\theta(q)$$

is an Ad^*-equivariant momentum map.

Proof. Remark that $(\phi_{\gamma_\xi(t)})^*\theta = \theta$ for $t \in \mathbb{R}$ and $\xi \in \mathfrak{g}$. Thus, due to the fact that $\phi_{\gamma_\xi(t)}$ is the flow of ξ_M we have that $\mathcal{L}_{\xi_M}\theta = 0$ which means

$$d(i_{\xi_M}\theta) = i_{\xi_M}\omega \quad \Rightarrow \quad dJ_\xi = i_{\xi_M}\omega$$

and J is a momentum map.
Now let us prove that J is Ad^*-equivariant by showing that $J(\phi_g(q)) = Ad^*_{g^{-1}}J(q)$ for every $q \in M$. It is equivalent to

$$J(\phi_g(q)) = Ad^*_{g^{-1}}J(q) \;\Leftrightarrow\; J_\xi(\phi_g(q)) = J(q)(Ad_{g^{-1}}\xi) \;\Leftrightarrow\; i_{\xi_M}\theta(\phi_g(q)) = i_{(Ad_{g^{-1}}\xi)_M}\theta(q)$$

for $\xi \in \mathfrak{g}$. The last equality holds since using lemma 3.56

$$\begin{aligned} i_{(Ad_{g^{-1}}\xi)_M}\theta(q) &= \theta(q)((Ad_{g^{-1}}\xi)_M(q)) = \theta(q)\left(T_{\phi_g(q)}\phi_{g^{-1}}(\xi_M(\phi_g(q)))\right) \\ &= (\phi_{g^{-1}})^*\theta(\phi_g(q))(\xi_M(\phi_g(q))) = \theta(\phi_g(q))\xi_M(\phi_g(q)) \\ &= i_{\xi_M}\theta(\phi_g(q)) \end{aligned}$$

□

Once the theorem is proved we are ready to study the cotangent lift of any action and show its properties.

Corollary 4.13. *Let M be a smooth manifold and $\phi : G \times M \to M$ be an action. Then,*

i. $\phi^{T^*} : M \times T^*M \to T^*M$ is a symplectic action.

ii. The map $J : T^*M \to \mathfrak{g}^*$ defined by

$$J(\alpha_q)(\xi) = J_\xi(\alpha_q) = \alpha_q(\xi_M(q)) \quad \text{for } \alpha_q \in T_q^*M \text{ and } \xi \in \mathfrak{g}$$

is an Ad^*-equivariant momentum map.

Proof. i. Recall the Definition 3.34 of the cotangent lift of an action and the Proposition 2.13. Then, it is clear that ϕ^{T^*} is a symplectic action.

ii. If $\pi_M : T^*M \to M$ denotes the canonical projection, remark that π_M is equivariant with respect to the actions ϕ^{T^*} and ϕ. Indeed, if $\alpha_q \in T_q^*M$ and $g \in G$,

$$\pi_M(\phi_g^{T^*}(\alpha_q)) = \pi_M(T_q^*\phi_{g^{-1}}(\alpha_q)) = \phi_g(q) = \phi_q(\pi_Q(\alpha_q))$$

From Proposition 3.49 we have that $T_{\alpha_q}\pi_M(\xi_{T^*M}(\alpha_q)) = \xi_M(\pi_M(\alpha_q))$, so if λ_M denotes the Liouville 1-form we have

$$\begin{aligned} i_{\xi_{T^*M}}\lambda_M(\alpha_q) &= \lambda_M(\alpha_q)(\xi_{T^*M}(\alpha_q)) = \alpha_q\left(T_{\alpha_q}\pi_M(\xi_{T^*M}(\alpha_q))\right) \\ &= \alpha_q(\xi_M(\pi_M(\alpha_q))) = \alpha_q(\xi_M(q)) \\ &= J_\xi(\alpha_q) \end{aligned}$$

Thus, using the previous theorem we can conclude that J is Ad^*-equivariant momentum map.

□

Remark 4.14. Note that the momentum map calculated in the Example 4.6 coincides with the momentum map of Corollary 4.13 assuming that ϕ is the usual action of $SO(3)$ on $\mathbb{R}^3$. Indeed,

$$J_\xi(q,v) =< v, \hat{\xi}_{\mathbb{R}^3}(q)) >= v(\xi \times q) = (q \times v)\xi \quad \Rightarrow J(q,v) = q \times v$$

As well, we have already proved in Example 4.8 that it is Ad^*-equivariant without using the previous result.

Under the same conditions as in Corollary 4.13, assume that X is a vector field on M. Then, we can define a real function on T^*M as follows,

$$\begin{array}{rccc} P(X): & T^*M & \to & \mathbb{R} \\ & \alpha_q & \to & \alpha_q(X(q)) \end{array}$$

Note that if $\xi \in \mathfrak{g}$ we have that

$$J_\xi = P(\xi_M)$$

Definition 4.15. *The map $P(X)$ is called the* momentum *corresponding to X.*

Take local charts $(U, \varphi \equiv (q^1, \ldots, q^n))$ in M and $(\pi_M^{-1}(U), \underline{\varphi} \equiv (q^1, \ldots, q^n, p_1, \ldots, p_n))$ in T^*M. Assume that the vector field X in these charts is given by

$$X = \sum_{i=1}^n X^i \frac{\partial}{\partial q^i}$$

Thus, if $\alpha_q \in T_q^*M$,

$$P(X)(\alpha_q) = \alpha_q(X(q)) = \alpha_q \left(\sum_{i=1}^n X^i(\pi_M(\alpha_q)) \frac{\partial}{\partial q^i}_{|\pi_M(\alpha_q)} \right) = \sum_{i=1}^n p_i X^i(\pi_M(\alpha_q))$$

Finally, we give some interesting examples of momentum maps associated to lifted actions.

Example 4.16. Consider the action $\phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ given by $\phi(t,q) = t + q$. It is easy to see that given $\xi \in T_0\mathbb{R}^n \cong \mathbb{R}^n$ the infinitesimal generator is $\xi_{\mathbb{R}^n}(q) = \xi$ for any $q \in \mathbb{R}^n$. Using the previous corollary, the cotangent lift of that action is symplectic and an Ad^*-equivariant map is given by

$$J_\xi(q^i, v_i) = \sum_{i=1}^n v_i dq^i(\xi)$$

Thus, $J = \sum_{i=1}^n v_i dq^i$ and the momentum map coincides with the linear momentum.

Example 4.17. Let $\phi : GL(n,\mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}^n$ be the action defined as $\phi(A,q) = Aq$. Given $A \in \mathfrak{gl}(n,\mathbb{R})$ its associated infinitesimal generator is $A_{\mathbb{R}^n}(q) = Aq$ for any $q \in \mathbb{R}^n$. Again, the cotangent lift of the action leads to a symplectic action whose associated Ad^*-equivariant momentum map is

$$J_A(q^i, v_i) = \left(\sum_{i=1}^n v_i dq^i \right)(Aq)$$

Example 4.18. Consider the left action of a Lie group G on itself (see Example 3.47) and its cotangent lift. It is a symplectic action and its momentum map satisfies

$$J_\xi(\alpha_g) = \alpha_g(T_e R_g \xi) = ((T_e R_g)^*(\alpha_g))(\xi)$$

for $g \in G$, $\alpha_g \in T_g^*G$ and $\xi \in \mathfrak{g}$. Thus, the momentum map is $J(\alpha_g) = (T_e R_g)^* \alpha_g$

As one can note, the previous result gives us a powerful tool to compute certain momentum maps. Furthermore, the corollary guarantees that the obtained momentum map is Ad^*-equivariant. Remark 4.14 is a good example of how this theorem makes the computations of the momentum map easier.

Chapter 5

Poisson manifolds

Poisson manifolds is the last topic that we need in order to enunciate and prove the Lie-Poisson reduction theorem. We have already worked on them since, as we shall see, symplectic manifolds are a particular type of Poisson manifolds. Not any Poisson manifold is sympletic, although they always admit a symplectic foliation that will be a key issue for our next discussions. Specifically, we will need Poisson manifolds associated with linear Poisson structures because the dual space of a real Lie algebra of finite dimension has such a structure. In addition, this particular type of Poisson manifolds verifies very interesting properties.

We introduce Poisson manifolds from the Poisson bracket, or equivalently, the Poisson 2-vector. Then, we study the particular case of symplectic manifolds and the linear Poisson structures on a real vector space of finite dimension which are known as Lie-Poisson structures. Later, we see some important results on Poisson morphisms and apply them to study Poisson vector fields. Finally, we prove that every Poisson manifold has a symplectic foliation associated with it and we describe the leaves of the foliation in the case of a Lie-Poisson structure.

5.1 Generalities on Poisson structures

There are two ways of describing the Poisson manifolds. From the dynamic point of view a Poisson manifold can be described from a bracket of functions verifying certain conditions. Otherwise, from the geometric point of view, Poisson manifolds are characterized from a 2-vector whose Schouten-Nijenhuis bracket with itself is zero. Here we begin with the dynamic definition and after that we prove that the other one is equivalent.

5.1.1 Poisson brackets

Definition 5.1. *A smooth manifold M is said to be a* Poisson manifold *if there exists an operation* $\{\ ,\ \} : \mathcal{F}(M) \times \mathcal{F}(M) \to \mathcal{F}(M)$ *such that*

i. $(\mathcal{F}(M), \{\ ,\ \})$ *is a Lie algebra, that is,* $\{\ ,\ \}$ *is* $\mathbb{R}$*-bilinear, skew-symmetric and satisfies the Jacobi identity.*

ii. $\{\ ,\ \}$ *verifies the Leibniz rule, that is,* $\{f, gh\} = \{f, g\}\, h + g\, \{f, h\}$ *for all functions* $f, g, h \in \mathcal{F}(M)$*.*

$\{\ ,\ \}$ *is known as the* Poisson bracket

Example 5.2. Any smooth manifold is a Poisson manifold with the trivial Poisson bracket $\{f, g\} = 0$ for every $f, g \in \mathcal{F}(M)$.

Example 5.3. If $M = \mathbb{R}^2$ we have the canonical Poisson bracket defined as

$$\{f, g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} \tag{5.1}$$

In Chapter 2 we introduced the Hamiltonian vector fields on a symplectic manifold. However, it is not necessary a symplectic manifold to define Hamiltonian vector fields but a Poisson manifold. For every $g \in \mathcal{F}(M)$ consider the map,

$$\begin{array}{rccc} X_g: & \mathcal{F}(M) & \to & \mathcal{F}(M) \\ & f & \to & X_g(f) = \{f, g\} \end{array} \tag{5.2}$$

For every $q \in M$, by the properties of the Poisson bracket, we have that $X_g(q)$ is $\mathbb{R}$-linear and satisfies the Leibniz rule. Therefore, $X_g(q) \in T_qM$ is a derivation and X_g defines a vector field on M.

Definition 5.4. *Let $(M, \{\ ,\ \})$ be a Poisson manifold and $f \in \mathcal{F}(M)$ be a function. The vector field X_f defined above is known as the* Hamiltonian vector field *of the function f.*

Further on we will prove that if M is a symplectic manifold, it is also a Poisson manifold and the Hamiltonian vector fields defined by the symplectic form coincide with the Hamiltonian vector fields defined by the Poisson bracket.

Using the properties that the Hamiltonian vector fields inherit from the Poisson bracket we have the following result.

Theorem 5.5. *Let $(M, \{\ ,\ \})$ be a Poisson manifold and X_f be a Hamiltonian vector field. If g and h are first integrals of X_f, i.e, $X_f(g) = X_f(h) = 0$, then $\{g, h\}$ is also a first integral of X_f.*

Proof. We have to prove that $X_f(\{g, h\}) = 0$, which is nothing else that the Jacobi identity,

$$X_f(\{g, h\}) = \{\{g, h\}\, f\} = -\{\{f, g\}\, h\} - \{\{h, f\}\, g\} = \{X_f(g), h\} - \{X_f(h), g\} = 0$$

□

Furthermore, we have a Lie algebra anti-morphism between the smooth functions on M and the vector fields on M.

Proposition 5.6. *Let $(M, \{\ ,\ \})$ be a Poisson manifold and X_f be the Hamiltonian vector field of the function $f \in \mathcal{F}(M)$. There exists a Lie algebra anti-morphism φ between the smooth functions on M and the vector fields on M defined by*

$$\begin{array}{rccc} \varphi: & (\mathcal{F}(M), \{\ ,\ \}) & \to & (\mathcal{X}(M), [\ ,\ ]) \\ & f & \to & X_f \end{array}$$

In particular,

$$[X_f, X_g] = -X_{\{f,g\}}$$

Proof. For any $f, g, h \in \mathcal{F}(M)$ we have the following equality

$$[X_f, X_g](h) = X_f(X_g(h)) - X_g(X_f(h)) = \{\{h, g\}\, f\} - \{\{h, f\}\, g\} = \{\{f, g\}\, h\} = -X_{\{f,g\}}(h)$$

Thus, $[X_f, X_g] = -X_{\{f,g\}}$. □

5.1.2 Poisson 2-vector

From the properties that the Poisson bracket satisfies, it is clear that it is a biderivation[1]. Thus, it defines a 2-vector $w : \Omega^1(M) \times \Omega^1(M) \to \mathcal{F}(M)$ which is characterized by the condition

$$\{f, g\} = w(df, dg) \tag{5.3}$$

Definition 5.7. *Let $(M, \{\ ,\ \})$ be a Poisson manifold. The 2-vector induced by $\{\ ,\ \}$ is known as the* Poisson structure *of M.*

[1] See Appendix A for a further explanation on these topics.

If $(U, \varphi = (q^1, \ldots, q^m))$ is a local chart on M, the local expression of the Poisson structure is

$$w = \sum_{i<j} w_{ij} \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial q^j} = \frac{1}{2} \sum_{i,j} w_{ij} \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial q^j} \tag{5.4}$$

where $w_{ij} = w(dq^i, dq^j) = \left\{q^i, q^j\right\}$ for all $i, j = 1, \ldots, m$. Therefore,

$$\{f, g\} = w(df, dg) = \sum_{i,j} w_{ij} \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial q^j} \tag{5.5}$$

It is not true that any 2-vector defines a Poisson structure. Indeed, if w is a Poisson structure, then the bracket defined by the previous equality (5.3) satisfies the Jacobi identity. Some calculations show that the induced bracket by a 2-vector w verifies the Jacobi identity if, and only if,

$$\sum_h \left(w_{hi} \frac{\partial w_{jk}}{\partial q^h} + w_{hj} \frac{\partial w_{ki}}{\partial q^h} + w_{hk} \frac{\partial w_{ij}}{\partial q^h} \right) = 0 \quad \forall i, j, k \tag{5.6}$$

Our purpose now is to determine when a 2-vector is a Poisson structure without checking that the equality (5.6) is satisfied. We will achieve it by using the Schouten-Nijenhuis bracket, that is, a $\mathbb{R}$-bilinear extension of the Lie derivative to an operation $[\ ,\ ] : \mathcal{X}^p(M) \times \mathcal{X}^q(M) \to \mathcal{X}^{p+q-1}(M)$. Here $\mathcal{X}^r(M)$ denotes the space of r-vectors on M (see Appendix A). Such an operation has the following properties:

i. For all $P \in \mathcal{X}^p(M)$ and $Q \in \mathcal{X}^q(M)$, $[P, Q] = (-1)^{pq}[Q, P]$.

ii. For all $P \in \mathcal{X}^p(M)$, $R \in \mathcal{X}^r(M)$ and $Q \in \mathcal{X}^q(M)$, $[P, Q \wedge R] = [P, Q] \wedge R + (-1)^{pq+q} Q \wedge [P, R]$.

iii. For all $P \in \mathcal{X}^p(M)$, $R \in \mathcal{X}^r(M)$ and $Q \in \mathcal{X}^q(M)$,
$(-1)^{p(r-1)}[P, [Q, R]] + (-1)^{q(p-1)}[Q, [R, P]] + (-1)^{r(q-1)}[R, [P, Q]] = 0$. *(Graded Jacobi identity)*

When $p = q = 2$, using the previous properties it can be shown that the local expression of the bracket of two 2-vectors $w = \sum_{i<j} w_{ij} \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial q^j}$ and $w' = \sum_{h<k} w'_{hk} \frac{\partial}{\partial q^h} \wedge \frac{\partial}{\partial q^k}$ is given by

$$[w, w'] = \sum_{i<j<k} \sum_h \left(w_{hi} \frac{\partial w'_{jk}}{\partial q^h} + w_{hj} \frac{\partial w'_{ki}}{\partial q^h} + w_{hk} \frac{\partial w'_{ij}}{\partial q^h} + w'_{hi} \frac{\partial w_{jk}}{\partial q^h} + w'_{hj} \frac{\partial w_{ki}}{\partial q^h} + w'_{hk} \frac{\partial w_{ij}}{\partial q^h} \right) \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial q^j} \wedge \frac{\partial}{\partial q^k} \tag{5.7}$$

Therefore, letting $w' = w$ a sufficient condition for 2-vector to be a Poisson structure is to

$$[w, w] = 0 \tag{5.8}$$

More details on these topics may be found in Appendix A.

Given a 2-vector w, it has associated an homomorphism $\#_w : T^*M \to TM$ that maps any covector $\alpha_q \in T^*_q M$ to a vector $\#_w(\alpha_q) \in T_q M$ such that for any $\beta_q \in T^*_q M$

$$\beta_q(\#_w(\alpha_q)) = w(\alpha_q, \beta_q)$$

In the same charts as before, if we denote by $\#_{w(q)}$ the restriction of $\#_w$ to $T^*_q M$ and $\alpha_q = \sum_{i=1}^m \alpha_i dq^i{}_{|q} \in T^*_q M$ one has

$$\#_{w(q)} \left(\sum_{i=1}^m \alpha_i dq^i{}_{|q} \right) = \sum_{i,j=1}^m w_{ij} \alpha_i \frac{\partial}{\partial q^j}_{|q}$$

Then, $\#_{w(q)}$ is linear.

By abuse of notation, we denote also by $\#_w$ the induced map between 1-forms and vector fields. If $\alpha \in \Omega^1(M)$ is a 1-form, the corresponding vector field $\#_w(\alpha) \in \mathcal{X}(M)$ is defined by $\#_w(\alpha)(q) = \#_w(\alpha(q))$, for $q \in M$. Remark that for any $f, g \in \mathcal{F}(M)$ we have

$$dg(\#_w(df))(q) = dg(q)(\#_w(df(q))) = w(df(q), dg(q)) = -X_f(g)(q) \tag{5.9}$$

Thus, $\#_w(df) = -X_f$.

5.1.3 Symplectic manifolds and Poisson structures

If (M,ω) is a symplectic manifold, we can define a Poisson bracket on M by

$$\{f,g\} = \omega(X_f, X_g) = i_{X_f}\omega(X_g) = df(X_g) = X_g(f) \tag{5.10}$$

where X_f and X_g are the Hamiltonian vector fields of the functions $f, g \in \mathcal{F}(M)$ with respect to the symplectic form ω. It is obvious that the bracket $\{\ ,\ \}$ is $\mathbb{R}$-bilinear and that satisfies the Leibniz rule. Let us prove the Jacobi identity. Using that the symplectic form is closed, given three vector fields $X_f, X_g, X_h \in \mathcal{X}(M)$ we have

$$\begin{aligned}
0 &= d\omega(X_f, X_g, X_h)\\
&= X_f(\omega(X_g, X_h)) + X_g(\omega(X_h, X_f)) + X_h(\omega(X_f, X_g))\\
&- \omega([X_f, X_g], X_h) - \omega([X_g, X_h], X_f) - \omega([X_h, X_f], X_g)\\
&= X_f(\{g,h\}) + X_g(\{h,f\}) + X_h(\{f,g\}) + [X_f, X_g](h) + [X_g, X_h](f) + [X_h, X_f](g)\\
&= \{\{g,h\}, f\} + \{\{h,f\}, g\} + \{\{f,g\}, h\}\\
&X_f(X_g(h)) - X_g(X_f(h)) + X_g(X_h(f)) - X_h(X_g(f)) + X_h(X_f(g)) - X_f(X_h(g))\\
&= -\left(\{\{g,h\}, f\} + \{\{h,f\}, g\} + \{\{f,g\}, h\}\right)
\end{aligned}$$

Furthermore, from the definition of the vector bundle isomorphism b_ω associated with the symplectic form ω, and the homomorphism $\#_w$ associated with the Poisson structure w induced by the symplectic structure (5.10), it is easy to show that,

$$\#_w = -b_\omega^{-1} \tag{5.11}$$

It means that the homomorphism $\#_w$ associated to a Poisson structure defined from a symplectic form is an isomorphism. In particular, the Hamiltonian vector fields associated to the symplectic form ω coincide, with the Hamiltonian vector fields defined by the Poisson bracket $\{\ ,\ \}$. In fact,

$$X_f = -\#_w(df) = b_\omega^{-1}(df) = X_f \text{ for any } f \in \mathcal{F}(M)$$

A more exhaustive explanation of these topics is found in Appendix A.

Let M be a smooth manifold and ω_M be the canonical symplectic structure of the cotangent bundle. Set a chart $(U, \varphi = (q^1, \ldots, q^m))$ on M and the corresponding induced chart $(\pi_Q^{-1}(U), \underline{\varphi} \equiv (q^1, \ldots, q^n, p_1, \ldots, p_n))$ on T^*M. As we have already seen in Chapter 2, the Hamiltonian vector fields are locally given by $X_f = \sum_{i=1}^n \left(\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i}\right)$. Thus, the local expression of the Poisson bracket induced by the canonical symplectic structure is

$$\{f,g\} = X_g(f) = \sum_{i=1}^n \left(\frac{\partial g}{\partial p_i}\frac{\partial f}{\partial q^i} - \frac{\partial g}{\partial q^i}\frac{\partial f}{\partial p_i}\right) \tag{5.12}$$

It is worth mentioning, that if (M,ω) is a symplectic manifold of dimension $2n$, then there exist local coordinates $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ on M such that the local expression of ω is

$$\omega = \sum_{i=1}^n dq^i \wedge dp_i \tag{5.13}$$

This is the well known Darboux theorem (see [AbMa]). The local coordinates in which the symplectic form takes its canonical form (5.13) are known *canonical coordinates*. It means that in a canonical coordinate system, any Poisson bracket induced by a symplectic structure is locally given by the above expression (5.12).

In the results that follow we study which properties verifies the Poisson bracket when it is defined from a symplectic structure.

Proposition 5.8. *If $\{\ ,\ \}$ is the Poisson structure associated with the symplectic manifold (M, ω), it is non-degenerate, i.e, for $q \in M$ we have that*

$$\{f, g\}(q) = 0 \ \ \forall g \in \mathcal{F}(M) \ \Leftrightarrow \ df(q) = 0$$

Proof. $\Rightarrow\rfloor$ For any $v \in T_qM$, consider the 1-form $\alpha \in T_q^*M$ given by $\alpha = b_{\omega(q)}(v)$. Let $g \in \mathcal{F}(M)$ be such that $\alpha = dg(q)$. Then,

$$df(q)(v) = df(q)\left(b^{-1}_{\omega(q)}(\alpha)\right) = df(q)\left(b^{-1}_{\omega(q)}(dg(q))\right) = df(q)\left(X_g(q)\right) = X_g(f)(q) = \{f, g\}(q) = 0$$

$\Leftarrow\rfloor$ For any $g \in \mathcal{F}(M)$,

$$\{f, g\}(q) = X_g(f)(q) = df(q)(X_g(q)) = 0$$

□

Proposition 5.9. *If $(M, \{\ ,\ \})$ is a Poisson manifold and $\{\ ,\ \}$ is non-degenerate, M is a symplectic manifold.*

Proof. Consider the following map,

$$b_q^{-1}: \begin{array}{ccc} T_q^*M & \rightarrow & T_qM \\ \alpha & \rightarrow & X_f(q) \end{array} \ \text{if } \alpha = df(q)$$

Remark that if $df(q) = 0$, then $X_f(q) = 0$. Thus, if $\alpha = df(q) = dg(q)$, one has that $d(f-g)(q) = 0$ which implies $X_{f-g}(q) = \{\ , f-g\}(q) = 0$. Therefore, from de $\mathbb{R}$-bilinearity of the Poisson bracket,

$$X_f(q) = \{\ , f\}(q) = \{\ , g\}(q) = X_g(q)$$

and b_q^{-1} is well defined. Moreover, since the bracket is non-degenerate,

$$ker\ b_q^{-1} = \left\{\alpha \in T_q^*M \mid X_f(q) = 0\right\} = \left\{df(q) \in T_q^*M \mid X_f(q) = 0\right\} = \left\{df(q) \in T_q^*M \mid df(q) = 0\right\} = 0$$

and b_q^{-1} is an isomorphism.

Now, define a 2-form w on M as follows. For $q \in M$, put

$$b_{w(q)} = b_q : T_qM \rightarrow T_q^*M$$

It is obvious that ω is a 2-form, that is, ω is $\mathbb{R}$-bilinear and skew-symmetric. Using Proposition 5.6 we show that it is closed as follows,

$$\begin{aligned} d\omega(X_f, X_g, X_h) &= X_f(\omega(X_g, X_h)) + X_g(\omega(X_h, X_f)) - X_h(\omega(X_f, X_g)) \\ &- \omega([X_f, X_g], X_h) - \omega([X_g, X_h], X_f) - \omega([X_h, X_f], X_g) \\ &= X_f(\{g, h\}) + X_g(\{h, f\}) + X_h(\{f, g\}) \\ &- \omega(-X_{\{f,g\}}, X_h) - \omega(-X_{\{g,h\}}, X_f) - \omega(-X_{\{h,f\}}, X_g) \\ &= 2\left(\{\{g, h\}\, f\} + \{\{h, f\}\, g\} + \{\{f, g\}\, h\}\right) = 0 \end{aligned}$$

for any $X_f, X_g, X_h \in \mathcal{X}(M)$. Finally, we have to show that ω is non-degenerate. In order to do that, it will be enough to prove that $b_{\omega(q)} : T_qM \rightarrow T_q^*M$ is a linear isomorphism. But this condition holds since $b_{w(q)} = b_q$. Thus ω is a symplectic form and (M, ω) is a symplectic manifold. □

Notice that the Poisson bracket defined by the symplectic structure associated with the non-degenerate Poisson structure, coincides with the original Poisson bracket $\{\ ,\ \}$. Indeed, for any two vector fields $X_f, X_g \in \mathcal{X}(M)$,

$$\omega(X_f, X_g) = i_{X_f}\omega(X_g) = df(X_g) = X_g(f) = \{f, g\}$$

The two previous definitions leads us to the following corollary.

Corollary 5.10. *Let $(M, \{\ ,\ \})$ be a Poisson manifold. The Poisson bracket $\{\ ,\ \}$ is non-degenerate if, and only if, $(M, \{\ ,\ \})$ is a symplectic manifold.*

As we pointed out before, we have that symplectic manifolds are privileged Poisson manifolds.

5.1.4 Linear Poisson structures

Now, we are going to study a particular type of Poisson manifolds. They have associated a special Poisson structure that yields to many interesting properties. In addition, as we will see throughout the section, the dual space of a Lie algebra admits such particular structure that we will need further on.

Definition 5.11. *A* linear Poisson structure *is a Poisson structure on a real vector space V such that for any couple of linear functions f and g defined on V, $\{f,g\}$ also is a linear function.*

From the local expression of the Poisson bracket (5.4) we deduce that the condition of being a linear Poisson structure is equivalent to its local components are linear, i.e,

$$w_{ij} = \sum_k c_{ij}^k q^k \quad c_{ij}^k \in \mathbb{R} \;\; \forall i,j,k \tag{5.14}$$

Notice that if f,g are two linear functions on a vector space V, then $f,g \in V^*$. Thus, if $\{\ ,\ \}$ is a linear Poisson structure $\{f,g\} \in V^*$. It allows us to define a Lie algebra structure over V^* as follows,

$$\begin{array}{rccc} [\ ,\ ]: & V^* \times V^* & \to & V^* \\ & (f,g) & \to & \{f,g\} \end{array}$$

Conversely, we have the next result.

Proposition 5.12. *Any Lie algebra structure $(V^*, [\ ,\ ])$ defined on the dual space of a vector space V, induces a linear Poisson structure on V.*

Proof. If $f \in \mathcal{F}(V)$ and $\alpha \in V$, then $df(\alpha) \in T^*_\alpha V \cong V^*$. Thus, we may define the following bracket of real functions on V

$$\{f,g\}(\alpha) = \alpha([df(\alpha), dg(\alpha)]) \quad \text{for } f,g \in \mathcal{F}(V) \tag{5.15}$$

It is easy to prove that $\{\ ,\ \}$ is skew-symmetric and satisfies the Leibniz rule. Therefore, $\{\ ,\ \}$ induces a 2-vector w on V. In fact, if $\{e^1, \dots e^n\}$ is a basis of V^* such that $[e^i, e^j] = \sum_k c_k^{ij} e^k$, then

$$w = \frac{1}{2} \sum_{i,j,k} c_k^{i,j} \frac{\partial}{\partial v^i} \wedge \frac{\partial}{\partial v^j}$$

where $\{v^i\}$ are the coordinates on V which are induced by the basis $\{e^i\}$. Now, using that $[\ ,\ ]$ is a Lie bracket on V we deduce that $[w,w] = 0$, which proves that w defines a linear Poisson structure on V.

□

Finally, the next corollary resumes what we have proved.

Corollary 5.13. *There exists a natural bijection between the linear Poisson structures on a real vector space V of finite dimension and Lie algebra structures on the dual space V^*.*

In the next definition we specialize the definition of linear Poisson structures to the Lie algebras associated with Lie groups.

Definition 5.14. *In the case of V^* being the Lie algebra $\mathfrak{g}$ of a Lie group G, the linear Poisson structure given by (5.15) is known as the* Lie-Poisson structure *of $\mathfrak{g}^*$.*

5.2 Poisson morphisms

As one can deduce from the first part of this chapter, the Hamiltonian vector fields play an important role on the theory that we are working up. Right now, our goal is to study such vector fields in a deeper way. For this purpose, we need some background about morphisms between Poisson manifolds.

Definition 5.15. *Let $(M, \{\ ,\ \})$ and $(M', \{\ ,\ \}')$ be two Poisson manifolds and $\varphi : M \to M'$ be a smooth map. If the pull-back $\varphi^* : \mathcal{F}(M') \to \mathcal{F}(M)$ is a Lie algebra homomorphism, i.e.,*

$$\{\varphi^* f, \varphi^* g\} = \varphi^* \{f, g\}' \qquad \text{for } f, g \in \mathcal{F}(M)$$

φ is said to be a Poisson morphism.

Proposition 5.16. *Let $(M, \{\ ,\ \})$ and $(M', \{\ ,\ \}')$ be two Poisson manifolds and $\varphi : M \to M'$ be a smooth map. If $\#_w$ and $\#_{w'}$ denote the corresponding induced homomorphisms by $\{\ ,\ \}$ and $\{\ ,\ \}'$, the following statements are equivalent:*

i. *φ is a Poisson morphism.*

ii. *For every $q \in M$, $T_q\varphi \circ \#_w \circ T^*_{\varphi(q)}\varphi = \#_{w'}$*

iii. *For any $q \in M$ and $\alpha', \beta' \in T^*_{\varphi(q)}M'$, the following equality holds,*

$$w(q)(T^*_{\varphi(q)}\varphi(\alpha'), T^*_{\varphi(q)}\varphi(\beta')) = w'(\varphi(q))(\alpha', \beta')$$

Proof. First of all remark that for every $\alpha' \in T^*_{\varphi(q)}M'$ such that locally $\alpha' = df(\varphi(q))$ with $f \in \mathcal{F}(M')$ and for every $v \in T_qM'$

$$T^*_{\varphi(q)}\varphi(\alpha')(v) = T^*_{\varphi(q)}\varphi(df(\varphi(q)))(v) = df(\varphi(q))\,(T_q\varphi(v)) = T_q(f \circ \varphi)(v)$$

Since it holds for any vector $v \in T_qM$ we conclude that $T^*_{\varphi(q)}\varphi(\alpha') = T_q(f \circ \varphi)$. Now, assume that $\alpha', \beta' \in T^*_{\varphi(q)}M'$ are such that $\alpha' = df(\varphi(q))$ and $\beta' = dg(\varphi(q))$ with $f, g \in \mathcal{F}(M')$.

i. $\Rightarrow$ *iii.* $\rfloor$

$$\begin{aligned} w(q)(T^*_{\varphi(q)}\varphi(\alpha'), T^*_{\varphi(q)}\varphi(\beta')) &= w(q)(T_q(f \circ \varphi), T_q(g \circ \varphi)) = w(q)(d(f \circ \varphi)(q), d(g \circ \varphi)(q)) \\ &= \{f \circ \varphi, g \circ \varphi\}(q) = \{\varphi^* f, \varphi^* g\}(q) = \varphi^* \{f, g\}'(q) = \{f, g\}\,\varphi(q) \\ &= w'(\varphi(q))(df(q), dg(q)) = w'(\varphi(q))(\alpha', \beta') \end{aligned}$$

iii. $\Rightarrow$ *ii.* $\rfloor$ Let us show that the following diagram is commutative

$$\begin{array}{ccc} T^*_{\varphi(q)}M' & \xrightarrow{T^*_{\varphi(q)}\varphi} & T^*_qM \\ {\scriptstyle \#_{w'}}\downarrow & & \downarrow{\scriptstyle \#_w} \\ T_{\varphi(q)}M' & \xleftarrow{T_q\varphi} & T_qM \end{array}$$

To be precise, what we will show is that $\beta'(T_q\varphi \circ \#_w \circ T^*_{\varphi(q)}\varphi(\alpha')) = \beta'(\#_{w'}(\alpha'))$. Indeed,

$$\begin{aligned} \beta'(T_q\varphi \circ \#_w \circ T^*_{\varphi(q)}\varphi(\alpha')) &= dg(\varphi(q))(T_q\varphi \circ \#_w \circ T^*_{\varphi(q)}\varphi(\alpha')) \\ &= T_q(g \circ \varphi)(\#_w \circ T^*_{\varphi(q)}\varphi(\alpha')) = w(q)(T^*_{\varphi(q)}\varphi(\alpha'), T_q(g \circ \varphi)) \\ &= w(q)(T^*_{\varphi(q)}\varphi(\alpha'), T^*_{\varphi(q)}\varphi(\beta')) = w'(\varphi(q))(\alpha', \beta') = \beta'(\#_{w'}(\alpha')) \end{aligned}$$

Now, the proofs *iii.* $\Rightarrow$ *i.* and *ii.* $\Rightarrow$ *iii.* are almost direct and are left as an exercise.

$\square$

Remark 5.17. Given a smooth map between two smooth manifolds $F : M \to M'$ and a point $q \in M$, F induces a linear map

$$\Lambda^k T_q F : \quad \Lambda^k T_q M \quad \to \quad \Lambda^k T_{F(q)} M'$$

Given $w_q \in \Lambda^k T_q M$ and $\alpha'^1_{F(q)}, \dots, \alpha'^k_{F(q)} \in T^*_{F(q)} M'$, the image of w_q by $\Lambda^k T_q F$ is

$$(\Lambda^k T_q F(w_q))(\alpha'^1_{F(q)}, \dots, \alpha'^k_{F(q)}) = w_q((T^*_{F(q)} F)(\alpha'^1_{F(q)}), \dots, (T^*_{F(q)} F)(\alpha'^k_{F(q)})).$$

Particularly, if F is a diffeomorphism, given a k-vector w on M, we can define a k-vector $\Lambda^k TFw$ on M' as follows,

$$\Lambda^k TFw(q') = \Lambda^k T_{F^{-1}(q')} F(w(F^{-1}(q'))) \quad \forall q' \in M' \tag{5.16}$$

Furthermore, it is obvious that the map φ of the previous proposition is a Poisson morphism if, and only if,

$$\Lambda^2 T_q \varphi(w(q)) = w'(\varphi(q)) \tag{5.17}$$

Finally, we come to the definition of Poisson isomorphisms and give two examples.

Definition 5.18. *Let $(M, \{\ ,\ \})$ and $(M', \{\ ,\ \}')$ be two Poisson manifolds. A Poisson morphism $\varphi : M \to M'$ is a* Poisson isomorphism *if it is a diffeomorphism.*

Example 5.19. If $\varphi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism, $\varphi^* : \mathfrak{h}^* \to \mathfrak{g}^*$ is a Poisson map when $\mathfrak{g}^*$ and $\mathfrak{h}^*$ are endowed with the Lie-Poisson structure.

Example 5.20. Any symplectomorphism is a Poisson isomorphism. In particular, if $\varphi : M \to M$ is a diffeomorphism, the cotangent lift $T^*\varphi : T^*M \to T^*M$ is a Poisson isomorphism.

5.2.1 Infinitesimal automorphims and Hamiltonian vector fields

In order to study the Hamiltonian vector fields of a Poisson manifold, we will define a more general type of vector fields that we will call Poisson vector fields. We will see that in particular, the Hamiltonian vector fields are always Poisson.

Definition 5.21. *Let $(M, \{\ ,\ \})$ be a Poisson manifold and w be the corresponding Poisson 2-vector. A vector field $X \in \mathcal{X}(M)$ is said to be a* Poisson vector field *if it is an* infinitesimal automorphism *of the Poisson structure, that is,*

$$\mathcal{L}_X w = 0$$

Remark that for any two functions $f, g \in \mathcal{F}(M)$ and any point $q \in M$ one has

$$\begin{aligned} \mathcal{L}_X w(df, dg)(q) &= X(w(df, dg))(q) - w(\mathcal{L}_X df, dg)(q) - w(df, \mathcal{L}_X dg)(q) \\ &= X(w(df, dg))(q) - w(dX(f), dg)(q) - w(df, dX(g))(q) \\ &= X\{f, g\}(q) - \{X(f), g\}(q) - \{f, X(g)\}(q) \end{aligned}$$

Thus, X is a Poisson vector field if, and only if,

$$X\{f, g\} = \{X(f), g\} + \{f, X(g)\} \tag{5.18}$$

Moreover we have the next result.

Proposition 5.22. *Let $(M, \{\ ,\ \})$ be a Poisson manifold and $X \in \mathcal{X}(M)$ be a vector field. The following statements are equivalent:*

i. X is a Poisson vector field.

ii. The flow φ_t of X consits on Poisson isomorphisms.

Proof. *i.* $\Rightarrow$ *ii.* $\rfloor$ Notice that for any $\alpha, \beta \in \Omega^1(M)$,

$$\begin{aligned}(\mathcal{L}_X(\Lambda^2 T\varphi_{t_0} w))(\alpha,\beta) &= X(\Lambda^2 T\varphi_{t_0} w(\alpha,\beta)) - (\Lambda^2 T\varphi_{t_0} w)(\mathcal{L}_X\alpha,\beta) - (\Lambda^2 T\varphi_{t_0} w)(\alpha,\mathcal{L}_X\beta)\\ &= X(w((T\varphi_{t_0})^*(\alpha),(T\varphi_{t_0})^*(\beta))) - w((T\varphi_{t_0})^*(\mathcal{L}_X\alpha),(T\varphi_{t_0})^*(\beta))\\ &- w((T\varphi_{t_0})^*(\alpha),(T\varphi_{t_0})^*(\mathcal{L}_X\beta))\end{aligned}$$

Thus, using that $(T\varphi_{t_0})^*(\mathcal{L}_X\gamma) = \mathcal{L}_X((T\varphi_{t_0})^*(\gamma))$ for any $\gamma \in \Omega^1(M)$, it follows that

$$\begin{aligned}(\mathcal{L}_X(\Lambda^2 T\varphi_{t_0} w))(\alpha,\beta) &= \mathcal{L}_X w((T\varphi_{t_0})^*(\alpha),(T\varphi_{t_0})^*(\beta))\\ &= (\Lambda^2 T\varphi_{t_0}(\mathcal{L}_X w))(\alpha,\beta)\end{aligned}$$

Therefore, $0 = \Lambda^2 T\varphi_{t_0}(\mathcal{L}_X w) = \mathcal{L}_X \Lambda^2 T\varphi_{t_0} w$ and if $q \in M$,

$$\begin{aligned}0 = \mathcal{L}_X \Lambda^2 T\varphi_{t_0} w(q) &= \frac{d}{dt}_{|t=0} \left(\Lambda^2 T_{\varphi_t(q)}\varphi_{-t}\right)(\Lambda^2 T_{\varphi_{t-t_0}(q)}\varphi_{t_0}(w(\varphi_{t-t_0}(q))))\\ &= \frac{d}{dt}_{|t=0} \left(\Lambda^2 T_{\varphi_{t-t_0}(q)}\varphi_{t_0-t}\right)(w(\varphi_{t-t_0}(q)))\\ &= \frac{d}{dt}_{|t=t_0} \left(\Lambda^2 T_{\varphi_t(q)}\varphi_{-t}\right)(w(\varphi_t(q)))\end{aligned}$$

Then,

$$\left(\Lambda^2 T_{\varphi_t(q)}\varphi_{-t}\right)(w(\varphi_t(q))) = \left(\Lambda^2 T_{\varphi_0(q)}\varphi_0\right)(w(\varphi_0(q))) = w(q) \quad \forall t,\ \forall q \in M$$

which means that

$$\Lambda^2 T_q\varphi_t(w(q)) = w(\varphi(q))$$

ii. $\Rightarrow$ *i.* $\rfloor$ Using (5.17) we have

$$\Lambda^2 T_{\varphi_t(q)}\varphi_{-t}(w(\varphi_t(q))) = w(q)$$

Hence,

$$\mathcal{L}_X w(q) = \frac{d}{dt}_{|t=0} T_{\varphi_t(q)}\varphi_{-t}(w(\varphi_t(q))) = \frac{d}{dt}_{|t=0} w(q) = 0$$

□

Now, let us focus on the Hamiltonian vector fields. If X_f is a Hamiltonian vector field, the equality (5.18) is just the Jacobi identity,

$$X_f\{h,g\} = \{\{h,g\},f\} = -\{\{f,h\},g\} - \{\{g,f\},h\} = \{X_f(h),g\} + \{h,X_f(g)\}$$

Thus, any Hamiltonian vector field is a Poisson vector field. In particular, the flow of a Hamiltonian vector field is made up of Poisson isomorphisms. Furthermore, we have the following result.

Proposition 5.23. *Let $(M,\{\ ,\ \})$ and $(M',\{\ ,\ \}')$ be two Poisson manifolds and $\varphi : M \to M'$ be a smooth map. The following conditions are equivalent:*

i. *φ is a Poisson morphism.*

ii. *For any $f' \in \mathcal{F}(M')$ and any $q \in M$, $T_q\varphi(X_{f'\circ\varphi}(q)) = X'_{f'}(\varphi(q))$*

Proof. We will prove that for any two functions $f', g' \in \mathcal{F}(M')$ one has the following equivalence

$$\{f'\circ\varphi, g'\circ\varphi\} = \{f',g'\}'\circ\varphi \quad\Leftrightarrow\quad T_q\varphi(X_{f'\circ\varphi}(q)) = X'_{f'}(\varphi(q)) \text{ for all } q \in M$$

Since the first statement implies that φ is a Poisson morphism, by Proposition 5.16, the result will be proved. Here we write down the right implication, and rephrasing it, we obtain the left one,

$$T_q\varphi(X_{f'\circ\varphi}(q))(g') = X_{f'\circ\varphi}(g'\circ\varphi)(q) = \{g'\circ\varphi, f'\circ\varphi\}(q) = \{g',f'\}'\circ\varphi(q) = X'_{f'}(\varphi(q))$$

□

5.3 Symplectic foliation of a Poisson manifold

This last section is devoted to prove that any Poisson manifold admits a completely integrable distribution whose leaves are symplectic manifolds. As well, we will describe the symplectic leaves in the case of the Poisson manifold being the dual space of the Lie algebra of a connected Lie group. Here, the Hamiltonian vector fields and their properties are crucial in order to define the distribution.

Definition 5.24. *Let $(M, \{\ ,\ \})$ be a Poisson manifold, w be the corresponding Poisson structure and $q \in M$ be a point. The image of $\#_{w(q)} = C_q$ is called the* characteristic space *at point q.*

Remark that the dimension of C_q coincides with the rank of $\#_w$ at q. Thus, the dimension of the characteristic space is always even. If rank $\#_{w(q)} = dim\ M$ we say that $\#$ is *non-degenerate* at the point q. As well, if the rank of $\#_{w(q)}$ does not depend on the point we say that w is a *regular Poisson structure.*

Example 5.25. Earlier, we have proved that a Poisson manifold M is symplectic, if and only if, the Poisson structure is non-degenerate. Hence, the symplectic manifolds have regular Poisson structures associated with them.

Another way of describing the characteristic space is in terms of the Hamiltonian vector fields,

$$C_q = \{X_f(q) \mid f \in \mathcal{F}(M)\} \tag{5.19}$$

It is immediate that the two definitions are equivalent.

The characteristic space C_q of a Poisson manifold M, induces a generalized distribution. Indeed,

$$q \in M \longrightarrow C_q \subseteq T_qM \tag{5.20}$$

is a generalized distribution since it is generated by the Hamiltonian vector fields. We call it the *generalized distribution* of the Poisson manifold and we denote it by $\mathcal{C}$. The next result proves that $\mathcal{C}$ is a generalized foliation on M and that the leaf of $\mathcal{C}$ through each point of M admits a symplectic structure. In order to prove this theorem, we will use some results on generalized distributions (for more information on these topics see Appendix B).

Theorem 5.26. *The characteristic distribution $\mathcal{C}$ of a Poisson manifold M is completely integrable and the Poisson structure induces symplectic structures on its leaves.*

Proof. First, we prove that $\mathcal{C}$ is invariant and by the generalized Frobenius theorem we will deduce that it is completely integrable. Since $C_q = \langle X_f(q) \mid f \in \mathcal{F}(M)\rangle$ it is enough to check that for any $f, g \in \mathcal{F}(M)$,

$$T_q\varphi_t^{X_f}(X_g(q)) \in \mathcal{C}(\varphi_t^{X_f}(q))$$

where $\varphi_t^{X_f}$ denotes the flow of the Hamiltonian vector field X_f. By Proposition 5.22 we know that $\varphi_t^{X_f}$ are Poisson isomorphisms and by Proposition 5.23 we have that

$$T_q\varphi_t^{X_f}(X_g(q)) = X_{g\circ(\varphi_t^{X_f})^{-1}}(\varphi_t^{X_f}(q)) \in \mathcal{C}(\varphi_t^{X_f}(q))$$

Thus, the characteristic distribution is completely integrable which implies that there exist a leaf through each point of M. Let us prove that such leaves have a symplectic structure. Let L be a leaf, we prove that $\{\ ,\ \}$ induces a Poisson structure on L as follows,

$$\begin{array}{rccc} \{\ ,\ \}_L: & \mathcal{F}(L)\times\mathcal{F}(L) & \to & \mathcal{F}(L) \\ & (f,g) & \to & \{f,g\}_L = \left\{\tilde{f},\tilde{g}\right\} \end{array}$$

where $\tilde{f}, \tilde{g} \in \mathcal{F}(M)$ are such that $\tilde{f}_{|L} = f$ and $\tilde{g}_{|L} = g$. If $q \in L$ we have,

$$\{g, f\}_L(q) = X_{\tilde{f}}(\tilde{g})(q) = (T_0\varphi_q^{X_{\tilde{f}}})(\tilde{g})$$

where $\varphi_q^{X_{\tilde{f}}}$ is the integral curve of the vector field $X_{\tilde{f}}$ with initial condition $q \in L$. In particular, if the initial condition is in L, the whole integral curve remains on L and $\{\ ,\ \}_L$ only depends on g. Likewise, permuting f and g we obtain that $\{\ ,\ \}_L$ only depends on f. Therefore, $\{\ ,\ \}_L$ is well defined. Moreover, easy calculations proves that $\{\ ,\ \}_L$ is $\mathbb{R}$-bilinear, skew-symmetric and satisfies the Jacobi identity and the Leibniz rule. Thus, $\{\ ,\ \}_L$ is a Poisson bracket on the leaf L. Finally, we will prove that such Poisson structure is non-degenerate and by Corollary 5.10 we will conclude that L is symplectic. For this purpose, first we prove that $\#_{w_L(q)} : T_q^*L \to T_qL$ is an isomorphism (here, w_L is the Poisson 2-vector on L). In fact,

$$T_qL = \left\{ X_{\tilde{f}}(q) \,|\tilde{f} \in \mathcal{F}(M) \right\} = \{X_f(q) \,|f \in \mathcal{F}(L)\} = \#_{w_L(q)}(T_q^*L)$$

proves that $\#_{w_L(q)}$ is exhaustive. Since $dim\ T_q^*L = dim\ T_qL$, $\#_{w_L(q)}$ is an isomorphism. Hence, $\{\ ,\ \}_L$ is non-degenerate which proves that L is symplectic.

□

Definition 5.27. *The leaves of the characteristic foliation are known as the* symplectic leaves *of the Poisson manifold M.*

Up to now, we have proved that the leaves of the foliation of a Poisson manifold are symplectic but we do not specify the symplectic form. Using that $\#_{w_L(q)}$ is bijective, for any two vectors $X, Y \in T_qL$ we have $X = \#_{w_L(q)}(\alpha)$ and $Y = \#_{w_L(q)}(\beta)$ for some $\alpha, \beta \in T_q^*L$. Then, we define the symplectic form ω in L as

$$\omega(X, Y) = \beta(X) = \beta(\#_{w_L(q)}(\alpha)) = w_L(q)(\alpha, \beta) \tag{5.21}$$

Example 5.28. Consider the 2-vector w in $\mathbb{R}^3$ given by

$$w = x\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + y\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} + z\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \tag{5.22}$$

Easily, using the Schouten-Niejenhuis bracket we have that w is a Poisson structure and $(\mathbb{R}^3, w)$ is a Poisson manifold. Remark that the Hamiltonian vector fields are given by

$$X_f = \frac{\partial f}{\partial x}\left(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right) + \frac{\partial f}{\partial y}\left(x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}\right) + \frac{\partial f}{\partial z}\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)$$

Therefore, the characteristic space is

$$C_{(x,y,z)} = \left\langle z\frac{\partial}{\partial y}_{|(x,y,z)} - y\frac{\partial}{\partial z}_{|(x,y,z)}, x\frac{\partial}{\partial z}_{|(x,y,z)} - z\frac{\partial}{\partial x}_{|(x,y,z)}, y\frac{\partial}{\partial x}_{|(x,y,z)} - x\frac{\partial}{\partial y}_{|(x,y,z)} \right\rangle \tag{5.23}$$

To simplify we note $C_{(x,y,z)} = \langle X_x(x,y,z), X_y(x,y,z), X_z(x,y,z)\rangle$, where X_x, X_y and X_z are the Hamiltonian vector fields of x, y and z respectively. It can be shown that the symplectic leaves of that foliation are

$$L_{|(x_0,y_0,z_0)} = \begin{cases} S^2_{r_0} & \text{if } (x_0,y_0,z_0) \neq (0,0,0) \\ (0,0,0) & \text{if } (x_0,y_0,z_0) = (0,0,0) \end{cases}$$

Here, $S^2_{r_0}$ denotes the sphere in $\mathbb{R}^3$ with center the origin and radius $\sqrt{r_0}$, where $r_0 = x_0^2 + y_0^2 + z_0^2$. A further explanation about the computation of the leaves is found in Appendix B.

Since $(\mathbb{R}^3, w)$ is a Poisson manifold, the leaves have symplectic structure. Let us find the induced symplectic form. If $u, v \in T_{(x,y,z)}S^2_{r_0}$ are two vectors, they are expressed locally as $u = u_1X_x + u_2X_y + u_3X_z$ and $v = v_1X_x + v_2X_y + v_3X_z$. We know that $\#_w : T^*\mathbb{R}^3 \to T\mathbb{R}^3$ sends dx (dy, dz respectively) to X_x (X_y, X_z respectively). Thus, if $\alpha = -u_1dx - u_2dy - u_3dz$ and $\beta = -v_1dx - v_2dy - v_3dz$, we have that $\#_{wS^2_{r_0}}(\alpha) = u =$

$u_1X_x(x,y,z)+u_2X_y(x,y,z)+u_3X_z(x,y,z)$ and $\#_{wS^2_{r_0}}(\beta)=v=v_1X_x(x,y,z)+v_2X_y(x,y,z)+v_3X_z(x,y,z)$. Then, the symplectic form ω is given by

$$\begin{aligned}\omega(u,v)(x,y,z)=\beta(u)(x,y,z) &= -v_1u_1X_x(x)-v_1u_2X_y(x)-v_1u_3X_z(x)\\ &\quad -v_2u_1X_x(y)-v_2u_2X_y(y)-v_2u_3X_z(y)\\ &\quad -v_3u_1X_x(z)-v_3u_2X_y(z)-v_3u_3X_z(z)\\ &= v_1u_2z-v_1u_3y-v_2u_1z+v_2u_3x+v_3u_1y-v_3u_2x\\ &= -x(v_3u_2-v_2u_3)-y(v_1u_3-v_3u_1)-z(v_2u_1-v_1u_2)\end{aligned}$$

Note that using the usual identification between $T_{(x,y,z)}\mathbb{R}^3\cong\mathbb{R}^3$, the symplectic form on $S^2_{r_0}$ is given by,

$$\omega(u,v)(x,y,z)=-(x,y,z)\cdot(u\times v)$$

5.3.1 Symplectic foliation of linear Poisson structures

This section is devoted to prove that the orbits of the coadjoint action of a connected Lie group G are the leaves of the symplectic foliation of the Lie-Poisson structure of $\mathfrak{g}^*$. Remember that the infinitesimal generator of the coadjoint action $Ad^*: G\times\mathfrak{g}^*\to\mathfrak{g}^*$ of a Lie group G associated to $\xi\in\mathfrak{g}$, is given by $\xi_{\mathfrak{g}^*}(\alpha)(\eta)=-\alpha[\xi,\eta]$ for $\alpha\in\mathfrak{g}^*$ and $\eta\in\mathfrak{g}$. Given two linear functions $\eta,\xi:\mathfrak{g}^*\to\mathbb{R}$, by duality, they can be considered as elements of $\mathfrak{g}$. Then, recalling section 5.1.4 we have that the Lie-Poisson structure on $\mathfrak{g}^*$ is given by $\{\xi,\eta\}(\alpha)=\alpha([\xi,\eta])$ for $\alpha\in\mathfrak{g}^*$. Hence, if X_ξ denotes the Hamiltonian vector field with respect to the Lie-Poisson structure of $\mathfrak{g}^*$, we have

$$X_\xi(\eta)(\alpha)=\{\eta,\xi\}(\alpha)=-\alpha([\xi,\eta])=\xi_{\mathfrak{g}^*}(\eta)(\alpha)$$

Therefore, the Hamiltonian vector field coincides with the infinitesimal generator of the coadjoint action associated to ξ, i.e,

$$X_\xi=\xi_{\mathfrak{g}^*} \tag{5.24}$$

It leads us to the following result.

Proposition 5.29. *The coadjoint action Ad^* of a connected Lie group G is a Poisson morphism on $\mathfrak{g}^*$.*

Proof. Given $\xi\in\mathfrak{g}$, by the previous equality 5.24, the flow $\varphi_t^{X_\xi}$ of the Hamiltonian vector field X_ξ coincides, with the flow $Ad_{\gamma_\xi(t)}$ of $\xi_{\mathfrak{g}^*}$. In addition, we know X_ξ is a Poisson vector field and by Proposition 5.22 we know that their flow is made up of Poisson isomorphims.

□

In view of this result, it is natural to try describing the characteristic space in terms of the infinitesimal generators. In fact, as $C_\alpha=\{X_\xi(\alpha)\mid\xi\in\mathcal{F}(\mathfrak{g}^*)\}$ it will be enough to prove that,

$$C_\alpha=\{X_\xi(\alpha)\mid\xi:\mathfrak{g}^*\to\mathbb{R}\text{ is linear}\} \tag{5.25}$$

Specifically, we will see that given a function $\xi\in\mathcal{F}(\mathfrak{g}^*)$, the local expression of its Hamiltonian vector field is $X_\xi=\sum_{i,j}\frac{\partial\xi}{\partial q^i}X_{q^j}$ where X_{q^i} are the Hamiltonian vector fields of the coordinate functions q^i. Indeed, if w_{ij} denote the components of the Lie-Poisson structure,

$$X_\xi=-\sum_{i,j}w_{ij}\frac{\partial\xi}{\partial q^i}\frac{\partial}{\partial q^j}=-\sum_{i,j}\frac{\partial\xi}{\partial q^i}\left(w_{ij}\frac{\partial}{\partial q^j}\right)=\sum_{i,j}\frac{\partial\xi}{\partial q^i}X_{q^j}$$

Now, if q^i are the coordinates on $\mathfrak{g}$ induced by the basis $\{\xi_i\}$, it follows that q^i is a linear function on $\mathfrak{g}^*$. Thus, $X_{q^i}=(\xi_i)_{g^*}$. Therefore, the characteristic space could be rewritten as,

$$C_\alpha=\{\xi_{\mathfrak{g}^*}(\alpha)\mid\xi\in\mathfrak{g}\} \tag{5.26}$$

It gives us the main tool to prove the result we referred at the beginning of this section.

Theorem 5.30. *The symplectic leaves of the Lie-Poisson structure on $\mathfrak{g}^*$ are the orbits of the coadjoint action Ad^* of a connected Lie group G whose Lie algebra is $\mathfrak{g}$.*

Proof. Denote by $G \cdot \eta = \left\{ Ad^*_g(\eta) \mid g \in G \right\}$ the orbit of the coadjoint action. Recall from Chapter 3 that its tangent space is given by,

$$T_\alpha(G \cdot \eta) = \{\xi_{\mathfrak{g}^*}(\alpha) \mid \xi \in \mathfrak{g}\}$$

Therefore, $T_\alpha(G \cdot \eta) = C_\alpha$, i.e, the tangent space to the orbit at α coincides with the characteristic space at α. Then, since the orbit is connected, we conclude that it is the corresponding leaf of $\mathcal{C}$ through α.

□

Again, we described the symplectic foliation but we did not say anything about its symplectic structure. If ω_η denotes such a symplectic structure in the leaf $G \cdot \eta$, then

$$\omega_\eta(\alpha)(\xi_{\mathfrak{g}^*}(\alpha), \zeta_{\mathfrak{g}^*}(\alpha)) = \omega_\eta(\alpha)(X_\xi(\alpha), X_\zeta(\alpha)) = -d\zeta(\alpha)(\xi_{\mathfrak{g}^*}(\alpha)) = -\xi_{\mathfrak{g}^*}(\alpha)(\zeta)$$

Example 5.31. Consider the coadjoint action of $SO(3)$ on $\mathbb{R}^3$. Given $\hat{\xi} \in \mathfrak{g}$, we known that the infinitesimal generator of the adjoint action under the identification $\mathfrak{so}(3) \cong \mathbb{R}^3$ is $\hat{\xi}_{\mathbb{R}^3}$. Thus, if $\hat{\eta} \in \mathfrak{g}$ and $\breve{\alpha} \in \mathfrak{g}^*$, the symplectic form of the leaves of the Lie-Poisson structure of $\mathfrak{so}^*(3)$ under the identifications $\mathfrak{so}(3) \cong \mathbb{R}^3$ via the hat map and $\mathfrak{so}^*(3) \cong \mathbb{R}^3$ via the breve map, is given by

$$\omega(\breve{\alpha})(\hat{\xi}_{\mathbb{R}^{3*}}(\breve{\alpha}), \hat{\eta}_{\mathbb{R}^{3*}}(\breve{\alpha})) = -\hat{\xi}_{\mathbb{R}^{3*}}(\breve{\alpha})(\hat{\eta}) = \breve{\alpha}([\hat{\xi}, \hat{\eta}]) = \breve{\alpha}(\widehat{\xi \times \eta}) = \alpha \cdot (\xi \times \eta)$$

We have that the symplectic leaves of $\mathfrak{so}^*(3)$ are the origin and the spheres centered on the origin. Indeed, we proved that the coadjoint action of $SO(3)$ under the previous identifications is the usual action of $SO(3)$ on $\mathbb{R}^3$. Moreover, we saw that the orbits of that action are the origin and the spheres centered in the origin. The previous theorem 5.30 gives us the result.

Furthermore, notice that the symplectic form is, up to sign, the symplectic form of the Example 5.28. In fact, the Lie-Poisson structure on the dual to the Lie algebra of a Lie group G can be also defined by $\{f, g\}(\alpha) = -\alpha([df(\alpha), dg(\alpha)])$ for $f, g \in \mathcal{F}(\mathfrak{g}^*)$ (see Remark 6.2). In such case, the symplectic form of $\mathfrak{so}^*(3)$ coincides with the symplectic form of the Example 5.28 which means that the Lie-Poisson structure of $\mathfrak{so}^*(3)$ is given by (5.22) and its characteristic distribution is (5.23).

Chapter 6

Lie-Poisson reduction theorem

At this point in the project, we notice that among all objects we have studied there are two spaces that stand out for their interesting properties. On the one hand, the cotangent bundle of any smooth manifold has a symplectic structure and hence, is a Poisson manifold. On the other hand, we know that the dual space of the Lie algebra of a Lie group always has associated a linear Poisson structure that we called the Lie Poisson structure. However, at no point during the project we have linked neither the spaces nor their structures. In fact, the Lie-Poisson reduction theorem is the culmination of the project and gives us such a result. Given a Lie group G, the theorem will relate its cotangent bundle T^*G with $\mathfrak{g}^*$ via a Poisson morphism. As a result of this, the dynamics on T^*G may be projected on $\mathfrak{g}^*$ and it is described with the half number of equations.

As one can expect, to prove this theorem we need most of the tools we have introduced. That is why we give a quick overview to the whole project emphasizing the main results we will use. Once it is done, the proof of the theorem is simple. Finally, we end by coming back one more time to the rigid body. Using the examples we saw and the theorem, we will give its geometric description and we will see that it coincides with the physical one. Then, we may conclude that our main objective is reached, and so on, the project is completed.

6.1 Lie-Poisson reduction theorem

Let us summarize the most important results that we need:
Assume that M is a smooth manifold, G is a Lie group, $\mathfrak{g}$ its Lie algebra and H is a Hamiltonian function. Let $(q^1, \dots, q^m)$ be the local coordinates on M and $(q^1, \dots, q^m, p_1, \dots p_m)$ be the corresponding coordinates on T^*M.

1. From Chapter 2, the cotangent bundle T^*M is endowed with the canonical symplectic structure $\omega_M = \sum_{i=1}^n dq^i \wedge dp_i$, so that, it is a symplectic manifold. Besides, we showed that the dynamical equations for the Hamiltonian function $H : T^*M \to \mathbb{R}$ are just the Hamilton equations,
$$\dot{q}^i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i} \qquad \text{for } i = 1, \dots, m$$
2. In Chapter 3, we proved that if $\phi : G \times M \to M$ and $\varphi : G \times N \to N$ are two actions, and $f : M \to N$ is an equivariant function with respect to them, then
$$T_q f(\xi_M(q)) = \xi_N(f(q)) \quad \text{for } \xi \in \mathfrak{g} \text{ and } q \in M$$
where ξ_M and ξ_N are the corresponding infinitesimal generators. As well, we showed that the infinitesimal generator of the action of a Lie group on itself by left translations
$$\begin{array}{rccc} L: & G \times G & \to & G \\ & (g,h) & \to & L_g(h) = g \cdot h \end{array}$$

associated with $\xi \in \mathfrak{g}$ is the right invariant vector field, i.e, $\xi_G(g) = T_e R_g(\xi)$, for all $g \in G$.

We also introduced the adjoint action of a Lie group on its Lie algebra and we gave the expression of the infinitesimal generator associated with $\xi \in \mathfrak{g}$, that is, $\xi_{\mathfrak{g}}(\eta) = [\xi, \eta]$ for $\xi, \eta \in \mathfrak{g}$. Then, we introduced the coadjoint action and described its infinitesimal generators which are given by $\xi_{\mathfrak{g}^*}(\eta)(\alpha) = -\alpha([\xi, \eta])$ for $\xi, \eta \in \mathfrak{g}$ and $\alpha \in \mathfrak{g}^*$.

3. Chapter 4 was suitable to show that given an action $\phi : G \times M \to M$, its cotangent lift is always a symplectic action that admits an Ad^*-equivariant momentum map $J : T^*M \to \mathfrak{g}^*$ given by,

$$J(\alpha_q)(\xi) = J_\xi(\alpha_q) = \alpha_q(\xi_M(q)) \quad \text{for } \alpha_q \in T_q^*M \text{ and } \xi \in \mathfrak{g}$$

Thus, the Hamiltoninan vector X_{J_ξ} is just the infinitesimal generator ξ_{T^*M}.

4. In Chapter 5, we saw that a Poisson structure w on M induces an homomorphism $\#_w : T^*M \to TM$ that is given by $\beta_q(\#_w(\alpha_q)) = w(\alpha_q, \beta_q)$ for $\alpha_q, \beta_q \in T_q^*M$. It allowed us to characterize the Poisson morphisms. Indeed, $\varphi : (M, \{\ ,\ \}) \to (M', \{\ ,\ \}')$ is a Poisson morphism, if, for every $q \in M$, $T_q\varphi \circ \#_w \circ T^*_{\varphi(q)}\varphi = \#_{w'}$.

In addition, we deduced that a symplectic manifold is always a Poisson manifold. In particular, we proved that the cotangent bundle of any smooth manifold admits a canonical Poisson bracket which is locally given by

$$\{f, g\} = X_g(f) = \sum_{i=1}^{n} \left(\frac{\partial g}{\partial p_i} \frac{\partial f}{\partial q^i} - \frac{\partial g}{\partial q^i} \frac{\partial f}{\partial p_i} \right)$$

On the other hand, we proved that $\mathfrak{g}^*$ admit a linear Poisson structure (the Lie-Poisson structure) which is defined by

$$\{f, g\}(\alpha) = \alpha([df(\alpha), dg(\alpha)])$$

for $f, g \in \mathcal{F}(\mathfrak{g}^*)$ and $\alpha \in \mathfrak{g}^*$. In addition, the Hamiltonian vector field $X_\xi^{\mathfrak{g}^*}$ of the linear function $\xi : \mathfrak{g}^* \to \mathbb{R}$ is just $\xi_{\mathfrak{g}^*}$.

Now, let us apply all those results to prove the theorem.

Theorem 6.1 (Lie-Poisson reduction). *Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. For any $\xi \in \mathfrak{g}$, denote by X_ξ the corresponding right-invariant vector field. Let $J : T^*G \to \mathfrak{g}^*$ be the map given by*

$$J(\alpha_g)(\xi) = \alpha_g(X_\xi(g)) \qquad \text{for } \alpha_g \in T_g^*G \text{ and } \xi \in \mathfrak{g} \tag{6.1}$$

then,

i. J is an exhaustive submersion.

ii. If on T^*G (respectively $\mathfrak{g}^*$) we consider the Poisson structure induced by the canonical symplectic structure (respectively, the Lie-Poisson structure), then J is a Poisson morphism.

iii. If $h : \mathfrak{g}^* \to \mathbb{R}$ and $H = h \circ J : T^*G \to \mathbb{R}$ are two Hamiltonian functions and $\gamma : I \to T^*G$ is a solution of the Hamilton equations for H, then $J \circ \gamma : I \to \mathfrak{g}^*$ is a solution of the Lie-Poisson equations for h.

Proof. i. It is sufficient to prove that $J_{|T_g^*G} : T_g^*G \to \mathfrak{g}^*$ is a linear isomorphism. It is clear that $J_{|T_g^*G} : T_g^*G \to \mathfrak{g}^*$ is a linear map. Furthermore, we have

$$\ker J_{|T_g^*G} = \left\{\alpha_g \in T_g^*G \mid J(\alpha_g)(\xi) = 0\ \forall \xi \in \mathfrak{g}\right\} = \left\{\alpha_g \in T_g^*G \mid \alpha_g(X_\xi(g)) = 0\ \forall \xi \in \mathfrak{g}\right\} = \left\{\alpha_g \in T_g^*G\ | \alpha_g = 0\right\} = 0$$

Moreover, $\dim T_g^*G = \dim \mathfrak{g} = \dim \mathfrak{g}^*$ so that $J_{|T_g^*G}$ is a linear isomorphism.

ii. Consider the action of G on itself by left translations

$$\begin{array}{rccc} L: & G\times G & \to & G \\ & (g,h) & \to & L_g(h)=g\cdot h \end{array}$$

Given $\xi\in\mathfrak{g}$, the infinitesimal generator of L associated with ξ is,

$$\xi_G(g)=T_eR_g(\xi)=X_\xi(g).$$

Thus, $\xi_G=X_\xi$. By Corollary 4.13 we know that $L^{T^*}:G\times T^*G\to T^*G$ is a symplectic action and admits an Ad^*-equivariant momentum map $\tilde{J}:T^*G\to\mathfrak{g}^*$ given by

$$\tilde{J}(\alpha_g)(\xi)=\alpha_g(\xi_G(g))=\alpha_q(X_\xi(g)) \text{ for } \alpha_g\in T^*_gG \text{ and } \xi\in\mathfrak{g}$$

Hence, $\tilde{J}=J$ and J is Ad^*-equivariant. Note that any $\xi\in\mathfrak{g}=(\mathfrak{g}^*)^*$ can be interpreted as a linear function $\xi:\mathfrak{g}^*\to\mathbb{R}$. Therefore, if $X^{T^*G}_{\xi\circ J}$ denotes the Hamiltonian vector field associated to $\xi\circ J$ with respect to the Poisson stucture of T^*G and $X^{\mathfrak{g}^*}_\xi$ denotes the Hamiltonian vector field associated to ξ with respect to the Lie-Poisson structure of $\mathfrak{g}^*$, we have

$$\begin{aligned} T_{\alpha_g}J(X^{T^*G}_{\xi\circ J}(\alpha_g)) &= T_{\alpha_g}J(X^{T^*G}_{J_\xi}(\alpha_g))=T_{\alpha_g}J(\xi_{T^*G}(\alpha_g)) \\ &= \xi_{\mathfrak{g}^*}(J(\alpha_g))=X^{\mathfrak{g}^*}_\xi(J(\alpha_g)) \end{aligned}$$

Let us use this fact to prove that for any $\alpha_g\in T^*_gG$, $T_{\alpha_g}J\circ\#_{T^*G}\circ T^*_{J(\alpha_g)}J=\#_{\mathfrak{g}^*}$, i.e, the following diagram is commutative,

$$\begin{array}{ccc} T^*_{J(\alpha_g)}\mathfrak{g}^* & \xrightarrow{T^*_{J(\alpha_g)}J} & T^*_{\alpha_g}(T^*G) \\ \#_{\mathfrak{g}^*}\downarrow & & \downarrow\#_{T^*G} \\ T_{J(\alpha_g)}\mathfrak{g}^* & \xleftarrow{T_{\alpha_g}J} & T_{\alpha_g}(T^*G) \end{array}$$

Let $\xi_1,\ldots,\xi_n$ be a basis of $\mathfrak{g}$. As before, $\xi_1,\ldots,\xi_n$ may be considered by duality as linear functions on $\mathfrak{g}^*$. Therefore, for any $\beta\in\mathfrak{g}^*$, $\{d\xi_1(\beta),\ldots,d\xi_n(\beta)\}$ is a basis of $T^*_\beta\mathfrak{g}^*$. Then, from the linearity of the involved functions it is sufficient to prove that the result holds for the elements of the basis, $d\xi_1(\beta),\ldots,d\xi_n(\beta)$. On the one hand,

$$\#_{\mathfrak{g}^*}(d\xi_i(J(\alpha_g)))=-X^{\mathfrak{g}^*}_{\xi_i}(J(\alpha_g))$$

On the other hand, for any $v\in T_{\alpha_g}(T^*G)$

$$T^*_{J(\alpha_g)}J(d\xi_i(J(\alpha_g))(v)=d\xi_i(J(\alpha_g)(T_{\alpha_g}J(v))=T_{\alpha_g}(\xi_i\circ J)(v)$$

So, $T^*_{J(\alpha_g)}(d\xi_i(J(\alpha_g))=T_{\alpha_g}(\xi_i\circ J)$. Then,

$$T_{\alpha_g}J\circ\#_{T^*G}\circ T^*_{J(\alpha_g)}J(d\xi_i(J(\alpha_g))=(T_{\alpha_g}J\circ\#_{T^*G})(T_{\alpha_g}(\xi_i\circ J))=(T_{\alpha_g}J\circ\#_{T^*G})(d(\xi_i\circ J)(\alpha_g))=-T_{\alpha_g}J(X^{T^*G}_{\xi_i\circ J}(\alpha_g))$$

Finally, as $\xi_1,\ldots,\xi_n$ are linear functions we conclude that

$$T_{\alpha_g}J\circ\#_{T^*G}\circ T^*_{J(\alpha_g)}J(d\xi_i(J(\alpha_g)))=-T_{\alpha_g}J(X^{T^*G}_{\xi_i\circ J}(\alpha_g))=-X^{\mathfrak{g}^*}_{\xi_i}(J(\alpha_g))=\#_{\mathfrak{g}^*}(d\xi_i(J(\alpha_g)))$$

iii. Using that J is a Poisson morphism we get,

$$\dot\gamma(t)=X^{T^*G}_{h\circ J}(\gamma(t)) \;\Rightarrow\; T_{\gamma(t)}J(\dot\gamma(t))=(T_{\gamma(t)}J)(X^{T^*G}_{h\circ J}(\gamma(t)) \;\Leftrightarrow\; (J\circ\gamma)'(t)=X^{\mathfrak{g}^*}_h(J(\gamma(t)))$$

Thus, if $\gamma:I\to T^*G$ is a solution of the Hamilton equations for $H=h\circ J$, then $\gamma\circ J:I\to\mathfrak{g}^*$ is a solution of the Lie-Poisson equations for h.

□

Remark 6.2. Its is noteworthy that here we have proved the "right" version of the Lie-Poisson reduction theorem. The "left" version is obtained when the momentum map J is defined in terms of the left-invariant vector fields. Then, the theorem states that J is a Poisson morphism between T^*G with the canonical Poisson structure and $\mathfrak{g}^*$ with the minus Lie-Poisson structure, that is,

$$\{f,g\}(\alpha) = -\alpha([df(\alpha), dg(\alpha)]) \quad \text{for } f,g \in \mathcal{F}(\mathfrak{g}^*) \tag{6.2}$$

The proof of this version is analogous to the one that we gave but considering right translations in stead of the left ones.

6.2 Rigid body

We are going to obtain the dynamic equations of the rigid body in geometric terms and to prove that they agree with the physical ones. Our first step must be identifying the configuration manifold.

Recall that a rigid body is any solid body in which the distance between any of two of its points remains constant over time. As we showed, it means that the equation that describes the position of a given particle X over time is $x(t) = R(t)X$, with $R(t) \in SO(3)$. Therefore, we have that this condition implies that the configuration space of the rigid body is $SO(3)$ with the Euler angles $(\varphi,\ \theta,\ \psi)$ as generalized coordinates.

Then, we may consider the corresponding coordinates $(\varphi,\ \theta,\ \psi,\ p_\varphi,\ p_\theta,\ p_\psi)$ on $T^*SO(3)$. Moreover, as we have seen in Chapter 5, the Poisson structure on a cotangent bundle induced by the canonical symplectic structure is given by (5.12). We will denote by $\{\ ,\ \}_{can}$ such a structure.

On the other hand, the Lie algebra $\mathfrak{so}(3)$ might be identified with $(\mathbb{R}^3, \times)$ via the hat map as we showed in (3.10). Under such an identification, we proved in Example 3.53 that the infinitesimal generator of the adjoint action of $SO(3)$ associated with $\hat{\xi} \in SO(3)$ is $\hat{\xi}_{\mathbb{R}^3}(x) = \xi \times x$ for $x \in \mathbb{R}^3$. As well, in Example 3.55 we saw that $\mathfrak{so}^*(3)$ is identified with $\mathbb{R}^3$ via the breve map. Therefore, the infinitesimal generator of the coadjoint action of $SO(3)$ is given by

$$\hat{\xi}_{\mathbb{R}^{3*}}(\hat{\eta})(\breve{\Pi}) = -\breve{\Pi}([\hat{\xi}, \hat{\eta}]) = -\breve{\Pi}(\widehat{\xi \times \eta}) = -\Pi \cdot (\xi \times \eta)$$

Hence, the minus Lie-Poisson bracket of $\mathfrak{so}^*(3)$ is

$$\left\{\hat{\xi}, \hat{\eta}\right\}(\breve{\Pi}) = -\Pi \cdot (\xi \times \eta)$$

Remark that the above brackets $\{\ ,\ \}_{can}$ and $\{\ ,\ \}$ coincide with the canonical bracket and the Lie-Poisson bracket defined in Chapter 1 for the rigid body.

Now, we may prove that the momentum map $J : T^*SO(3) \to \mathfrak{so}^*(3)$ given by (6.1) is just the map J of Theorem 1.1. Indeed, taking into account that the left-invariant vector fields of $SO(3)$ obtained from the canonical basis on $\mathbb{R}^3 \cong \mathfrak{so}(3)$ are

$$\begin{aligned}
X_1 &= \cos\psi \frac{\partial}{\partial\theta} + \frac{\sin\psi}{\sin\theta}\frac{\partial}{\partial\varphi} - \frac{\sin\psi\cos\theta}{\sin\theta}\frac{\partial}{\partial\psi} \\
X_2 &= -\sin\psi \frac{\partial}{\partial\theta} + \frac{\cos\psi}{\sin\theta}\frac{\partial}{\partial\varphi} - \frac{\cos\psi\cos\theta}{\sin\theta}\frac{\partial}{\partial\psi} \\
X_3 &= \frac{\partial}{\partial\psi}
\end{aligned}$$

one has that the corresponding momentum map is

$$\begin{array}{cccc}
J: & T^*SO(3) & \longrightarrow & \mathfrak{so}^*(3) \\
 & p_\varphi \frac{\partial}{\partial\varphi} + p_\theta + \frac{\partial}{\partial\theta} + p_\psi \frac{\partial}{\partial\psi} & \to & \frac{1}{\sin\theta}\left((p_\varphi - p_\psi\cos\theta)\sin\psi + p_\theta\sin\theta\cos\psi\right)d\Pi_1 \\
 & & & +\frac{1}{\sin\theta}\left((p_\varphi - p_\psi\cos\theta)\cos\psi - p_\theta\sin\theta\sin\psi\right)d\Pi_2 + p_\psi d\Pi_3
\end{array}$$

where Π_1, Π_2, Π_3 are the local coordinates on $\mathfrak{so}^*(3)$. Clearly, J coincides with the momentum map of Chapter 1. Therefore, using the left version of the Lie-Poisson reduction theorem, we deduce that such a map is a Poisson morphism, i.e,

$$\{f, g\} \circ J = \{f \circ J, g \circ J\}_{can}$$

As well, if $h : \mathfrak{g}^* \to \mathbb{R}$ and $H = h \circ J$ are Hamiltonian functions and $\gamma : I \to T^*G$ is a solution of the Hamilton equations for H, then $J \circ \gamma : I \to \mathfrak{g}^*$ is a solution of the Lie-Poisson equations for h, which is equivalent to the fact that

$$\dot{\Pi}_i = \{\Pi_i, h\}$$

holds if,

$$\begin{aligned} \dot{\varphi} &= \frac{\partial H}{\partial p_\varphi}, & \dot{p_\varphi} &= -\frac{\partial H}{\partial \varphi}, \\ \dot{\theta} &= \frac{\partial H}{\partial p_\theta}, & \dot{p_\theta} &= -\frac{\partial H}{\partial \theta}, \\ \dot{\psi} &= \frac{\partial H}{\partial p_\psi}, & \dot{p_\psi} &= -\frac{\partial H}{\partial \psi}. \end{aligned}$$

Notice that above statements are nothing but Theorem 1.1 from Chapter 1. It proves the equivalence between the geometric formulation and the physics one and leaves us with a powerful tool to keep working on.

Appendix A

Schouten-Nijenhuis bracket

A.1 k-vectors and k-forms

Let M be a manifold of dimension m and k be an integer such that $2 \leq k \leq m$. Consider the vector bundle

$$\Lambda^k TM = \bigcup_{q \in M} \Lambda^k T_q M = \bigcup_{q \in M} \left\{ \varphi_q : T_q^* M \times \overset{k}{\ldots} \times T_q^* M \to \mathbb{R} \mid \varphi_q \text{ is } \mathbb{R}\text{-multilinear and skew-symmetric} \right\} \tag{A.1}$$

If $(U, \varphi = (q^1, \ldots, q^m))$ is a local chart in M, $\{ \frac{\partial}{\partial q^{i_1}}_{|q} \wedge \ldots \wedge \frac{\partial}{\partial q^{i_k}}_{|q} \mid 1 \leq i_1 < \ldots < i_k \leq m \}$ is a basis of $\Lambda^k T_q M$. It implies that $\Lambda^k TM$ is a smooth manifold of dimension $m + \binom{m}{k}$. A smooth function $P : M \to \Lambda^k TM$ such that $P(q) \in \Lambda^k T_q M$ for all $q \in M$ is said to be a *k-vector*. In local coordinates a k-vector is given by,

$$P = \sum_{i_1 < \ldots < i_k} P_{i_1 \ldots i_k} \frac{\partial}{\partial q^{i_1}} \wedge \ldots \wedge \frac{\partial}{\partial q^{i_k}} \tag{A.2}$$

where $P_{i_1 \ldots i_k}$ are local smooth functions. From now on, $\mathcal{X}^k(M)$ will denote the space of k-vectors. Remark that $\mathcal{X}^0(M) = \mathcal{F}(M)$ and $\mathcal{X}^1(M) = \mathcal{X}(M)$.

The sections of the dual bundle to $\Lambda^k T^* M$ are known as *k-forms*. Likewise, a k-form is a smooth function $\alpha : M \to \Lambda^k T^* M$ such that $\alpha(q) \in \Lambda^k T_q^* M$ for all $q \in M$, where

$$\Lambda^k T^* M = \bigcup_{q \in M} \Lambda^k T_q^* M = \bigcup_{q \in M} \left\{ \varphi_q : T_q M \times \overset{k}{\ldots} \times T_q M \to \mathbb{R} \mid \varphi_q \text{ is } \mathbb{R}\text{-multilinear and skew-symmetric} \right\} \tag{A.3}$$

In the previous chart, $\{ dq^{i_1}_{|q} \wedge \ldots \wedge dq^{i_k}_{|q} \mid 1 \leq i_1 < \ldots < i_k \leq m \}$ is a basis of $\Lambda^k T_q^* M$, which means that $\Lambda^k T^* M$ is a smooth manifold of dimension $m + \binom{m}{k}$. If α is a k-form, it is locally expressed as

$$\alpha = \sum_{i_1 < \ldots < i_k} \alpha_{i_1 \ldots i_k} dq^{i_1} \wedge \ldots \wedge q^{i_k}$$

where $\alpha_{i_1 \ldots i_k}$ are again local smooth functions. Then, the pairing $\alpha(P)$ is a function given by

$$\alpha(P) = \sum_{i_1 < \ldots < i_k} \alpha_{i_1 \ldots i_k} P_{i_1 \ldots i_k}$$

Given a k-vector P, consider the map $P : \Omega^1(M) \times \overset{k}{\ldots} \times \Omega^1(M) \to \mathcal{F}(M)$ defined as

$$\begin{array}{rccc} P(\alpha_1, \ldots, \alpha_k) : & M & \to & \mathbb{R} \\ & q & \to & P(\alpha_1(q), \ldots, \alpha_k(q)) \end{array}$$

where $\alpha_i \in \Omega^1(M)$. Thus, one may regard a k-vector on M as a $\mathcal{F}(M)$-multilinear skew-symmetric map $P : \Omega^1(M)\times \overset{k}{\ldots} \times\Omega^1(M) \to \mathcal{F}(M)$. Moreover, with this interpretation of the k-vector P, one can define a $\mathbb{R}$-multilinear skew-symmetric map $P : \mathcal{F}(M)\times \overset{k}{\ldots} \times\mathcal{F}(M) \to \mathcal{F}(M)$ as follows

$$\begin{array}{rccc} P(f_1,\ldots,f_n): & M & \to & \mathbb{R} \\ & q & \to & P(df_1(q),\ldots,df_k(q)) \end{array} \tag{A.4}$$

where $f_i \in \mathcal{F}(M)$. Remark that since $d(fg) = fdg + gdf$, the map P satisfies the Leibniz rule. Such a map is said to be *multiderivation*. Conversely, a multiderivation P always define a k-vector by $P(f_1,\ldots,f_k) := P(df_1,\ldots,df_k)(q)$ (see [Duf] for a proof).

A.1.1 2-vectors

We specialize our discussions in the particular case when $k = 2$. According to the previous definitions, a 2-vector is a $\mathcal{F}(M)$-bilinear skew-symmetric map $w : \Omega^1(M)\times\Omega^1(M) \to \mathcal{F}(M)$ or, equivalently, a biderivation $\{\ ,\ \} : \mathcal{F}(M)\times\mathcal{F}(M) \to \mathcal{F}(M)$ such that

$$w(df,dg) = \{f,g\} \tag{A.5}$$

Any 2-vector define a smooth fiber map $\# : T^*M \to TM$. Indeed, for every $\alpha_q \in T_q^*M$, $\#(\alpha_q) \in T_qM$ is defined as

$$\beta_q(\#(\alpha_q)) = w(q)(\alpha_q,\beta_q) \quad \text{for } \beta_q \in T_q^*M \tag{A.6}$$

Likewise, the map $\#$ might be seen as a morphism of $\mathcal{F}(M)$-modules $\# : \Omega^1(M) \to \mathcal{X}(M)$ defined as

$$\begin{array}{rccc} \#(\alpha): & M & \to & TM \\ & q & \to & \#(\alpha(q)) \end{array} \tag{A.7}$$

for $\alpha \in \Omega^1(M)$.

With this definition, $\#$ can be extended to a morphism of $\mathcal{F}(M)$-modules, $\# : \Omega^k(M) \to \mathcal{X}^k(M)$. Given $\alpha \in \Omega^k(M)$, $\#(\alpha) \in \mathcal{X}^k(M)$ is such that

$$\begin{array}{rccc} \#(\alpha): & \Omega^1(M)\times \overset{k}{\ldots} \times\Omega^1(M) & \to & \mathcal{F}(M) \\ & (\alpha_1,\ldots,\alpha_k) & \to & (-1)^k\alpha(\#(\alpha_1),\ldots,\#(\alpha_k)) \end{array} \tag{A.8}$$

It is easy to check that,

$$\#(\alpha_1\wedge\ldots\wedge\alpha_k) = \#(\alpha_1)\wedge\ldots\wedge\#(\alpha_k) \quad \text{for } \alpha_i \in \Omega^1(M) \tag{A.9}$$

The *rank* of a 2-vector w in a point $q \in M$ is defined to be the rank of the linear map $\#_{|T_q^*M} : T_q^*M \to T_qM$. Remark that $w(q) : T_q^*M \times T_q^*M \to \mathbb{R}$ might be seen as a 2-form on T_q^*M whose rank coincides with the rank of $\#_{|T_q^*M}$. Therefore we conclude that the rank of a 2-vector is always even.

The particular case of a symplectic manifold

Let (M,ω) be a symplectic manifold. One can define a 2-vector on M as follows,

$$\begin{array}{rccc} \{\ ,\ \}: & \mathcal{F}(M)\times\mathcal{F}(M) & \to & \mathcal{F}(M) \\ & (f,g) & \to & \{f,g\} = \omega(X_f,X_g) \end{array} \tag{A.10}$$

where X_f and X_g are the Hamiltonian vector fields of the functions f and g. Moreover, we have that the 2-vector w satisfies $w(df,dg) = \{f,g\} = \omega(X_f,X_g)$ and we can define the morphism of $\mathcal{F}(M)$-modules $\# : \Omega^1(M) \to \mathcal{X}(M)$.

Let b_ω be the corresponding isomorphism of $\mathcal{F}(M)$-modules associated to the symplectic form ω,

$$b_\omega: \begin{array}{ccc} \mathcal{X}(M) & \to & \Omega^1(M) \\ X & \to & i_X\omega \end{array} \tag{A.11}$$

We are going to show that

$$\# = -b_\omega^{-1}. \tag{A.12}$$

For every $f, g \in \mathcal{F}(M)$,

$$(\#(df))(g) = dg(\#(df)) = w(df, dg) = \omega(X_f, X_g) = -\omega(X_g, X_f) = -i_{X_g}\omega(X_f) = -X_f(g)$$

and $\#(df) = -X_f = -b_\omega^{-1}(df)$. Thus, for any $\alpha \in \Omega^1(M)$ such that $\alpha = \sum_i \alpha_i dg_i$ for $\alpha_i,\ g_i \in \mathcal{F}(M)$,

$$\#(\alpha) = \sum_i \alpha_i \#(dg_i) = -\sum_i \alpha_i b_\omega^{-1}(dg_i) = -b_\omega^{-1}(\sum_i \alpha_i dg_i) = -b_\omega^{-1}(\alpha)$$

Furthermore, if we consider the extension of $\#$ to $\Omega^2(M)$, i.e., $\# : \Omega^2(M) \to \mathcal{X}^2(M)$ we have that

$$\#(\omega) = w \tag{A.13}$$

Indeed, since $\#(\omega)$ and w belong to $\mathcal{X}^2(M)$ it is enough to check that $\#(\omega)(df, dg) = w(df, dg)$ for any $f, g \in \mathcal{F}(M)$,

$$\#(\omega)(df, dg) = \omega(\#(df), \#(dg)) = \omega(X_f, X_g) = w(df, dg).$$

A.2 Lie derivative

As it is known, the Lie derivative of a 1-form $\alpha \in \Omega^1(M)$ with respect to a vector field $X \in \mathcal{X}(M)$ is the 1-form given by

$$\mathcal{L}_X\alpha = i_X d\alpha + d i_X \alpha \tag{A.14}$$

or, in other words,

$$(\mathcal{L}_X\alpha)(Y) = X(\alpha(Y)) - \alpha[X, Y] \quad \text{for } Y \in \mathcal{X}(M) \tag{A.15}$$

Now, let us introduce the Lie derivative of a k-vector $P \in \mathcal{X}^k(M)$ with respect to a vector field $X \in \mathcal{X}(M)$. $\mathcal{L}_X P$ is the k-vector defined as

$$\mathcal{L}_X P = [X, P]: \begin{array}{ccc} \Omega^1(M) \times \overset{k}{\cdots} \times \Omega^1(M) & \to & \mathcal{F}(M) \\ (\alpha_1, \ldots, \alpha_k) & \to & \mathcal{L}_X P(\alpha_1, \ldots, \alpha_k) \end{array} \tag{A.16}$$

where $\mathcal{L}_X P(\alpha_1, \ldots, \alpha_k) = X(P(\alpha_1, \ldots, \alpha_k)) - \sum_{i=1}^k P(\alpha_1, \ldots, \mathcal{L}_X\alpha_i, \ldots, \alpha_k)$. Taking into account that P is a k-vector and that X is a vector field, it is almost direct to prove that $\mathcal{L}_X P$ is skew-symmetric and $\mathcal{F}(M)$-multilinear. For instance, let us check that $\mathcal{L}_X P$ is $\mathcal{F}(M)$-multilinear. Let $\beta, \alpha_1, \ldots, \alpha_k \in \Omega^1(M)$ be 1-forms and $f \in \mathcal{F}(M)$ be a function, then

$$\begin{aligned}
\mathcal{L}_X P(\beta + f\alpha_1, \ldots, \alpha_k) =& X(P(\beta + f\alpha_1, \ldots, \alpha_k)) - P(\mathcal{L}_X(\beta + f\alpha_1), \alpha_2, \ldots, \alpha_k) \\
& - \sum_{i=2}^{k} P(\beta + f\alpha_1, \ldots, \mathcal{L}_X \alpha_i, \ldots, \alpha_k) \\
& = X(P(\beta, \ldots, \alpha_k)) + X(fP(\alpha_1, \ldots, \alpha_k)) - P(\mathcal{L}_X \beta, \alpha_2, \ldots, \alpha_k) \\
& - P(X(f)\alpha_1 + f\mathcal{L}_X \alpha_1, \alpha_2 \ldots, \alpha_k) - \sum_{i=2}^{k} P(\beta, \ldots, \mathcal{L}_X \alpha_i, \ldots, \alpha_k) \\
& - \sum_{i=2}^{k} P(f\alpha_1, \ldots, \mathcal{L}_X \alpha_i, \ldots, \alpha_k) \\
& = X(P(\beta, \ldots, \alpha_k)) + fX(P(\alpha_1, \ldots, \alpha_k)) + P(\alpha_1, \ldots, \alpha_k)X(f) \\
& - P(\mathcal{L}_X \beta, \alpha_2, \ldots, \alpha_k) - X(f)P(\alpha_1, \ldots, \alpha_k) - fP(\mathcal{L}_X \alpha_1, \alpha_2 \ldots, \alpha_k) \\
& - \sum_{i=2}^{k} P(\beta, \ldots, \mathcal{L}_X \alpha_i, \ldots, \alpha_k) - f\sum_{i=2}^{k} P(\alpha_1, \ldots, \mathcal{L}_X \alpha_i, \ldots, \alpha_k) \\
& = \mathcal{L}_X P(\beta, \ldots, \alpha_k) + f\mathcal{L}_X P(\alpha_1, \ldots, \alpha_k)
\end{aligned}$$

It can be also checked that if $X_1, \ldots X_k \in \mathcal{X}(M)$ are vector fields, then

$$\mathcal{L}_X(X_1 \wedge \ldots \wedge X_k) = \sum_{i=1}^{k} X_1 \wedge \ldots \wedge [X, X_i] \wedge \ldots \wedge X_k \tag{A.17}$$

So far, we have given the algebraic approach to the Lie derivative of a multivector P with respect to X. Next, we give the dynamic definition in terms of the flow ϕ of X

$$(\mathcal{L}_X P)(q) = \frac{d}{dt}_{|t=0} (T_{\phi_t(q)}\phi_{-t})(P(\phi_t(q))$$

A.2.1 Schouten-Nijenhuis bracket

Once we have had a quick overview on the Lie derivative of a k-vector and we have obtained the formula (A.17), it is natural to extend it to an operation between q-vectors and p-vectors. In fact, there is a unique $\mathbb{R}$-bilinear extension of the Lie derivative to an operation

$$[\ ,\ ]: \quad \mathcal{X}^p(M) \times \mathcal{X}^q(M) \quad \to \quad \mathcal{X}^{p+q-1}(M) \tag{A.18}$$

satisfying the following properties:

i. For all $P \in \mathcal{X}^p(M)$ and $Q \in \mathcal{X}^q(M)$, $[P, Q] = (-1)^{pq}[Q, P]$.

ii. For all $P \in \mathcal{X}^p(M)$, $R \in \mathcal{X}^r(M)$ and $Q \in \mathcal{X}^q(M)$, $[P, Q \wedge R] = [P, Q] \wedge R + (-1)^{pq+q} Q \wedge [P, R]$.

iii. For all $P \in \mathcal{X}^p(M)$, $R \in \mathcal{X}^r(M)$ and $Q \in \mathcal{X}^q(M)$,
$(-1)^{p(r-1)}[P, [Q, R]] + (-1)^{q(p-1)}[Q[R, P]] + (-1)^{r(q-1)}[R, [P, Q]]$. *(Graded Jacobi identity)*

Such an operation is called the *Schouten-Nijenhuis bracket*.

In order to prove this result, first we will prove that the Schouten-Nijenhuis bracket is a local operation, that is, $[P, Q]_{|U}$ only depends on $P_{|U}$ and $Q_{|U}$. Due to the fact that $[\ ,\ ]$ is skew-symmetric it will be enough to prove that $[P, Q_1](x_0) = [P, Q_2](x_0)$ if $Q_{1|U} = Q_{2|U}$ in a neighbourhood U of x_0.

Consider a bump function f such that $f = 1$ in a compact neighbourhood of x_0 contained in U and $f = 0$ outside U. Thus, from ii.

$$[P, fQ_i](x_0) = [P, f] \wedge Q_i(x_0) + f(x_0)[P, Q_i](x_0) = [P, Q_i](x_0) \quad \text{for } i = 1, 2$$

Since $fQ_1 = fQ_2$, we conclude that $[P, Q_1](x_0) = [P, Q_2](x_0)$. Now, using that the bracket has local character we can work on a local chart.

Remark that for any p-vector P and any $Q_1, \ldots, Q_q \in \mathcal{X}(M)$, from ii. we deduce that,

$$[Q_1 \wedge \ldots \wedge Q_q, P] = \sum_{i=1}^{q} (-1)^{i+1} Q_1 \wedge \ldots \wedge \hat{Q}_i \wedge \ldots \wedge Q_q \wedge [Q_i, P]$$

and, for any $P_1, \ldots, P_p \in \mathcal{X}(M)$

$$[P_1 \wedge \ldots \wedge P_p, Q_1 \wedge \ldots \wedge Q_q] = (-1)^{p+1} \sum_{i=0}^{p} \sum_{j=0}^{q} (-1)^{j+i} [P_i, Q_j] \wedge P_1 \wedge \ldots \wedge \hat{P}_i \wedge \ldots \wedge P_p \wedge Q_1 \wedge \ldots \wedge \hat{Q}_j \wedge \ldots \wedge Q_q$$

Moreover,

$$\begin{aligned}
[P_1 \wedge \ldots \wedge P_p, fQ_1 \wedge \ldots \wedge Q_q] &= [P_1 \wedge \ldots \wedge P_p, f] \wedge Q_1 \wedge \ldots \wedge Q_q + f[P_1 \wedge \ldots \wedge P_p, Q_1 \wedge \ldots \wedge Q_q] \\
&= \sum_{i=1}^{p} (-1)^{i+1} P_i(f) P_1 \wedge \ldots \wedge P_p \wedge Q_1 \wedge \ldots \wedge Q_q + f[P_1 \wedge \ldots \wedge P_p, Q_1 \wedge \ldots \wedge Q_q]
\end{aligned}$$

Let $(U, \varphi \equiv (x^1, \ldots, x^n))$ be a chart on M and $(\tau_M^{-1}(U), \overline{\varphi} \equiv (x^1, \ldots, x^n, \dot{x}^1, \ldots, \dot{x}^n))$ be the corresponding chart on TM. Any p-vector will be given by $P = \sum_{i_1 < \ldots < i_p} P_{i_1 \ldots i_p} \frac{\partial}{\partial x^{i_1}} \wedge \ldots \wedge \frac{\partial}{\partial x^{i_p}}$ and any q-vector by $Q = \sum_{i_1 < \ldots < i_q} Q_{i_1 \ldots i_q} \frac{\partial}{\partial x^{i_1}} \wedge \ldots \wedge \frac{\partial}{\partial x^{i_q}}$. Then, from the previous equalities it is clear how the bracket operates on any p-vector and q-vector. It proves the uniqueness in every local coordinate system. Thus, we deduce the result. Note that the graded Jacobi identity for the Schouten-Nijenhuis bracket follows using the fact that the standard Lie bracket of vector fields satisfies the Jacobi identity.

Let us study the particular case $p = q = 2$. Assume that,

$$P = \frac{1}{2} \sum_{i,j} P_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \qquad Q = \frac{1}{2} \sum_{h,k} Q_{hk} \frac{\partial}{\partial x_h} \wedge \frac{\partial}{\partial x_k}$$

Then,

$$\begin{aligned}
[P, Q] &= \frac{1}{4} \sum_{i,j,h,k} [P_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, Q_{hk} \frac{\partial}{\partial x_h} \wedge \frac{\partial}{\partial x_k}] \\
&= \frac{1}{4} \sum_{i,j,h,k} \left([P_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, Q_{hk}] \wedge \frac{\partial}{\partial x_h} \wedge \frac{\partial}{\partial x_k} + Q_{hk} [P_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_h} \wedge \frac{\partial}{\partial x_k}] \right) \\
&= \frac{1}{4} \sum_{i,j,h,k} \left([Q_{hk}, P_{ij}] \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_h} \wedge \frac{\partial}{\partial x_k} + P_{ij} [Q_{hk}, \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}] \wedge \frac{\partial}{\partial x_h} \wedge \frac{\partial}{\partial x_k} \right. \\
&\left. + Q_{hk} [\frac{\partial}{\partial x_h} \wedge \frac{\partial}{\partial x_k}, P_{ij}] \wedge \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} + Q_{hk} P_{ij} [\frac{\partial}{\partial x_h} \wedge \frac{\partial}{\partial x_k}, \wedge \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}] \right) \\
&= \frac{1}{4} \sum_{i,j,h,k} \left(P_{ij} [Q_{hk}, \frac{\partial}{\partial x_i}] \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_h} \wedge \frac{\partial}{\partial x_k} - P_{ij} [Q_{hk}, \frac{\partial}{\partial x_j}] \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_h} \wedge \frac{\partial}{\partial x_k} \right. \\
&\left. + Q_{hk} [P_{ij}, \frac{\partial}{\partial x_h}] \frac{\partial}{\partial x_k} \wedge \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} - Q_{hk} [P_{ij}, \frac{\partial}{\partial x_k}] \frac{\partial}{\partial x_h} \wedge \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \right) \\
&= \frac{1}{2} \sum_{i,j,h,k} \left(P_{ij} \frac{\partial Q_{hk}}{\partial x_i} \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_h} \wedge \frac{\partial}{\partial x_k} + Q_{hk} \frac{\partial P_{ij}}{\partial x_h} \frac{\partial}{\partial x_k} \wedge \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \right)
\end{aligned}$$

Hence,

$$[P,Q] = \sum_{i<j<k} \sum_{h} \left(P_{hi} \frac{\partial Q_{jk}}{\partial x_h} + P_{hj} \frac{\partial Q_{ki}}{\partial x_h} + P_{hk} \frac{\partial Q_{ij}}{\partial x_h} + Q_{hk} \frac{\partial P_{ij}}{\partial x_h} + Q_{hj} \frac{\partial P_{ki}}{\partial x_h} + Q_{hi} \frac{\partial P_{jk}}{\partial x_h} \right) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_k}$$

Appendix B

Distributions

B.1 Regular distributions

Assume that M is a smooth manifold of dimension m. A *regular distribution* of dimension k on M is a map $q \in M \to \mathcal{D}(q) \subseteq T_qM$ verifying:

i. $\mathcal{D}(q)$ is a subvector space of T_qM such that $dim\, \mathcal{D}(q) = k$.

ii. For every $q \in M$ there exists an open neighbourhood U of q, and k smooth vector fields $X_1, \dots X_k$ defined on U such that

$$\mathcal{D}(q) = \langle X_1(q), \dots, X_k(q) \rangle$$

A vector field $X \in \mathcal{X}(M)$ is said to be *tangent* to $\mathcal{D}$ if $X(q) \in D(q)$ for every $q \in M$.

Given a distribution $\mathcal{D}$ and a submanifold $i : N \to M$, N is said to be an *integral submanifold* of $\mathcal{D}$ if for every $q \in N$

$$(T_q i)(T_q N) \subseteq D(q)$$

If for any $q \in M$ there exists a local chart $(U, \varphi \equiv (q^1, \dots, q^k, q^{k+1}, \dots, q^m))$, with $q \in U$, such that

$$\mathcal{D}(q) = \left\langle \frac{\partial}{\partial q^1}_{|q}, \dots, \frac{\partial}{\partial q^k}_{|q} \right\rangle \quad \forall q \in U$$

then, $\mathcal{D}$ is said to be *completely integrable*. Furthermore, if $\mathcal{D}$ is completely integrable, then for any $q \in M$ there exists a maximal connected integral submanifold L_q such that $q \in L_q$. That is, L_q is an integral submanifold and if there exists a connected manifold N such that it is an integral submanifold of $\mathcal{D}$ and $q \in N$, then $N \subseteq L_q$. Indeed, taking the previous charts and assuming that $\varphi(q) = (q_1^0, \dots, q_m^0)$, we have that

$$\left\{ \varphi^{-1}(q^1, \dots q^k, q_{k+1}^0 \dots q_m^0) \mid (q_1, \dots, q^k, q_{k+1}^0 \dots q_m^0) \in \varphi(U) \right\}$$

is an integral submanifold of $\mathcal{D}$ which contains the point q. The maximal connected integral submanifold L_q is the known as the *leaf* of $\mathcal{D}$ through q.

If for any two vector fields $X, Y \in \mathcal{X}(M)$ tangents to $\mathcal{D}$, $[X, Y]$ is also tangent to $\mathcal{D}$, the distribution is said to be *involutive*. Thus, since $[\frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^j}] = 0$ it follows that any completely integrable distribution is involutive. As well, the *Frobenius theorem* proves that the converse also holds, i.e, a distribution $\mathcal{D}$ is involutive if and only if, it is completely integrable.

B.2 Generalized distributions

Let M be a smooth manifold of dimension m. A *generalized distribution* (locally finitely generated) on M is a map $q \in M \to \mathcal{D}(q) \subseteq T_qM$ verifying:

i. $\mathcal{D}(q)$ is a subvector space of T_qM.

ii. For every $q \in M$ there exists an open neighbourhood U of q, and a finite set of smooth vector fields $X_1, \ldots X_k$ defined on U such that

$$\mathcal{D}(q) = \langle X_1(q), \ldots, X_k(q) \rangle$$

Remark that the unique hypothesis that is missing with respect to the regular distributions is that we are not imposing any condition to the dimension.

The definitions of *integrable submanifold* and *involutive* distribution are the same for a generalized distribution. However, since there is no constraint on the dimension we have to rephrase the definition of a completely integrable distribution. In fact, given a generalized distribution $\mathcal{D}$, it is said to be *completely integrable* if for every $q \in M$ there exists maximal connected integral submanifold L_q of $\mathcal{D}$ such that $q \in L_q$. Just as before, L_q is known as the *leaf* of $\mathcal{D}$ through q. While no constraints are imposed on the dimension, the dimension of the leaves L_q may change depending on the point q. Again, we have that integrability implies involutibility. Indeed, if $X, Y \in \mathcal{X}(M)$ are two vector fields tangent to a distribution $\mathcal{D}$, and L_q is the leaf of $\mathcal{D}$ through q, then, since $T_xL_q = D(x)$ for all $x \in L_q$, $X_{|L_q}$ and $Y_{|L_q}$ are tangents to L_q. Therefore,

$$[X, Y](q) = [X_{|L_q}, Y_{|L_q}](q) \in T_qL_q = D(q)$$

Nonetheless, the reciprocal of the Frobenius theorem is not longer true and an involutive generalized distribution $\mathcal{D}$ might not be completely integrable. Furthermore, in the case of $\mathcal{D}$ being a generalized distribution a new definition arises. $\mathcal{D}$ is said to be *invariant* if for every vector field X tangent to $\mathcal{D}$, the following equality holds

$$T_q\phi_t(\mathcal{D}(q)) = \mathcal{D}(\phi_t(q)) \quad \text{for all } t \text{ and } q \in M$$

where ϕ is the flow of X.

This new definition leads to the *generalized Frobenius theorem* which states that a generalized distribution $\mathcal{D}$ is completely integrable if and only if, it is invariant. Thus, there exist leaves through every point of the manifold M, if and only if, the distribution is invariant.

Let us give two examples that illustrate the previous ideas.

Counterexample

As we have pointed out, the regular Frobenius theorem might not be satisfied when dealing with general distributions. In this counterexample we are giving a distribution which is involutive but not completely integrable. The distribution is defined on $\mathbb{R}^2$ by

$$(x, y) \in \mathbb{R}^2 \longrightarrow \mathcal{D}(x, y) = \left\langle \frac{\partial}{\partial x}_{|(x,y)}, \varphi(x) \frac{\partial}{\partial y}_{|(x,y)} \right\rangle \subseteq T_{(x,y)}\mathbb{R}^2$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is smooth and verifies

$$\varphi(x) = \begin{cases} \varphi(x) > 0 & \text{if } x > 0 \\ \varphi(x) = 0 & \text{if } x \leq 0 \end{cases}$$

First of all notice that $\mathcal{D}$ does not have constant dimension since

i. $\forall (x, y) \in \mathbb{R}^2$ such that $x > 0$, $dim\ \mathcal{D}(x, y) = 2$ and $\mathcal{D}(x, y) = T_{(x,y)}\mathbb{R}^2$

ii. $\forall (x, y) \in \mathbb{R}^2$ such that $x \leq 0$, $dim\ \mathcal{D}(x, y) = 1$ and $\mathcal{D}(x, y) = \left\langle \frac{\partial}{\partial x}_{|(x,y)} \right\rangle$

So, $\mathcal{D}$ is a generalized distribution. Moreover, it is involutive. In fact,

$$\left[\frac{\partial}{\partial x}, \varphi(x)\frac{\partial}{\partial y}\right](x,y) = \frac{d\varphi}{dx}_{|x}\frac{\partial}{\partial y}_{|(x,y)} = \begin{cases} \frac{d\ln\varphi}{dx}_{|x}\varphi(x)\frac{\partial}{\partial y}_{|(x,y)} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

Now, we are going to compute the integral submanifolds in order to see that $\mathcal{D}$ is not completely integrable. Let (x_0, y_0) be a point on $\mathbb{R}^2$ and suppose that L is a connected maximal integral submanifold of $\mathcal{D}$ through (x_0, y_0). We distinguish cases:

1. $x_0 > 0$. In such case, L verifies that $T_{(x,y)}L = \mathcal{D}(x,y)$ for all $(x,y) \in L$ and in particular $T_{(x_0,y_0)}L = \mathcal{D}(x_0,y_0)$. Thus, $dim\ L = dim\ \mathcal{D}(x_0,y_0) = 2$ and L is an open subset of $\mathbb{R}^2$. Moreover, since L is maximal and $\mathcal{D}(x,y) = T_{(x,y)}\mathbb{R}^2$ for any (x,y) with $x > 0$ we have that

   $$\left\{(x,y) \in \mathbb{R}^2 \mid x > 0\right\} \subseteq L$$

   As well, the points of the axis OY can not belong to L because there $dim\ \mathcal{D}(0,y) = 1$. Therefore, using the hypothesis that L is connected we obtain

   $$L = \left\{(x,y) \in \mathbb{R}^2 \mid x > 0\right\}$$

2. $x_0 < 0$. With the same argument as before we conclude that $dim\ L = dim\ \mathcal{D}(x_0,y_0) = 1$ and $T_{(x,y)}L = \left\langle \frac{\partial}{\partial x}_{|(x,y)} \right\rangle$. It means that L must be a parallel line to the OX axis. On the other hand, taking into account that $dim\mathcal{D}(x,y) = 2$ for $(x,y) \in \mathbb{R}^2$ such that $x > 0$ we deduce that

   $$L \subseteq \left\{(x,y) \in \mathbb{R}^2 \mid x \leq 0\right\}$$

   Finally, as L is a manifold without boundary, L can not intersect the OY axis and

   $$L = \left\{(x,y) \in \mathbb{R}^2 \mid x < 0\right\}$$

3. $x_0 = 0$. Proceeding as in the previous case, we have that $dim\ L = 1$, $T_{(x,y)}L = \left\langle \frac{\partial}{\partial x}_{|(x,y)} \right\rangle$ and consequently, L is a parallel line to the OX axis contained in the half plane $x \leq 0$. Since the point $(0, y_0)$ must belong to L, L has to be of the form

   $$L = \left\{(x,y_0) \in \mathbb{R}^2 \mid x \leq 0\right\}$$

   which is not possible since the leaf of $\mathcal{D}$ through a point is a manifold without boundary.

So far, we have seen that $\mathcal{D}$ is not completely integrable, and by the generalized Frobenius theorem it implies that $\mathcal{D}$ is not invariant. Let us check it.

First, compute the flows of the involved vector fields. Easily, $\phi_t(x,y) = (x+t, y)$ is the flow of $\frac{\partial}{\partial x}$ and $\psi_t(x,y) = (x, \varphi(x)t + y)$ is the flow of $\varphi(x)\frac{\partial}{\partial y}$. Remark that,

$$T_{(x_0,y_0)}\phi_{t_0}\left(\varphi(x_0)\frac{\partial}{\partial y}_{|(x_0,y_0)}\right) = \varphi(x_0)\frac{\partial}{\partial y}_{|(x_0+t_0,y_0)}$$

If $t_0 = -x_0$ with $x_0 > 0$, it is clear that $T_{(x_0,y_0)}\phi_{-x_0}\left(\varphi(x_0)\frac{\partial}{\partial y}_{|(x_0,y_0)}\right) \notin \mathcal{D}(\phi_{-x_0}(x_0,y_0)) = \mathcal{D}(0,y_0)$ and $\mathcal{D}$ is not invariant.

Example

Consider the distribution on $\mathbb{R}^3$ given by

$$\mathcal{D}(x,y,z) = \left\langle x\frac{\partial}{\partial z}_{|(x,y,z)} - z\frac{\partial}{\partial x}_{|(x,y,z)}, y\frac{\partial}{\partial x}_{|(x,y,z)} - x\frac{\partial}{\partial y}_{|(x,y,z)}, z\frac{\partial}{\partial y}_{|(x,y,z)} - y\frac{\partial}{\partial z}_{|(x,y,z)} \right\rangle \subseteq T_{(x,y,z)}\mathbb{R}^3$$

It is clear that $dim\ \mathcal{D}(x_0,y_0,z_0) = 2$ for all $(x_0,y_0,z_0) \in \mathbb{R}^3$ such that $(x_0,y_0,z_0) \neq 0$ and $dim\ \mathcal{D}(0,0,0) = 0$. It means that the leaf of $\mathcal{D}$ through $(0,0,0)$ is the point itself. Moreover, if we denote by $X = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}$, $Y = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$ and $Z = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}$, one can see that

$$D(x,y,z) = \begin{cases} \langle X, Y \rangle & \text{if } z = 0, x \neq 0 \\ \langle Y, Z \rangle & \text{if } z = 0, x = 0 \\ \langle X, Y \rangle & \text{if } z \neq 0 \end{cases} \qquad \forall (x,y,z) \neq 0 \tag{B.1}$$

Consider the map on $\mathbb{R}^3$ given by,

$$\begin{array}{cccc} F: & \mathbb{R}^3 & \rightarrow & \mathbb{R} \\ & (x,y,z) & \rightarrow & x^2+y^2+z^2 \end{array}$$

and a point $(x_0,y_0,z_0) \in \mathbb{R}^3 - \{0\}$ such that $r_0 = x_0^2 + y_0^2 + z_0^2$. Then, $F^{-1}(r_0) = S^2_{r_0}$, where $S^2_{r_0}$ denotes the sphere of radius $\sqrt{r_0}$ centered in the origin. It is easy to see that $T_{(x,y,z)}S^2_{r_0} = ker\ T_{(x,y,z)}F = \mathcal{D}(x,y,z)$ for $(x,y,z) \in S^2_{r_0}$. Thus, if L denotes the leaf of $\mathcal{D}$ through (x_0,y_0,z_0), we have that $S^2_{r_0} \subseteq L$ because L is maximal.

On the other hand, using that $T_{(x,y,z)}F(v) = 0$ for any $v \in \mathcal{D}(x,y,z) = T_{(x,y,z)}L$ and the fact that L is connected, we conclude that $F_{|L}$ is constant. Thus, $F_{|L}(x,y,z) = F(x_0,y_0,z_0) = r_0$ for any $(x,y,z) \in L$ and $L \subseteq S^2_{r_0}$. Therefore,

$$L = S^2_{r_0}$$

Since there exist a leaf through each point, $\mathcal{D}$ is completely integrable and invariant.

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