MASTER'S DEGREE THESIS

Master of Science in Advanced Mathematics and Mathematical Engineering

Title: ***Lagrangian Lie subalgebroids of the canonical symplectic Lie algebroid*.**

Author: Mònica Aymerich Valls

Advisor: Juan Carlos Marrero (Universidad de La Laguna) and Narciso Román Roy (UPC)

Department: Matemàtica Aplicada IV

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Facultat de Matemàtiques i Estadística

UNIVERSITAT POLITÈCNICA DE CATALUNYA

Agraïments

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Introduction

"Everything is a Lagrangian submanifold"

A. Weinstein [14]

It is well-known that symplectic geometry plays an important role in the mathematical description of Classical Mechanics. In fact, the phase space of a Hamiltonian system may be identified with the cotangent bundle T^*Q of the configuration space Q. So, using the canonical symplectic structure of T^*Q and the Hamiltonian function $H : T^*Q \to \mathbb{R}$ one may obtain a vector field on T^*Q, the Hamiltonian vector field of H. The integral curves of this vector field are just the solutions of the Hamilton equations (see [1]).

On the other hand, given a Hamiltonian system with a Hamiltonian function $H : T^*Q \to \mathbb{R}$, knowing one solution of the (partial) Hamilton-Jacobi equation simplifies the search of trajectories of the Hamiltonian vector field. It is well-known that the solutions of such an equation live in Lagrangian submanifolds of T^*Q (see [1], [10]). More generally, one may see that a Lagrangian foliation on the cotangent bundle T^*Q may be associated with a complete solution of the Hamilton-Jacobi equation (see [3]). Thus, knowing the local structure of the Lagrangian submanifolds of the cotangent bundle is useful to find solutions of the Hamilton-Jacobi equation. The description of this local structure may be found, for instance, in [10].

So, Lagrangian submanifolds are a very interesting class of submanifolds of a symplectic manifold. In fact, apart from the previous arguments the following facts justify such a statement:

- A geometric formulation of Lagrangian and Hamiltonian Mechanics may be developed using Lagrangian submanifolds of special symplectic manifolds (see [18], [19]).
- Lagrangian submanifolds play an important role in the geometric description of variational mechanical systems subjected to nonholonomic constraints (or vakonomic systems) (see [9]).
- A completely integrable mechanical system admits a Lagrangian foliation on its phase space of momenta whose leaves are torus of dimension n, n being the dimension of the configuration space (see [2]).
- A geometric formulation of discrete constrained Lagrangian mechanics may be developed using Lagrangian submanifolds of symplectic groupoids. This theory may be applied to discretize several concrete problems in optimal control (see, for instance, [8], [16]).

In another direction, if our Hamiltonian function $H : T^*Q \to \mathbb{R}$ is invariant under the action of a symmetry group G which acts freely and properly on the configuration manifold

Q, then a new Hamiltonian function h may be defined on the reduced phase space T^*Q/G. Thus, we have a new dynamical system T^*Q/G and the integration of this system allows us, in some cases, to recover the dynamics of the original system through a process of reconstruction (see [14] [12]). We remark that T^*Q/G is not, in general, a symplectic manifold but a linear Poisson manifold. In fact, T^*Q/G is a vector bundle over Q/G and the dual bundle TQ/G admits a Lie algebroid structure. TQ/G is the Atiyah algebroid associated with the principal G-bundle, $p : Q \to Q/G$ (see [13]). The Lie algebroid structure on TQ/G induces, in a natural way, the linear Poisson structure on T^*Q/G. T^*Q/G is called the Sternberg-Weinstein phase space (see [17] for more details).

Recently, in the context of Lie algebroids de León, Marrero and Martínez in [6] have developed a Lagrangian and Hamiltonian formulation of Classical Mechanics. Given a Lie algebroid A over M one may define their A-tangent bundle to A and A^* as the prolongation of A over the projections of A and A^* on M, respectively (see [15] [6] for more details). In the particular case when the Lie algebroid is the tangent bundle to M they coincide with $T(TM)$ and $T(T^*M)$, respectively.

In the A-tangent bundle to A, the mathematical formulation of Lagrangian mechanics is developed in an analogous way as its classical formulation on the tangent bundle. Likewise, thanks to the existence of a linear Poisson structure on A^*, we can provide the A-tangent bundle to A^*, $\mathcal{T}^A A^*$, with a symplectic section and Hamiltonian mechanics is build there as in the cotangent bundle. Note that thanks to the fact that $\mathcal{T}^A A^*$ has a symplectic structure, we can consider Lagrangian Lie subalgebroids which may be very useful for the Hamilton-Jacobi theory, as in the classical case.

This is just the aim of this master thesis, describing the Lagrangian Lie subalgebroids of $\mathcal{T}^A A^*$ in order to apply the obtained results in the Hamilton-Jacobi theory for Hamiltonian systems on linear Poisson manifolds. We follow the scheme that [10] uses in the description of Lagrangian submanifolds on the cotangent bundle. Thus, the first step is to describe the local structure of Lagrangian Lie subalgebroids L of $\mathcal{T}^A A^*$ such that its base manifold C is fibered over a submanifold of M and after that consider the general case. In this master thesis we just consider the first case. Furthermore, as we will see, if L is fibered over C, then C is coisotropic on A^*. Therefore, for the local description of the Lagrangian Lie subalgebroids it will be necessary to give the local description of the coisotropic subbundles of A^*. Notice that A^*, in general, is not symplectic but Poisson, so we will need to use the coisotropic calculus in Poisson geometry (see [21]). Additionally, given that locally any fibered Lagrangian submanifold of T^*M is affine, we focus on the case in which C is an affine subbundle, and let the other cases for further work.

Summarizing, the aim of this master thesis is the local description of the Lagrangian Lie subalgebroids of $\mathcal{T}^A A^*$ which are fibered over an affine subbundle of A^*. The project is structured as follows. In Chapter 1 we discuss some basic aspects about the Lie algebroid theory and the Poisson geometry. In particular, we recall that there exists a one-to-one correspondence between Lie algebroid structures on a vector bundle and linear Poisson structures on the dual bundle. In Chapter 2, we present some basic aspects of the Lagrangian (respectively, coisotropic) calculus in symplectic (respectively, Poisson) geometry and we discuss the local structure of Lagrangian submanifolds of the cotangent bundle T^*M which are fibered over a submanifold N of M. We finish the chapter by proving that any Lagrangian Lie subalgebroid of $\mathcal{T}^A A^*$ is fibered over a coisotropic submanifold of A^*. This leads to Chapter 3 devoted to study the local structure of the coisotropic affine subbundles of A^*. Then, in Chapter 4 we obtain a local model for a certain type of Lagrangian Lie subalgebroids of $\mathcal{T}^A A^*$ which are fibered over coisotropic affine subbunldes of A^*. Up to our knowledge, the results in Chapter 3 and 4 are new in the literature although, it should be noted that, in [4] the author obtains a one-to-one correspondence between coisotropic submanifolds of a Poisson manifold M and Lagrangian Lie subalgebroids of the cotangent Lie algebroid T^*M. The Master thesis ends with our conclusions and description of future research directions.

Chapter 1

Linear Poisson structures and Lie Algebroids

One of the main objectives of this project is to study the Lagrangian subbundles of the A-tangent bundle to the dual bundle of a Lie algebroid A. To do so, it is essential to introduce the notion of a Lie algebroid and its prolongation over the projection τ_{A^*}, as well as present the basic concepts of the Lagrangian and coisotropic calculus on Poisson manifolds. We devote the first chapter to give the background on Poisson geometry and Lie algebroids, while the Lagrangian and coisotropic calculus will be introduced on the next chapter.

We begin by recalling the basic notions of Poisson geometry with special emphasis on linear Poisson structures (see [5], [10], [20]). Then we move on to the concept of a Lie algebroid and we introduce the linear Poisson structure on the dual bundle A^* of a Lie algebroid (see [6], [13], [15]). Finally, we end up by studying the prolongation of a Lie algebroid over the projection τ_{A^*} and focus on its Lie algebroid structure (see [6], [15]).

1.1 Poisson manifolds

Definition 1.1. *A smooth manifold M is said to be a* Poisson manifold *if there exists an operation $\{\ ,\ \}: \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ known as* Poisson bracket*, such that*

i. *$(\mathcal{C}^\infty(M), \{\ ,\ \})$ is a Lie algebra.*

ii. *$\{\ ,\ \}$ verifies the Leibniz rule, that is, $\{ff', g\} = f\{f', g\} + f'\{f, g\}$ for every $f, f', g \in \mathcal{C}^\infty(M)$.*

Such a bracket induces a 2-vector w by $\{f, g\} = w(df, dg)$ for every $f, g \in \mathcal{C}^\infty(M)$. An equivalent definition of a Poisson manifold is a pair (M, w) satisfying that $[w, w] = 0$, where $[\ ,\ ]$ denotes the Schouten-Nijenhuis bracket. If $(q^1, \cdots, q^m)$ are local coordinates on M one has that the local expresion of w is

$$w = \sum_{i<j} w_{ij} \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial q^j} \tag{1.1}$$

where $w_{ij} = w(dq^i, dq^j) = \{q^i, q^j\}$ for all $i, j = 1, \ldots, m$.

The 2-vector w induces a vector bundle morphism $\#_w : T^*M \to TM$ by

$$\beta_q(\#_w(\alpha_q)) = w(\alpha_q, \beta_q), \quad \text{for } \alpha_q, \beta_q \in T_q^*M \tag{1.2}$$

As in the symplectic case, in a Poisson manifold there is also a distingusihed type of vector fields known as *Hamiltonian vector fields* that have associated a potential function with them. Given $f \in \mathcal{C}^\infty(M)$, the Hamiltonian vector field X_f associated with f is defined by

$$X_f = \{\ , f\} \tag{1.3}$$

Or, if we denote again by $\#_w : \Omega^1(M) \to \mathcal{X}(M)$ the induced morphism of $\mathcal{C}^\infty(M)$-modules, the Hamiltonian vector fields are given by $X_f = -\#(df)$. From the Jacobi identity one easily has

$$[X_f, X_g] = -X_{\{f,g\}}. \tag{1.4}$$

Definition 1.2. *Given two Poisson manifolds* $(M_1, \{\ ,\ \}_1)$ *and* $(M_2, \{\ ,\ \}_2)$ *and a smooth map* $\varphi : M_1 \to M_2$*, then* φ *is said to be a* Poisson morphism *if* $\{f, g\}_2 \circ \varphi = \{f \circ \varphi, g \circ \varphi\}_1$ *for every* $f, g \in \mathcal{C}^\infty(M_2)$.

It is easy to see that an equivalent condition for φ to be a Poisson morphism is

$$w_1(q)(T^*_{\varphi(q)}\varphi(\alpha), T^*_{\varphi(q)}\varphi(\beta)) = w_2(\varphi(q))(\alpha, \beta) \tag{1.5}$$

for every $\alpha, \beta \in T^*_{\varphi(q)}M_2$ where w_1 and w_2 are the corresponding Poisson bivectors,.

Hamiltonian vector fields are always *Poisson vector fields*, that is, their flows consist on Poisson morphisms. Another usual characterization of a Poisson vector field X is $\mathcal{L}_X w = 0$, where $\mathcal{L}_X$ is the Lie derivative with respect to X.

Symplectic manifolds

A pair (M, ω) where M is a smooth manifold and ω is a 2-form is said to be a *symplectic manifold* if ω is closed and non-degegnerated. Remark that in such a case, ω induces a isomorphism of vector bundles $\flat_\omega : TM \to T^*M$ by

$$\flat_\omega(v) = i_v\omega(x) \ \text{ if } \ v \in T_xM$$

Given a smooth function $f \in \mathcal{C}^\infty(M)$, due to $\flat_\omega$ is an isomorphism, there also exists the notion of *Hamiltonian vector field* associated to a f as the vector field X_f given by $\flat_w(X_f) = df$. It allows us to define a Poisson bracket on M as follows:

$$\{f, g\} = \omega(X_f, X_g) \tag{1.6}$$

Thus, every symplectic manifold is a Poisson manifold. Moreover, since $\#_w = -\flat_\omega^{-1}$, we have that $\#_w$ is an isomorphism, so that, the Poisson structure is non-degegnerated . In fact, we have the following relation between Poisson and symplectic structures (see [14]).

Proposition 1.3. *A Poisson bracket* $\{\ ,\ \}$ *is non-degegnerated if, and only if,* $(M, \{\ ,\ \})$ *is a symplectic manifold.*

Symplectic foliation of a Poisson manifold

Let (M, w) be a Poisson manifold. Consider the *characteristic space* at point q given by $\#_{w(q)}(T^*_qM) = C_q$, or, analagously by $C_q = \{X_f(q) \mid f \in \mathcal{C}^\infty(M)\}$. Observe that the dimension of C_q coincides with the rank of $\#_w$ at q. Thus, the dimension of the characteristic space is always even. If rank$\#_{w(q)} = \dim M$ we say that $\#$ is *non-degenerate* at the point q. As well, if the rank of $\#_{w(q)}$ does not depend on the point we say that w is a *regular Poisson structure.*

The characteristic space C_q of a Poisson manifold M induces a generalized distribution which is generated by the Hamiltonian vector fields.We call it the *generalized distribution* of the Poisson manifold and we denote it by $\mathcal{C}$. Moreover we have the following result (see [20] for the details).

Theorem 1.4. *The characteristic distribution $\mathcal{C}$ of a Poisson manifold M is completely integrable and the Poisson structure induces a symplectic structure on each leaf.*

Proof. (Sketch) Using the generalized Frobenius theorem and some properties of the Hamiltonian vector fields one concludes that $\mathcal{C}$ is a foliation.
Let L be a leaf. We prove that $\{\ ,\ \}$ induces a Poisson structure on L as follows,

$$\begin{array}{rccc} \{\ ,\ \}_L: & \mathcal{C}^\infty(L)\times\mathcal{C}^\infty(L) & \to & \mathcal{C}^\infty(L) \\ & (f,g) & \to & \{f,g\}_L = \left\{\tilde{f},\tilde{g}\right\} \end{array}$$

where $\tilde{f},\tilde{g} \in \mathcal{C}^\infty(M)$ are such that $\tilde{f}_{|L} = f$ and $\tilde{g}_{|L} = g$. If $q \in L$ we have,

$$\{g,f\}_L(q) = X_{\tilde{f}}(\tilde{g})(q) = (\frac{d}{dt}_{|t=0}\varphi_q^{X_{\tilde{f}}}(t))(\tilde{g})$$

where $\varphi_q^{X_{\tilde{f}}}$ is the integral curve of the vector field $X_{\tilde{f}}$ with initial condition $q \in L$. In particular, if the initial condition is in L, the whole integral curve remains on L and $\{\ ,\ \}_L$ only depends on g. Likewise, we obtain that $\{\ ,\ \}_L$ only depends on f and $\{\ ,\ \}_L$ is well defined. Easy calculations prove that $\{\ ,\ \}_L$ is $\mathbb{R}$-bilinear, skew-symmetric and satisfies the Jacobi identity and the Leibniz rule. Thus, $\{\ ,\ \}_L$ is a Poisson bracket on L. Finally,

$$T_qL = \left\{X_{\tilde{f}}(q)\ |\tilde{f} \in \mathcal{C}^\infty(M)\right\} = \{X_f(q)\ |f \in \mathcal{C}^\infty(L)\} = \#_L(T_q^*L).$$

This proves that $\#_L : T_q^*L \to T_qL$ is an isomorphism. □

The leaves of the characteristic foliation are known as the *symplectic leaves* of the Poisson manifold. The symplectic form on a leaf L is given by

$$\omega(X,Y) = \beta(X) = \beta(\#_L(\alpha)) = w_L(q)(\alpha,\beta) \tag{1.7}$$

where $X = \#_L(\alpha)$ and $Y = \#_L(\beta)$ with $\alpha,\beta \in T^*M$.

Example 1.5. Given a connected smooth manifold M, the cotangent bundle T^*M is always a symplectic manifold and its characteristic foliation has a unique leaf, namely, T^*M.

Linear Poisson structures

Definition 1.6. [14] *A* linear Poisson structure *is a Poisson structure on a real vector space V such that for every couple of linear functions f and g defined on V, $\{f,g\}$ also is a linear function.*

From the local expression of the Poisson bracket we deduce that the condition of being a linear Poisson structure is equivalent to the local components of the Poisson 2-vector to be linear.

If f, g are two linear functions on a vector space V, then $f, g \in V^*$. Thus, if $\{\ ,\ \}$ is a linear Poisson structure on V it induces a Lie algebra structure over V^* by $\{f, g\} \in V^*$. Conversely, if $(V^*, [\ ,\ ])$ is a Lie algebra one can define a Poisson bracket by

$$\{f, g\}(\alpha) = \alpha([df(\alpha), dg(\alpha)]), \quad \text{for } f, g \in \mathcal{C}^\infty(V) \text{ and } \alpha \in V \tag{1.8}$$

It is immediate to see that $\{\ ,\ \}$ is linear, skew-symmetric and satisfies the Leibniz rule. Thus, there exists a 2-vector w on V^* such that $w(df, dg) = \{f, g\}$ for $f, g \in \mathcal{C}^\infty(V^*)$. To check that $\{\ ,\ \}$ is a Poisson bracket one only has to verify that $[w, w] = 0$

In short, there exists a natural bijection between the linear Poisson structures on a real vector space V of finite dimension and Lie algebra structures on the dual space V^*. In the case of V^* being the Lie algebra $\mathfrak{g}$ of a Lie group G, the linear Poisson structure is known as the *Lie-Poisson* structure.

1.2 Generalities on Lie Algebroids

In this section we present some generalities on Lie algebroids (for more details see [13]).

Definition 1.7. *A* Lie Algebroid *is a triple* $(\tau_A, [\![\ ,\ ]\!], \rho)$ *such that*

i. $\tau_A : A \to M$ *is a vector bundle.*

ii. $(\Gamma(A), [\![\ ,\ ]\!])$ *is a Lie algebra.*

iii. $\rho : A \to TM$ *is morphism of vector bundles, the* anchor map*, that induces a Lie algebra homomorphism* $\rho : \Gamma(A) \to \mathcal{X}(M)$ *satisfying the compatibility condition:*

$$[\![X_1, fX_2]\!] = f[\![X_1, X_2]\!] + \rho(X_1)(f)X_2 \ \text{ for } \ f \in \mathcal{C}^\infty(M),\ X_1, X_2 \in \Gamma(A) \tag{1.9}$$

In this context, as it is done in a manifold, it is posible to define an exterior algebra calculus. Sections of τ_A play the role of vector fields, and sections of the dual bundle $\tau_{A^*} : A^* \to M$ are like 1-forms. Likewise, the algebra $\bigoplus_k \Gamma(\Lambda^k A^*)$ plays the role of the algebra of the differential forms and it is possible to define a differential operator $d^A : \Gamma(\Lambda^k A^*) \to \Gamma(\Lambda^{k+1} A^*)$ as

$$\begin{aligned} d^A\phi(X_0, \cdots, X_k) = & \sum_{i=0}^{k} (-1)^i \rho(X_i)(\phi(X_0, \cdots, \hat{X}_i, \cdots, X_k)) \\ & + \sum_{i<j} (-1)^{i+j} \phi([\![X_i, X_j]\!], X_0, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_k) \end{aligned}$$

where $X_0, \cdots X_k \in \Gamma(A)$ and $\phi \in \Gamma(\Lambda^k A^*)$. From the properties of the Lie algebroid it follows that d^A is a cohomology operator (that is, $(d^A)^2 = 0$) and $d^A(\alpha \wedge \beta) = d^A\alpha \wedge \beta + (-1)^k \alpha \wedge d^A\beta$, for $\alpha \in \Gamma(\Lambda^k A^*)$ and $\beta \in \Gamma(\Lambda^r A^*)$. Conversely, it is posible to recover the Lie algebroid structure of A from the existence of an exterior differential on $\Gamma(\Lambda^\bullet A^*)$. Indeed, given a vector bundle $\tau_A : A \to M$ one can define the anchor and the Lie bracket as follows:

i. $\rho(X)f = (d^A f)(X)$ for $X \in \Gamma(A)$ and $f \in \mathcal{C}^\infty(M)$

ii. $i_{[\![X,Y]\!]}\theta = \rho(X)\theta(Y) - \rho(Y)\theta(X) - d^A\theta(X,Y)$ for $X, Y \in \Gamma(A)$ and $\theta \in \Gamma(A^*)$.

Given $X \in \Gamma(A)$, we can define an operator $\mathcal{L}^A_X : \Gamma(\Lambda^k A^*) \to \Gamma(\Lambda^k A^*)$ that plays the role of the Lie derivative by means of the Cartan identity:

$$\mathcal{L}^A_X \theta = i_X d^A \theta + d^A i_X \theta$$

for $\theta \in \Gamma(\Lambda^k A^*)$. One has the following identities:

i. $d^A \circ \mathcal{L}^A_X = \mathcal{L}^A_X \circ d^A$.

ii. $\mathcal{L}^A_X i_Y - i_X \mathcal{L}^A_Y = i_{[\![X,Y]\!]}$.

iii. $\mathcal{L}^A_X \mathcal{L}^A_Y - \mathcal{L}^A_Y \mathcal{L}^A_X = \mathcal{L}^A_{[\![X,Y]\!]}$.

We may consider two type of distinguished functions on a vector bundle. First, given a function $f \in \mathcal{C}^\infty(M)$ one may define a function $\tilde{f}$ on A by $\tilde{f} = f \circ \tau_A$. This type of functions are known as *basic functions*. Furthermore, every section θ of the dual bundle $\tau_{A^*} : A^* \to M$ may be regarded as a *linear function* $\hat{\theta}$ on A in the following sense

$$\hat{\theta}_{|A_q} = \theta(q), \quad \forall q \in M.$$

Remark that $\Omega^1(A)$ is locally generated by the differentials of basic and linear functions.

Supose that $(q^1, \cdots, q^m)$ are local coordinates on M and $\{e_1, \cdots, e_n\}$ is a local basis of sections of the bundle. Then, every $y \in A$ is expressed as $y = y^1 e_1(\tau_A(y)) + \cdots + y^n e_n(\tau_A(y))$, so that, we have local coordinates $\left\{q^i, y^\alpha\right\}^{1 \le \alpha \le n}_{1 \le i \le m}$ on A.
Once we have local coordinates, we define the *structure functions* ρ^i_α and $C^\gamma_{\alpha\beta}$ as

$$\rho(e_\alpha) = \rho^i_\alpha \frac{\partial}{\partial q^i}, \qquad [\![e_\alpha, e_\beta]\!] = C^\gamma_{\alpha\beta} e_\gamma \tag{1.10}$$

From 1.7 these functions must satisfy the *structure equations* written below:

$$\rho^j_\alpha \frac{\partial \rho^i_\beta}{\partial q^j} - \rho^j_\beta \frac{\partial \rho^i_\alpha}{\partial q^j} = \rho^i_\gamma C^\gamma_{\alpha\beta} \tag{1.11}$$

$$\sum_{\text{ciclic } \alpha,\beta,\gamma} \left(\rho^i_\alpha \frac{\partial C^\eta_{\beta\gamma}}{\partial q^i} + C^\eta_{\alpha\mu} C^\mu_{\beta\gamma} \right) = 0 \tag{1.12}$$

Hence,

$$d^A q^i = \rho^i_\alpha e^\alpha, \qquad d^A e^\alpha = -\frac{1}{2} C^\gamma_{\alpha\beta} e^\beta \wedge e^\gamma \tag{1.13}$$

and if, $\theta = \theta_\alpha e^\alpha$,

$$d^A f = \frac{\partial f}{\partial q^i} \rho^i_\alpha e^\alpha, \qquad d^A \theta = \left(\frac{\partial \theta_\gamma}{\partial q^i} \rho^i_\beta - \frac{1}{2} \theta_\alpha C^\gamma_{\alpha\beta} \right) e^\beta \wedge e^\gamma \tag{1.14}$$

where $\{e^\alpha\}$ is the dual basis of $\{e_\alpha\}$.

Example 1.8. i. **Tangent bundle** The standard example of a Lie algebroid is the tangent bundle of a manifold M. In this case, the space of sections is just the set of vector fields on M and the Lie algebra structure on $\Gamma(TM) \equiv \mathcal{X}(M)$ is induced by the standard Lie bracket of vector fields on M. The anchor map is the identity.

ii. **Lie algebra** Let $\mathfrak{g}$ be a Lie algebra and $M = \{q\}$ be a unique point. One has a Lie algebra structure on $\Gamma(\mathfrak{g})$ induced by the Lie algebra structure of $\mathfrak{g}$. Furthermore, $TM = \{0\}$ and one may consider the anchor map $\rho = 0$. Thus, $(\mathfrak{g}, [\ ,\ ], \rho)$ is a Lie algebroid over q.

iii. **Action Lie Algebroid** Let $\phi : G \times M \to M$ be a action of a Lie group G on a manifold M. It induces a Lie algebra antihomomorphism between $\mathfrak{g}$ and $\mathcal{X}(M)$ by

$$\begin{array}{ccc} \mathfrak{g} & \to & \mathcal{X}(M) \\ \xi & \to & \xi_M \end{array} \tag{1.15}$$

where ξ_M is the infinitesimal generator of the action corresponding to ξ. Consider the vector bundle $M \times \mathfrak{g} \to M$. The sections given by $\bar{\eta} : q \in M \to (q, \eta) \in M \times \mathfrak{g}$ for $\eta \in \mathfrak{g}$ span $\Gamma(M \times \mathfrak{g})$. Thus, we may define a Lie algebra structure on $\Gamma(M \times \mathfrak{g})$ as

$$[\![\bar{\eta}, \bar{\nu}]\!]_{M\times\mathfrak{g}}(q) = (q, [\eta, \nu]) = \overline{[\eta, \nu]}$$

and an anchor $\rho : M \times \mathfrak{g} \to TM$ by $\rho(q, \xi) = -\xi_M(q)$.

iv. **Atiyah Algebroid** Now assume that G acts free and properly on M and denote by $\pi : M \to \hat{M} = M/G$ the associated principal bundle. The tangent lift of the action gives a free an proper action of G on TM and $\widehat{TM} = TM/G$ is a quotient manifold. Thus, we can consider the fibration $\tau : \widehat{TM} \to \hat{M}$ given by $\tau([v_q]) = \pi(q)$. It can be proved that τ is a vector bundle whose fiber over a point $\pi(q) \in \hat{M}$ is isomorphic to T_qM.

Now, let us provide $\widehat{TM}$ with a Lie algebroid structure. First, it can be seen that the sections of $\widehat{TM}$ are identified with the Lie subalgebra of the G-invariant vector fields

$$\Gamma(\widehat{TM}) = \{X \in \mathcal{X}(M) \mid X \text{ is G-invariant}\} = \mathcal{X}(M)^G.$$

Thus, the bracket on $\widehat{TM}$ is just the bracket of vector fields. Furthermore the anchor $\rho : \widehat{TM} \to T\hat{M}$ is given by $\rho([v_q]) = T_q\pi(v_q)$ and since the G-invariant vector fields are π-projectable it follows that ρ is a Lie algebra homomorphism satisfying the compatibility condition.

Remark 1.9. Notice that from the definition of the Atiyah algebroid one may recover the Lie algebroid structure of the tangent bundle by choosing G to be the trivial group. As well, if we set $M = G$ and consider the left action of G on TG we recover the example of a Lie algebra. Furthermore, this example could also be recovered from the example of the action Lie algebroid choosing $M = \{q\}$.

The dual bundle

Given a Lie algebroid $(\tau_A, [\![\ ,\ ]\!], \rho)$, the dual bundle $\tau_{A^*} : A^* \to M$ is not in general a Lie algebroid. It is enough to consider the cotangent bundle of a manifold M. However, in the contagent bundle T^*M one has a symplectic structure in a natural way. In general, the dual bundle of a Lie algebroid is not necesarely symplectic, but Poisson. Let us construct its Poisson structure.

We have seen that given a vector bundle $\tau_A : A \to M$, there are two types of functions whose differentials span $\Omega^1(A)$. Thus, in the case of dual bundle $\tau_{A^*} : A^* \to M$, the basic functions that are given by $\tilde{f} = f \circ \tau_{A^*}$ for $f \in \mathcal{C}^\infty(M)$ and the linear functions defined by $\hat{X}(a^*) = a^*(X\tau_{A^*}(a^*))$ for $X \in \Gamma(A)$, $a^* \in A$, span $\Omega^1(A^*)$. Therefore, it suffices to define the Poisson bracket for that functions:

$$\{\tilde{f}, \tilde{g}\} = 0, \qquad \{\tilde{f}, \hat{X}\} = \widetilde{\rho(X)f}, \qquad \{\hat{X}, \hat{Y}\} = -\widehat{[\![X, Y]\!]} \tag{1.16}$$

It is an easy exercise to verify that it defines a Poisson bracket. Remark that $\{\ ,\ \}$ is not only a Poisson bracket, but a linear Poisson bracket.

If $\{q^i, y_\alpha\}$ are local coordinates on A^*, then

$$\{\tilde{q}^i, \tilde{q}^j\} = 0, \qquad \{\tilde{q}^i, \hat{y}_\alpha\} = \rho^i_\alpha, \qquad \{\hat{y}_\alpha, \hat{y}_\beta\} = -y_\gamma C^\gamma_{\alpha\beta} \tag{1.17}$$

This leads to the Poisson bivector

$$w = -\frac{1}{2} C^\gamma_{\alpha\beta} y_\gamma \frac{\partial}{\partial y_\alpha} \wedge \frac{\partial}{\partial y_\beta} + \rho^i_\alpha \frac{\partial}{\partial y_\alpha} \wedge \frac{\partial}{\partial q^i}. \tag{1.18}$$

Vertical and complete lifts

Let us introduce two canonical operations that we have on a Lie algebroid A. The first one is a consequence of A being a vector bundle and the second one is obtained using the Lie algebroid structure of A (for more details, see [6], [15]).

On the one hand, given a section $X \in \Gamma(A)$ we define its *vertical lift* as the vector field $X^v \in \mathcal{X}(A)$ given by

$$X^v(a) = X(q)^v_a, \text{ for } a \in A_q \tag{1.19}$$

where ${}^v_a : A_q \to T_a A_q$ is the vertical isomorphism given by

$$b^v_a = \frac{d}{dt}_{|t=0}(a + tb).$$

On the other hand, given $X \in \Gamma(A)$ we define its *complete lift to* A as the unique vector field $X^c \in \mathcal{X}(A)$ such that

i. $X^c(\tilde{f}) = \widetilde{\rho(X)(f)}$ for every $f \in \mathcal{C}^\infty(M)$.

ii. $X^c(\hat{\alpha}) = \widehat{\mathcal{L}^A_X \alpha}$ for every $\alpha \in \Gamma(A^*)$.

Remark that from the first condition it follows that X^c is τ_A-projectable to $\rho(X)$. With these definitions we have the following properties:

$$[X^c, Y^c] = [\![X, Y]\!]^c \qquad [X^c, Y^v] = [\![X, Y]\!]^v \qquad [X^v, Y^v] = 0 \tag{1.20}$$

for $X, Y \in \Gamma(A)$

Likewise, we can also define the *complete lift to* A^* as the unique vector field $X^{*c} \in \mathcal{X}(A^*)$ characterized by

i. $X^{*c}(\tilde{f}) = \widetilde{\rho(X)(f)}$ for every $f \in \mathcal{C}^\infty(M)$.

ii. $X^{*c}(\hat{Y}) = \widehat{\mathcal{L}_X^A Y}$ for every $Y \in \Gamma(A)$.

As before, the first condition implies that X^{*c} is τ_{A^*}-projectable to $\rho(X)$.

Finally, it is also possible to define a vertical lift on A^*. Indeed, given $\alpha \in \Gamma(A^*)$, its *vertical lift* is the vector field $\alpha^v \in \mathcal{X}(A^*)$ given by $\alpha^v(a^*) = \alpha(q)^v_{a^*}$, for $a^* \in A_q$, where the isomorphism $\,^v_a$ is defined in analogous way as before. Thus, we have defined vertical and complete lifts on A^* and properties (1.20) may be reformulated as follows:

$$[X^{*c}, Y^{*c}] = [\![X, Y]\!]^{*c} \qquad [X^c, \alpha^v] = (\mathcal{L}_X \alpha)^v \qquad [\alpha^v, \beta^v] = 0 \tag{1.21}$$

for $X, Y \in \Gamma(A)$ and $\alpha, \beta \in \Gamma(A^*)$.

Morphisms of Lie algebroids and subalgebroids

Given a morphism of vector bundles (F, f) between two Lie algebroids $(\tau_B, [\![\ ,\ ]\!]_B, \rho_B)$ and $(\tau_A, [\![\ ,\ ]\!]_A, \rho_A)$

$$\begin{array}{ccc} B & \xrightarrow{F} & A \\ {\scriptstyle \tau_B}\downarrow & & \downarrow{\scriptstyle \tau_A} \\ M & \xrightarrow[f]{} & M' \end{array}$$

(F, f) is said to be a *morphism of Lie algebroids* if

$$d^B((F, f)^*\theta) = (F, f)^*(d^A\theta) \text{ for } \theta \in \Gamma(\Lambda^k A^*)\ \forall k$$

where $(F, f)^*\theta$ is the section of $\Lambda^k B^* \to M$ given by

$$((F, f)^*\theta)(p)(a_1, \cdots, a_k) = \theta(f(p))(F(a_1), \cdots, F(a_k)),$$

for $a_1, \cdots, a_k \in B_p$ for $p \in M$.

In particular, a Lie algebroid morphism preserves the anchor and the bracket of projectable sections. In fact, an equivalent definition of morphism of Lie algebroids could be given in terms of the bracket and the anchor. For more details see [13]. In some cases, we also have a definition trought the dual bundle and its Poisson structure as follows.

Proposition 1.10. *Let $(\tau_A, [\![\ ,\ ]\!]_A, \rho_A)$ and $(\tau_B, [\![\ ,\ ]\!]_B,\ \rho_B)$ be Lie algebroids and (F, f) be morphism of vector bundles. Then, (F, f) is a Lie algebroid morphism if, and only if, f is a diffeomorphism and $(F, f)^*$ is a Poisson morphism.*

In particular, if A and B have the same basis and $f = Id$, the condition reduces to (F, Id) be a Poisson morphism.

In the particular case of $(F, f) = (j, i)$ being a monomorphism of vector bunldes with i an inmersion, we say that $(\tau_B, [\![\ ,\ ]\!]_B, \rho_B)$ is a Lie subalgebroid of $(\tau_A, [\![\ ,\ ]\!]_A, \rho_A)$. An alternative definition is obtained as follows.

Definition 1.11. *Let $(\tau_A, [\![\ ,\ ]\!]_A, \rho_A)$ be a Lie algebroid over M and N be a submanifold of M. A* Lie subalgebroid *of A over N is a vector subbundle B of A over N*

$$\begin{array}{ccc} B & \xrightarrow{j} & A \\ {\scriptstyle \tau_B}\downarrow & & \downarrow{\scriptstyle \tau_A} \\ N & \xrightarrow[i]{} & M \end{array}$$

such that,

i. $\rho_B = \rho_{A|B} : B \to TN$ *is well defined.*

ii. Given $X, Y \in \Gamma(B)$ and $\tilde{X}, \tilde{Y} \in \Gamma(A)$ extensions of X, Y respectively, we have that $([\![\tilde{X}, \tilde{Y}]\!]_A)_{|N} \in \Gamma(B)$.

Example 1.12. i. **Tangent bundle** Let N be a submanifold of M. Then, it follows easily that TN is a Lie subalgebroid of TM.

Another non trivial examples of Lie subalgebroids of the tangent bundle of a manifold are the foliations. In fact, a completely integrable distribution $\mathcal{F}$ on a manifold M equiped with the bracket of vector fields is a Lie algebroid since $\tau_{|\mathcal{F}} : \mathcal{F} \to M$ is a vector bundle and if $\mathcal{F}$ is a foliation, $(\Gamma(\mathcal{F}), [\ ,\ ])$ is a Lie algebra. Moreover, it is easy to prove that the inclusion $\mathcal{F} \to TM$ is a Lie algebroid monomorphism.

In a similar way, if N is a submanifold of M and $\mathcal{F}_N$ is a foliation on N then $\mathcal{F}_N$ is a Lie subalgebroid of $TM \to M$.

ii. **Lie algebra** Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ be a Lie subalgebra. If we consider the Lie algebroids induced by $\mathfrak{g}$ and $\mathfrak{h}$ over a point, then $\mathfrak{h}$ is a Lie subalgebroid of $\mathfrak{g}$.

iii. **Action Lie algebroid** Let $M \times \mathfrak{g} \to M$ be an action Lie algebroid and let N be a submanifold of M. Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$ such that the infinitesimal generators of the elements of $\mathfrak{h}$ are tangent to N, i.e,

$$\begin{array}{ccc} \mathfrak{h} & \to & \mathcal{X}(N) \\ \xi & \to & \xi_N \end{array}$$

is well defined. Thus, it follows that the action Lie algebroid $N \times \mathfrak{h} \to N$ is a Lie subalgebroid of $M \times \mathfrak{g} \to M$.

iv. **Atiyah algebroid** Suppose that the Lie group G acts free and properly on M and denote by $\pi : M \to \hat{M} = M/G$ the associated G-bundle. Let N be a G-invariant submanifold of M and $\mathcal{F}_N$ be a G-invariant foliation over N. We may consider the vector bundle $\widehat{\mathcal{F}_N} = \mathcal{F}_N/G \to \hat{N} = N/G$ and endow it with a Lie algebroid structure. The sections of $\widehat{\mathcal{F}_N}$ are

$$\Gamma(\widehat{\mathcal{F}_N}) = \{X \in \mathcal{X}(N) \mid X \text{ is } G\text{-invariant and } X(q) \in \mathcal{F}_N(q)\ \forall q \in N\}.$$

Thus, the standard bracket of vector fields on N induces a Lie algebra structure on $\Gamma(\widehat{\mathcal{F}_N})$. The anchor map is the canonical inclusion of $\widehat{\mathcal{F}_N}$ on $T\hat{N}$ and $\widehat{\mathcal{F}_N}$ is a Lie subalgebroid of $\widehat{TM} = TM/G \to \hat{M} = M/G$.

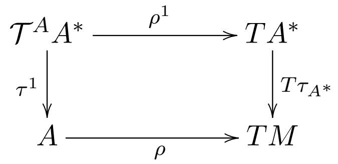
1.3 A-tangent bundle of the dual bundle of a Lie algebroid

In this section we present some results of the A tangent bundle to A^*, A being a Lie algebroid (for more details see [6]).

Let $(\tau_A, [\![\ ,\ ]\!], \rho)$ be a Lie algebroid over M and consider the vector bundle over A^* given by

$$\mathcal{T}^A A^* = \bigcup_{\alpha \in A^*} T^A_\alpha A^* = \bigcup_{\alpha \in A^*} \{(a, v) \in A \times T_\alpha A^* \mid \rho(a) = T_\alpha \tau_{A^*}(v)\}$$

where $\tau_{A^*} : A^* \to M$ is the canonical projection. $\mathcal{T}^A A^*$ is known as the *prolongation of* A *over* τ_{A^*}, or, in short, the *A-tangent bundle to* A^*. If we denote by $\tau^1 : \mathcal{T}^A A^* \to A$ and $\rho^1 : \mathcal{T}^A A^* \to TA^*$ the projections on the first and the second factor respectively, then the following square is commutative



We anticipate that ρ^1 is the anchor that endows $\mathcal{T}^A A^*$ with a Lie algebroid structure.

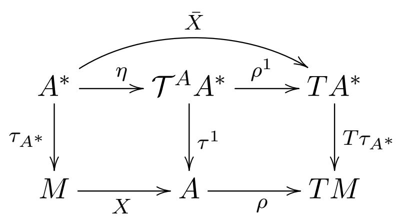
If rank $A = n$, it is easy to see that $\mathcal{T}^A A^*$ is a vector bundle of rank $2n$. Let (q^i, y_α) be local coordinates on A^* induced by a local basis of sections $\{e^\alpha\}$, then

$$\tilde{e}_\alpha = (e_\alpha \circ \tau_{A^*}, \rho^i_\alpha \frac{\partial}{\partial q^i}) \qquad\qquad \bar{e}_\alpha = (0, \frac{\partial}{\partial y_\alpha}) \tag{1.22}$$

are a local basis of sections of $\Gamma(\mathcal{T}^A A^*)$, where $\{e_\alpha\}$ is the dual basis of $\{e^\alpha\}$. Thus, any $w \in \mathcal{T}^A A^*$ is written as $w = z^\alpha \tilde{e}_\alpha(\tau(w)) + v_\alpha \bar{e}_\alpha(\tau(w))$ with $\tau : \mathcal{T}^A A^* \to A^*$ the vector bundle projection. We have local coordinates $(q^i, y_\alpha; z^\alpha, v_\alpha)$ on $\mathcal{T}^A A^*$ where (q^i, y_α) are the coordinates on A^* of $\tau(w)$.

Sections and lifts on the A-tangent bundle to A^*

We say that a section $\eta \in \Gamma(\mathcal{T}^A A^*)$ is *projectable* if there exists a section $X \in \Gamma(A)$ and a vector field $\bar{X} \in \mathcal{X}(A^*)$ τ_{A^*}-projectable over $\rho(X)$, such that $\eta = (X \circ \tau_{A^*}, \bar{X})$. The following diagram illustrates the above situation.



Using the vertical and complet lifts on A^* one may define projectable sections of $\mathcal{T}^A A^*$. Indeed, given a section $\alpha \in \Gamma(A^*)$ one may define its *vertical lift on* $\mathcal{T}^A A^*$ as the section $\alpha^{\mathbf{v}} \in \Gamma(\mathcal{T}^A A^*)$ given by

$$\alpha^{\mathbf{v}} = (0, \alpha^v). \tag{1.23}$$

If $\alpha \in \Gamma(A^*)$, $X \in \Gamma(A)$ and $f \in \mathcal{C}^\infty(M)$, we have

$$\rho^1(\alpha^{\mathbf{v}})(\tilde{f}) = 0 \qquad\qquad \rho^1(\alpha^{\mathbf{v}})(\hat{X}) = \widetilde{\alpha(X)} \tag{1.24}$$

Similarly, given a section $X \in \Gamma(A)$, one may define its *complete lift on* $\mathcal{T}^A A^*$ as the section $X^{*\mathbf{c}} \in \Gamma(\mathcal{T}^A A^*)$ given by

$$X^{*\mathbf{c}} = (X \circ \tau_{A^*}, X^{*c}). \tag{1.25}$$

If $X, Y \in \Gamma(A)$ and $f \in \mathcal{C}^\infty(M)$, we have

$$\rho^1(X^{*\mathbf{c}})(\tilde{f}) = \widetilde{\rho(X)f} \qquad\qquad \rho^1(X^{*\mathbf{c}})(\hat{Y}) = \widehat{[\![X, Y]\!]} \tag{1.26}$$

Let $\theta \in \Gamma(A^*)$ and $X \in \Gamma(A)$ be sections such that in the previous local system of coordinates are expressed as

$$\theta = \theta_\alpha e^\alpha \qquad\qquad X = X^\alpha e_\alpha,$$

then,

$$\theta^{\mathbf{v}} = \theta_\alpha \bar{e}_\alpha \qquad\qquad X^{*\mathbf{c}} = X^\alpha \tilde{e}_\alpha - \left(\rho^i_\alpha \frac{\partial X^\beta}{\partial q^i} y_\beta + C^\gamma_{\alpha\beta} y_\gamma X^\beta\right) \bar{e}_\alpha$$

It enables us to ensure that given a local basis of sections X_i of $\Gamma(A)$ and its dual basis α_i of $\Gamma(A^*)$, their complete and vertical lifts are a local basis of $\Gamma(\mathcal{T}^A A^*)$. In particular there always exists a local basis of $\Gamma(\mathcal{T}^A A^*)$ of projectable sections.

Lie algebroid structure of $\mathcal{T}^A A^*$

In order to turn $\mathcal{T}^A A^*$ into a Lie algebroid we have to define an anchor and a bracket of sections such that the conditions in 1.7 hold. We have already mentioned that the projection into the second member $\rho^1 : \mathcal{T}^A A^* \to TA^*$ will be the anchor of $\mathcal{T}^A A^*$. Futhermore, since there always exists a local basis of $\Gamma(\mathcal{T}^A A^*)$ of projectable sections, it is enough to define the bracket for such sections. Given two projectable sections $(X, \bar{X}), (Y, \bar{Y}) \in \Gamma(\mathcal{T}^A A^*)$, we define

$$[\![(X, \bar{X}), (Y, \bar{Y})]\!]_{\mathcal{T}^A A^*} = ([\![X, Y]\!], [\bar{X}, \bar{Y}]). \tag{1.27}$$

Remark that $[\![\ ,\ ]\!]_{\mathcal{T}^A A^*}$ is well defined and endows $\Gamma(\mathcal{T}^A A^*)$ with a Lie algebra structure. Indeed,

$$\rho([\![X, Y]\!]) = [\rho(X), \rho(Y)] = [T\tau_{A^*}(\bar{X}), T\tau_{A^*}(\bar{Y})] = T\tau_{A^*}[\bar{X}, \bar{Y}]$$

proves that is well defined and from the properties of $[\![\ ,\ ]\!]$ and $[\ ,\ ]$ it follows that $(\Gamma(\mathcal{T}^A A^*), [\![\ ,\ ]\!]_{\mathcal{T}^A A^*})$ is a Lie algebra. In addition, from (1.21) we deduce that

$$[\![X^{*\mathbf{c}}, Y^{*\mathbf{c}}]\!]_{\mathcal{T}^A A^*} = [\![X, Y]\!]^{*\mathbf{c}} \qquad [\![X^{*\mathbf{c}}, \alpha^{\mathbf{v}}]\!]_{\mathcal{T}^A A^*} = (\mathcal{L}^A_X \alpha)^{\mathbf{v}} \qquad [\![\alpha^{\mathbf{v}}, \beta^{\mathbf{v}}]\!]_{\mathcal{T}^A A^*} = 0 \tag{1.28}$$

for $X, Y \in \Gamma(A)$ and $\alpha, \beta \in \Gamma(A^*)$

Now, let us see that ρ^1 is a Lie algebra homomorphism and verifies the compatibility condition. On the one hand, if $f \in \mathcal{C}^\infty(M)$ and $(X, \bar{X}), (Y, \bar{Y}) \in \Gamma(\mathcal{T}^A A^*)$ are projectable sections

$$\begin{aligned}
[\![(X, \bar{X}), f(Y, \bar{Y})]\!]_{\mathcal{T}^A A^*} =& ([\![X, fY]\!]_A, [\bar{X}, \tilde{f}\bar{Y}]) \\
&= (f[\![X, Y]\!]_A + \rho(X)(f)Y, \tilde{f}[\bar{X}, \bar{Y}] + T\tau_{A^*}\bar{X}(f)\bar{Y}) \\
&= f[\![(X, \bar{X}), (Y, \bar{Y})]\!]_{\mathcal{T}^A A^*} + \rho^1((X, \bar{X}))(f)(Y, \bar{Y})
\end{aligned}$$

On the other hand,

$$\rho^1([\![(X,\bar{X}),(Y,\bar{Y})]\!]_{\mathcal{T}^A A^*}) = [\bar{X},\bar{Y}] = [\rho^1((X,\bar{X})),\rho^1((Y,\bar{Y}))]$$

proves that ρ^1 defines an anchor map on $\mathcal{T}^A A^*$.

If $\{\tilde{e}_\alpha, \bar{e}_\alpha\}$ are the local basis of $\Gamma(\mathcal{T}^A A^*)$ given in (1.22) it follows easily

$$\rho^1(\tilde{e}_\alpha) = \rho^i_\alpha \frac{\partial}{\partial q^i} \qquad \rho^1(\bar{e}_\alpha) = \frac{\partial}{\partial y_\alpha} \tag{1.29}$$

and

$$[\![\tilde{e}_\alpha, \tilde{e}_\beta]\!]_{\mathcal{T}^A A^*} = C^\gamma_{\alpha\beta}\tilde{e}_\gamma \qquad [\![\tilde{e}_\alpha, \bar{e}_\beta]\!]_{\mathcal{T}^A A^*} = 0 \qquad [\![\bar{e}_\alpha, \bar{e}_\beta]\!]_{\mathcal{T}^A A^*} = 0 \tag{1.30}$$

where ρ^i_α and $C^\gamma_{\alpha\beta}$ are the structure functions of A. Thus,

$$d^{\mathcal{T}^A A^*} f = \rho^i_\alpha \frac{\partial f}{\partial q^i}\tilde{e}^\alpha + \frac{\partial f}{\partial y_\alpha}\bar{e}^\alpha \qquad d^{\mathcal{T}^A A^*}\tilde{e}^\gamma = -\frac{1}{2}C^\gamma_{\alpha\beta}\tilde{e}^\alpha \wedge \tilde{e}^\beta \qquad d^{\mathcal{T}^A A^*}\bar{e}^\gamma = 0 \tag{1.31}$$

for $f \in \mathcal{C}^\infty(A^*)$ and $\{\tilde{e}^\alpha, \bar{e}^\alpha\}$ the dual basis of $\{\tilde{e}_\alpha, \bar{e}_\alpha\}$.

Examples 1.13. i. **Tangent bundle** In the case of $A = TM$ one may identify $\mathcal{T}^A A^*$ with $T(T^*M)$ with the standard Lie algebroid structure.

ii. **Lie algebra** Let $\mathfrak{g}$ be a real Lie algebra of finite dimension. Then, $\mathfrak{g}$ is a Lie algebroid over a single point $M = \{q\}$. Moreover, using that the anchor map of $\mathfrak{g}$ is zero, it is easy to prove that $\mathcal{T}^{\mathfrak{g}}\mathfrak{g}^*$ may be identificate with the trivial vector bundle pr_1 : $\mathfrak{g}^* \times (\mathfrak{g} \times \mathfrak{g}^*) \to \mathfrak{g}^*$. Under this identification the anchor map is given by

$$\begin{array}{rccc} \rho^1: & \mathfrak{g}^* \times (\mathfrak{g} \times \mathfrak{g}^*) & \to & T\mathfrak{g}^* \cong \mathfrak{g}^* \times \mathfrak{g}^* \\ & (\mu,(\xi,\alpha)) & \to & (\mu,\alpha) \end{array}$$

and the Lie bracket of two constant sections $(\xi,\alpha),(\eta,\beta) \in \mathfrak{g} \times \mathfrak{g}^*$ is

$$[\![(\xi,\alpha),(\eta,\beta)]\!]_{\mathcal{T}^{\mathfrak{g}}\mathfrak{g}^*} = ([\xi,\eta],0).$$

iii. **Action Lie Algebroid** Let $A = M \times \mathfrak{g} \to M$ be an action Lie algebroid over M. If $(q,\mu) \in M \times \mathfrak{g}^*$ it follows that

$$\mathcal{T}^A_{(q,\mu)}A^* = \{((q,\eta),(X_q,\alpha)) \in M \times \mathfrak{g} \times T_qM \times \mathfrak{g}^* \mid -\eta_M(q) = X_q\} \cong \mathfrak{g} \times \mathfrak{g}^*.$$

Then, $\mathcal{T}^A A^*$ may be identified with the trivial vector bundle $(M \times \mathfrak{g}^*) \times (\mathfrak{g} \times \mathfrak{g}^*) \to M \times \mathfrak{g}^*$. Under this identification, the anchor map $\rho^1 : (M \times \mathfrak{g}^*) \times (\mathfrak{g} \times \mathfrak{g}^*) \to TM \times T\mathfrak{g}^* \cong TM \times (\mathfrak{g}^* \times \mathfrak{g}^*)$ is given by

$$\rho^1((q,\mu)(\xi,\alpha)) = (-\xi_M(q),\mu,\alpha).$$

Moreover, the Lie bracket of two constant sections $(\xi,\alpha),(\eta,\beta) \in \mathfrak{g} \times \mathfrak{g}^*$ is

$$[\![(\xi,\alpha),(\eta,\beta)]\!]_{\mathcal{T}^A A^*} = ([\xi,\eta],0).$$

iv. **Atiyah algebroid** Let us describe the A-tangent bundle to A^* in the case of A being an Atiyah algebroid induced by a trivial principal G-bundle $\pi : G \times M \to M$. In such case, by left trivialization we have that the Atiyah algebroid is the vector bundle $\tau : \mathfrak{g} \times TM \to M$. Thus, if $X \in \mathcal{X}(M)$ and $\xi \in \mathfrak{g}$ then we may consider the sections $X^\xi : M \to \mathfrak{g} \times TM$ of the Atiyah algebroid given by $X^\xi(q) = (\xi, X(q))$, for $q \in M$. Moreover, if $(\llbracket\ ,\ \rrbracket, \rho)$ is the Lie algebroid structure on the Atiyah algebroid, we have that,

$$\llbracket X^\xi, Y^\eta \rrbracket = ([X, Y]_{TM}, [\xi, \eta]_{\mathfrak{g}}) = [X, Y]^{[\xi,\eta]}$$

and $\rho(X^\xi) = X$. On the other hand, if $(\mu, \beta_q) \in \mathfrak{g}^* \times T_q^*M$ then the fiber of $\mathcal{T}^A A^*$ over (μ, β_q) is

$$\mathcal{T}^A_{(\mu,\beta_q)} A^* = \left\{ \left((\eta, u_q), (\alpha, X_{\beta_q}) \right) \in \mathfrak{g} \times T_qM \times \mathfrak{g}^* \times T_{\beta_q}(T^*M) \mid u_q = T_{\beta_q}\tau(X_{\beta_q}) \right\}.$$

This implies that $\mathcal{T}^A_{(\mu,\beta_q)} A^*$ may be identified with the vector space $(\mathfrak{g} \times \mathfrak{g}^*) \times T_{\beta_q}(T^*M)$. Thus, the Lie algebroid $\mathcal{T}^A A^*$ may be identified with the vector bundle $\mathfrak{g}^* \times (\mathfrak{g} \times \mathfrak{g}^*) \times T(T^*M) \to \mathfrak{g}^* \times T^*M$ whose vector bundle projection is

$$(\mu, ((\xi, \alpha), X_{\beta_q})) \to (\mu, \beta_q)$$

for $(\mu, ((\xi, \alpha), X_{\beta_q})) \in \mathfrak{g}^* \times (\mathfrak{g} \times \mathfrak{g}^*) \times T(T^*M)$. Therefore, if $(\xi, \alpha) \in \mathfrak{g} \times \mathfrak{g}^*$ and $X \in \mathcal{X}(T^*M)$ then one may consider the section $((\xi, \alpha), X)$ given by

$$((\xi, \alpha), X)(\mu, \beta_q) = (\mu, ((\xi, \alpha), X(\beta_q))), \text{ for } (\mu, \beta_q) \in \mathfrak{g}^* \times T_q^*M.$$

Moreover,

$$\llbracket ((\xi, \alpha), X), ((\eta, \beta), Y) \rrbracket = (([\xi, \eta], 0), [X, Y])$$

and the anchor map $\rho^1 : \mathfrak{g}^* \times (\mathfrak{g} \times \mathfrak{g}^*) \times T(T^*M) \to \mathfrak{g}^* \times \mathfrak{g}^* \times T(T^*M)$ is defined as

$$\rho^1(\mu, ((\xi, \alpha), X)) = ((\mu, \alpha), X)$$

Remark 1.14. We saw that in the case of A being the tangent bundle to a manifold M, the A-tangent bundle to A^* coincides with $T(T^*M)$. Nevertheless, it is also true that $T(TM)$ admits a Lie algebroid structure. In fact, generally, given a Lie algebroid A, one may define the prolongation of the Lie algebroid over the projection τ_A in the very same way we did for τ_{A^*},

$$\mathcal{T}^A A = \{(a, v) \in A \times TA \mid \rho(a) = T\tau_A(v)\}$$

$\mathcal{T}^A A$ is known as the *A-tangent bundle to A* and can be also endowed with a Lie algebroid structure. For more details on such a construction we remit the reader to [15].

The construction of the A-tangent bundle to A and A^* are just particular cases of the more general theory of prolongation of a Lie algebroid over a fibration than can be found in [6], [11].

Symplectic section of $\mathcal{T}^A A^*$

As we know, if A is the standard Lie algebroid $\tau_M : TM \to M$, then the A-tangent bundle to A^* is just the standard Lie algebroid $\tau_{T^*M} : T(T^*M) \to T^*M$. It is well known that $T(T^*M)$ admits a symplectic vector bundle structure that inherits from the symplectic structure of T^*M. We are going to see that such a structure that $T(T^*M)$ has in a natural way is not an special case, but $\mathcal{T}^A A^*$ can be always endowed with a symplectic vector bundle structure with a procedure similar to that used for $T(T^*M)$.

Definition 1.15. *The* Liouville section *of the A-tangent bundle to A^*, $\mathcal{T}^A A^*$, is the section λ defined as*

$$\lambda(\alpha)(X) = \alpha(\tau^1(X)), \quad \forall \alpha \in A^* \text{ and } X \in \mathcal{T}^A A^*$$

Notice that for every $X \in \Gamma(A)$ and $a^* \in A^*$,

$$\lambda(X^{*\mathbf{c}})(a^*) = a^*(\tau^1(X^{*\mathbf{c}}(a^*))) = a^*(X(\tau_{A^*}(a^*))) = \hat{X}(a^*) \tag{1.32}$$

so, $\lambda(X^{*\mathbf{c}}) = \hat{X}$. Likewise, it is easy to see that for every $\alpha \in \Gamma(A^*)$,

$$\lambda(\alpha^{\mathbf{v}}) = 0. \tag{1.33}$$

Now, in an analogous way that the canonical symplectic form is defined from the Liouville 1-form on the cotangent bundle, we introduce the 2-section Ω on $\mathcal{T}^A A^*$ as

$$\Omega = -d^{\mathcal{T}^A A^*}\lambda \tag{1.34}$$

which is symplectic as proves the following result.

Proposition 1.16. [5] *Ω is a non-degegnerated 2-section of $\mathcal{T}^A A^*$ such that $d^{\mathcal{T}^A A^*}\Omega = 0$.*

Proof. It is enough to notice that if $\{\tilde{e}^\alpha, \bar{e}^\alpha\}$ denotes the dual basis of $\{\tilde{e}_\alpha, \bar{e}_\alpha\}$ induced by the local coordinates (q^i, y_α) described in (1.22), then

$$\lambda(q^i, y_\alpha) = y_\alpha \tilde{e}^\alpha$$

so that,

$$\Omega = \tilde{e}^\alpha \wedge \bar{e}^\alpha + \frac{1}{2} C^\gamma_{\alpha\beta} \tilde{e}^\alpha \wedge \tilde{e}^\beta.$$

Now, it is straightforward to check that Ω is non-degegnerated and $d^{\mathcal{T}^A A^*}\Omega = 0$. □

From (1.34),(1.32) and (1.33) we have that

$$\Omega(X^{*\mathbf{c}}, Y^{*\mathbf{c}}) = -\widehat{[\![X, Y]\!]} \qquad \Omega(X^{*\mathbf{c}}, \alpha^{\mathbf{v}}) = \widetilde{\alpha(X)} \qquad \Omega(\alpha^{\mathbf{v}}, \beta^{\mathbf{v}}) = 0 \tag{1.35}$$

for $X, Y \in \Gamma(A)$ and $\alpha, \beta \in \Gamma(A^*)$.

In the same way that the canonical symplectic structure of the cotangent bundle allows us to define Hamiltonian vector fields, the symplectic section we just defined allows us to define Hamiltonian sections of $\mathcal{T}^A A^*$. Indeed, let $f : A^* \to \mathbb{R}$ be a Hamiltonian function, then, since Ω is non-degegnerated, there exists a unique section $\mathcal{H}_f$ verifying

$$i_{\mathcal{H}_f}\Omega = d^{\mathcal{T}^A A^*} f. \tag{1.36}$$

If $X, Y \in \Gamma(A)$ and $\alpha \in \Gamma(A^*)$ then, from (1.24), (1.26) and (1.35) it follows that

$$\Omega(\mathcal{H}_{\hat{X}}, Y^{*\mathbf{c}}) = d^{\mathcal{T}^A A^*}\hat{X}(Y^{*\mathbf{c}}) = \rho^1(Y^{*\mathbf{c}})\hat{X} = -\widehat{[\![X, Y]\!]}$$
$$\Omega(\mathcal{H}_{\hat{X}}, \alpha^{\mathbf{v}}) = d^{\mathcal{T}^A A^*}\hat{X}(\alpha^{\mathbf{v}}) = \rho^1(\alpha^{\mathbf{v}})\hat{X} = \widetilde{\alpha(X)}.$$

Hence, $\mathcal{H}_{\hat{X}} = X^{*\mathbf{c}}$. Likewise one proves that $\mathcal{H}_{\tilde{f}} = (-d^A f)^{\mathbf{v}}$.

Once we have the expression of the Hamiltonian sections of the basic and linear functions on A^* it is posible to recover the linear Poisson structure of A^* from the symplectic section in $\mathcal{T}^A A^*$. Indeed, using (1.35) and (1.16), we conclude that

$$\{f, g\} = \Omega(\mathcal{H}_f, \mathcal{H}_g) \tag{1.37}$$

In the local basis of sections $\{\tilde{e}_\alpha, \bar{e}_\alpha\}$ of $\Gamma(\mathcal{T}^A A^*)$ we have that the local expression of the Hamiltonian sections is

$$\mathcal{H}_f = \frac{\partial f}{\partial y_\alpha}\tilde{e}_\alpha + \left(C^\gamma_{\alpha\beta}\frac{\partial f}{\partial y_\beta} + \rho^i_\alpha \frac{\partial f}{\partial q^i}\right)\bar{e}_\alpha.$$

Thus, the vector field $\rho^1(\mathcal{H}_f)$ is

$$\rho^1(\mathcal{H}_f) = \rho^i_\alpha \frac{\partial f}{\partial y_\alpha}\frac{\partial}{\partial q^i} + \left(C^\gamma_{\alpha\beta}\frac{\partial f}{\partial y_\beta} + \rho^i_\alpha \frac{\partial f}{\partial q^i}\right)\frac{\partial}{\partial y_\alpha}$$

that is, the Hamiltonian vector field X_f associated to f with respect to the linear Poisson structure of A^*.

Example 1.17. Next we obtain explicit expressions of the symplectic section of the A-tangent bundle to A^* when A is an action Lie algebroid or an Atyiah algebroid. The other examples are just particular cases.

i. **Action Lie algebroid** Let $A = \mathfrak{g} \times M \to M$ be an action Lie algebroid and λ be the Liouville section of $\mathcal{T}^A A^* \cong (M \times \mathfrak{g}^*) \times (\mathfrak{g} \times \mathfrak{g}^*)$. Then, for $(q, \mu) \in M \times \mathfrak{g}^*$ and $(\eta, \alpha) \in \mathfrak{g} \times \mathfrak{g}^*$,

$$\lambda(q, \mu)(\eta, \alpha) = \mu(\eta).$$

Thus, the symplectic section Ω is

$$\begin{aligned}\Omega(q, \mu)\left((\eta, \alpha), (\nu, \beta)\right) &= -\left(-\rho^1(\eta, \alpha)\lambda(\nu, \beta)(q, \mu)\right.\\ &\quad \left.+\rho^1(\nu, \beta)\lambda(\eta, \alpha)(q, \mu) - \lambda([\eta, \nu], 0)(q, \mu)\right)\\ &= \alpha(\nu) - \beta(\eta) + \mu([\eta, \nu]).\end{aligned}$$

ii. **Atiyah algebroid** Let $A = \mathfrak{g} \times TM \to M$ be an Atiyah algebroid and λ be the Liouville section of $\mathcal{T}^A A^* \cong \mathfrak{g}^* \times (\mathfrak{g} \times \mathfrak{g}^*) \times T(T^*M) \to \mathfrak{g}^* \times T^*M$. Then,

$$\lambda(\mu, \beta_q)((\eta, \alpha), X) = \mu(\eta) + \beta_q(T\pi(X))$$

for $(\mu, \beta_q) \in \mathfrak{g}^* \times T^*_q M$ and $((\eta, \alpha), X) \in \mathcal{T}^A_{(\mu, \beta_q)} A^*$.

Remark that $\lambda = \lambda_{\mathfrak{g}^*} + \lambda_M$ where $\lambda_{\mathfrak{g}^*}$ is the Liouville section of the algebroid $\mathfrak{g}$ and λ_M is the Liouville 1-form of the cotangent bundle of M. Hence,

$$\Omega = -d^{\mathcal{T}^A A^*}\lambda_{\mathfrak{g}^*} - d^{\mathcal{T}^A A^*}\lambda_M = \Omega_{\mathfrak{g}^*} + \omega \tag{1.38}$$

Remark 1.18. [Mechanics on Lie Algebroids] The Hamiltonian formalism on Classical Mechanics is developed on the cotagent bundle of a manifold, as well as, the Lagrangian formalism is developed on the tangent bundle. As we have seen, in the context of Lie algebroids the A-tangent bundle to A^* plays the role of $T(T^*M)$ and the A-tangent bundle to A plays the role of $T(TM)$. Nevertheless, the parallelism is even larger as it is also possible to develop the Hamiltonian and Lagrangian formalism in $\mathcal{T}^A A^*$ and $\mathcal{T}^A A$ respectively.

On the one hand, we defined the symplectic section of $\mathcal{T}^A A^*$ and we introduced the Hamiltonian sections. It allows us to give a Hamiltoninan description of the mechanics as it is done in [6]. On the other hand, as it is done in [15] one can work on the A-tangent bundle to A, to introduce the Liouville section and the vertical endomorphism and to develop the Lagrangian formalism. Further, in [6] the authors define a Legendre transformation that links both formalisms.

Chapter 2

Lagrangian and coisotropic calculus in symplectic and Poisson geometry

This chapter is dedicated to motivate and understand the object of study of this master thesis. We first recall the definitions of an isotropic (respectively, coisotropic and Lagrangian) submanifolds of a symplectic manifold and give some basic properties in order to understand the following results. We turn then to study the local structure of the Lagrangian submanifolds of cotangent bundle which are fibered over a submanifold of the basis. This is a known result that can be found in [10] for example, which will be our starting point to describe the local structure of the Lagrangian Lie subalgebroids of the A^*-tangent bundle.

In the second part of the chapter we redefine the notion of a coisotropic submanifold of a Poisson manifold and we apply this theory to the particular case when the Poisson manifold is the dual bundle to a Lie algebroid. We end up by proving a first result on Lagrangian Lie subalgebroids which states that the base manifold of every Lagrangian Lie subalgebroid of the A-tangent bundle to A^* is coisotropic in A^*.

2.1 Lagrangian calculus in symplectic geometry

Recall that a *symplectic manifold* is a pair (M, ω), where M is a smooth manifold and ω is 2-form which is closed and non-degenerate, or, equivalently, the morphism of vector bundles $\flat_\omega : TM \to T^*M$ induced by ω,

$$\flat_\omega(v) = i_v\omega(x) \text{ if } v \in T_xM$$

is an isomorphism. By abuse of notation we denote again by $\flat_\omega : \mathcal{X}(M) \to \Omega^1(M)$ the corresponding isomorphism of $\mathcal{C}^\infty(M)$-modules. As we know, the *Hamiltonian vector field* X_f associated with a real $\mathcal{C}^\infty$-function f is given by $\flat_\omega(X_f) = df$.

Given two symplectic manifolds (M_1, ω_1) and (M_2, ω_2) we say that a diffeomorphism $f : M_1 \to M_2$ is a *symplectomorphism* if $f^*\omega_2 = \omega_1$. In particular, the cotangent lift of a diffeomorphism is always a symplectomorphism. We remark that if $f : M_1 \to M_2$ is a symplectomorphism and we consider the non-degenerate Poisson structures on M_1 and M_2 then f is a Poisson isomorphism (see Section 1.1).

Likewise, a vector field is said to be a *symplectic vector field* if its flow consists on local symplectomorphisms. Another usual characterizations of a symplectic vector field X are

i. $\mathcal{L}_X\omega = 0$.

ii. $d(\flat_\omega(X)) = 0$.

In particular, any Hamiltonian vector field is a symplectic vector field and it is clear that symplectic vector fields are Poisson vector fields.

Symplectic vector spaces

One of the goals of this first part is to overcome to the definitions of isotropic, coisotropic and Lagrangian submanifolds of a symplectic manifold. While the notion of symplectic manifold is well known, it can not be said the same, in general, about the other concepts, so we need to introduce them carefully.

Recall that the notion of a symplectic manifold is a generalization of the definition of a symplectic vector space, which roughly speaking, "glue" symplectic vector spaces togheter in a bundle. The submanifolds of a symplectic manifold we want to study are constructed in a similar way, so that, it is necessary to start with the definition of an isotropic (respectively, coisotropic and Lagrangian) subspace of a symplectic vector space.

Definition 2.1. *A* symplectic structure *on a vector space V is a 2-form $\omega : V \times V \to \mathbb{R}$ on V which is non-degegnerated.*

If ω is a symplectic structure on V, then $\dim V = 2n$ and $\omega^n = \omega\wedge \overset{n}{\cdots} \wedge\omega \neq 0$, or equivalently, the linear map $\flat_\omega : V \to V^*$ defined by

$$\flat_\omega(u)(v) = \omega(u, v), \text{ for } u, v \in V$$

is an isomorphism.

Definition 2.2. *Let W be a subspace of a symplectic vector space (V, ω). Then, the* symplectic othogonal *$W^\perp$ of W is the subspace of V given by*

$$W^\perp = \{x \in V \mid \omega(x, y) = 0\ \forall y \in W\}.$$

Some useful properties of the orthogonal are listed below.

Proposition 2.3. *i.* $(W^\perp)^\perp = W$.

ii. $\dim W + \dim W^\perp = \dim V$.

iii. $\flat(W) = (W^\perp)^0$ *and* $\flat(W^\perp) = W^0$.

iv. $W_1 \subset W_2 \Rightarrow W_1^\perp \supset W_2^\perp$.

v. $(W_1 \cap W_2)^\perp = W_1^\perp + W_2^\perp$.

Denote by ω_W the 2-form induced by ω on the vector subspace W. In general, ω_W is not symplectic anymore and it has kernel,

$$\ker \omega_W = \{x \in W \mid \flat_\omega(x) = 0\} = W \cap W^\perp$$

Definition 2.4. *A vector subspace W of a symplectic vector space (V, ω) is said to be* *i.* isotropic *if* $\omega_W = 0$, *i.e,* $W \subset W^{\perp}$.

ii. coisotropic *if* $\omega_{W^{\perp}} = 0$, *i.e,* $W^{\perp} \subset W$.

iii. Lagrangian *if* $W = W^{\perp}$.

iv. symplectic *if* $W \cap W^{\perp} = \{0\}$.

We remark that a Lagrangian subspace is isotropic and coisotropic at the same time.

Suppose that $\dim V = 2n$ and $\dim W = s$, then the following equality holds

$$\text{rank}\, \flat_W - \text{rank}\, \flat_{W^{\perp}} = 2(s-n),$$

where $\flat_W$ and $\flat_{W^{\perp}}$ are the restrictions of $\flat_\omega$ to W and $W^{\perp}$, respectively. Using this relation, we deduce,

Proposition 2.5. *i. If W is isotropic, then $s < n$*

ii. If W is coisotropic, then $s > n$

iii. If W is Lagrangian, then $s = n$

iv. If W is isotropic and $V \subset W$ is a vector subspace, then V is isotropic.

v. If W is coisotropic and V is a vector space such that $W \subset V$, then V is coisotropic.

Special submanifolds of a symplectic manifold

Now the idea is to take advantatge from the previous definitions and give the notion of an isotropic, coisotropic and Lagrangian submanifold.

As above, given a submanifold $N \overset{i}{\hookrightarrow} M$ of a symplectic manifold (M, ω), we denote by $\omega_N = i^*\omega$ the 2-form induced by ω in N, which in general is degenerate. If its kernel

$$\mathcal{D}_N = \ker \omega_N = TN \cap T^{\perp}N$$

has constant rank, then it defines a completely integrable distribution on N. In fact, since ω_N is a closed form, the result follows.

Definition 2.6. *Let N be a submanifold of the symplectic manifold (M, ω). N is said to be* isotropic *(resp.* coisotropic, Lagrangian*) at a point $q \in N$ if T_qN is an isotropic (resp. coisotropic, Lagrangian) subspace of $(T_qM, \omega(q))$ in the sense defined in 2.4.*
We say that N is an isotropic submanifold *(resp.* coisotropic, Lagrangian*) if it is isotropic (resp. coisotropic Lagrangian) at every point.*

From this definition it is obvious that the relations 2.5 are still satisfied.

2.1.1 Local structure of Lagrangian submanifolds on the cotangent bundle

Below we summarize the most relevant and known results on the local structure of Lagrangian submanifolds on cotangent bundles fibered over a submanifold of the basis (see [10] for more details).

As it is already known, the cotangent bundle of a manifold M is a symplectic manifold equiped with the 2-form $\omega = -d\lambda$, where λ is the *Liouville 1-form*, that is, the unique 1-form satisfying $\beta^*\lambda = \beta$, for any 1-form $\beta \in \Omega^1(M)$. If $(q^1, \cdots, q^m)$ are local coordinates on M and, $(q^1, \cdots, q^m, p_1, \cdots, p_m)$ are the induced coordinates on T^*M, one has that

$$\omega = dq^i \wedge dp_i$$

The first example of a Lagrangian submanifold of the cotangent bundle fibered over the base manifold is given by the following result.

Proposition 2.7. *Let β be a 1-form defined over a manifold M. $\beta(M)$ is a Lagrangian submanifold of T^*M if, and only if, β is closed.*

Proof. Of course, β is an injective immersion and $\dim \beta(M) = \frac{1}{2} \dim T^*M$. Thus, it is enough to see that $\beta(M)$ is isotropic if, and only if, β is closed. Indeed, if $\omega = -d\lambda$ denotes the canonical symplectic structure of T^*M, then,

$$\omega_{\beta(M)} = \beta^*\omega = -d\beta^*\lambda = -d\beta.$$

Using this relation we deduce the result. □

Now, let us study a very important set of Lagrangian submanifolds of the cotangent bundle T^*M. Suppose that N is a submanifold of M and $F \in \mathcal{C}^\infty(N)$ is a smooth function on N. If π_M is the canonical projection of T^*M on M and j denotes the projection $j : T^*_N M \to T^*N$, we define

$$L = j^{-1}(dF(N)) = \{\alpha \in T^*M \mid \pi_M(\alpha) = q \in N,\ \alpha(v) = dF(v)\ \forall v \in T_qN\}. \tag{2.1}$$

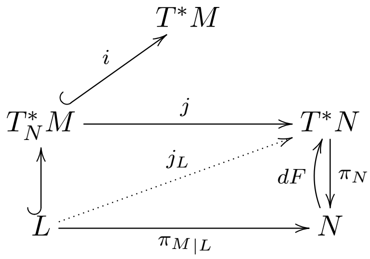
With this definition, L is a Lagrangian submanifold of T^*M. Indeed, choose adapted coordinates to N in such way that $(q^1, \cdots, q^m, p_1, \cdots p_m)$ are local coordinates on T^*M and $(q^1, \cdots, q^n)$ are coordinates on N with $n \leq m$. Then, the local expression of L is

$$L = \left\{ (q^1, \cdots, q^m, p_1, \cdots p_m) \mid q^{n+i} = 0,\ p_j = \frac{\partial F}{\partial q^j} \text{ for } i = 1, \cdots, m-n,\ j = 1, \cdots, n \right\}.$$

Thus, it follows that $\dim L = \frac{1}{2} \dim T^*M$. Moreover, taking into account the local expression of the canonical symplectic structure ω of the cotangent bundle it is obvious that $\omega_N = 0$. In other words, since L is isotropic and its dimension is a half of the dimension of the ambient space, L is Lagrangian. It is noteworthy that this result could also be obtained intrinsically as it is done in [10], but the proof is long and we prefer to omit it here.

The importance of this example lies in the fact that every Lagrangian submanifold of the cotangent bundle fibered over a submanifold of the basis can be described locally by this procedure.

Theorem 2.8. [Local structure of fibered Lagrangian submanifolds on the cotangent bundle] *Let L be a Lagrangian submanifold of T^*M which is fibered over a submanifold N of M and $j : T^*_N M \to T^*N$ be the projection. Then, for every $\xi \in L$, there exists an open neighbourhood V and a function $F \in \mathcal{C}^\infty(N)$ such that V is an open subset of $j^{-1}(dF(N))$:*



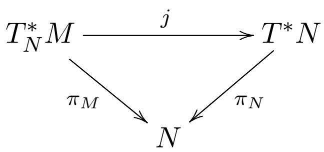
Proof. Using that L is fibered over N, we deduce that L is a submanifold of T_N^*M. Hence, if $\alpha \in L$

$$T_\alpha L \subset T_\alpha(T_N^*M) \;\;\Rightarrow\;\; T_\alpha^\perp(T_N^*M) \subset T_\alpha^\perp L = T_\alpha L$$

Thus, from Proposition 2.5 it follows that that T_N^*M is coisotropic in T^*M.
Denote by λ_M and λ_N the Liouville 1-forms on T^*M and T^*N respectively and by i the inclusion of T_N^*M into T^*M. For every $\alpha \in T_N^*M$ and every $v \in T_\alpha(T_N^*M)$ one has,

$$\begin{aligned} j^*\lambda_N(\alpha)(v) &= \lambda_N(j(\alpha))(T_\alpha j(v)) = j(\alpha)(T_\alpha(\pi_N \circ j)(v)) \\ &= j(\alpha)(T_\alpha \pi_M(v)) = \alpha(T_\alpha \pi_M(v)) = \lambda_M(\alpha)(v) \\ &= i^*\lambda_M(\alpha)(v) \end{aligned}$$

where we have used the commutativity of the following diagram:



(2.2)

Hence, $i^*\omega_M = j^*\omega_N$.
Now, denote by $\hat{\omega} = j^*\omega_N$ the 2-form induced by j on T_N^*M. As $\hat{\omega}_{|(T_N^*M)^\perp} = 0$ because T_N^*M is coisotropic, from the non degeneracy of ω_N we conclude,

$$v \in T_\alpha^\perp(T_N^*M) \;\Rightarrow\; \flat_{\hat{\omega}}(v) = 0 \;\Rightarrow \flat_{j^*\omega_N}(v) = \flat_{\omega_N}(T_\alpha j(v)) = 0 \;\Rightarrow\; v \in \ker T_\alpha j$$

Furthermore, if we set $\dim M = m$ and $\dim N = n$, then, using that j is a surjective submersion it follows that

$$\begin{aligned} \dim(\ker T_\alpha j) &= \dim T_\alpha(T_N^*M) - \dim T_{j(\alpha)}(T^*N) = m - n \\ \dim T_\alpha^\perp(T_N^*M) &= \dim T_\alpha(T^*M) - \dim T_\alpha(T_N^*M) = m - n \end{aligned}$$

so that, $\ker T_\alpha j = T_\alpha^\perp(T_N^*M) \subset T_\alpha L$. Thus, if j_L is the restriction of j to L, we have that

$$\operatorname{rank} j_L = \dim T_\alpha L - \dim\ker T_\alpha j = \dim N$$

i.e, j has constant rank. Therefore, we can choose an open subset V of L such that $\xi \in V$ and $j_L(V)$ is a n-dimensional submanifold of T^*N.
In addition, $j_L(V)$ is a Lagrangian submanifold of T^*N. First, remark that j is defined as

$$j(\alpha)(v) = \alpha(v), \;\; \forall \alpha \in T_q^*M, \pi(\alpha) = q \in N \text{ and } \forall v \in T_qN.$$

Thus, since $i^*\omega_M = j^*\omega_N$ and L is Lagrangian on T^*M, we have that $j_L^*\omega_N = 0$, so that, $j_L(V)$ is Lagrangian on (T^*N, ω_N).
Also, since $\pi_{M|L} : L \to N$ is a fibration, we have that $j_L(V)$ is fibered on N. Now, using that $\dim j_L(V) = n$, it folllows that $\pi_{N|j_L(V)} : j_L(V) \to N$ is a local diffeomorphism. Restricting V if necessary, we can obtain that $j_L(V)$ intersects each fiber of T^*N transversally at a unique point and we can define a 1-form β on N such that

$$\beta(N) = j_L(V).$$

Finally, by Proposition 2.7 we conclude that β is closed and by Poincaré Lemma there exists a function $F \in \mathcal{C}^\infty(N)$ such that $\beta = dF$ locally. Restricting V more if necesary, we obtain

$$V \subset j^{-1}(dF(N))$$

□

Using Theorem 2.8, we deduce the following result.

Corollary 2.9. *Let L be a Lagrangian submanifold of T^*M which is fibered over a submanifold N of M. Then, L is locally an affine subbundle of $\pi_M : T^*M \to M$.*

2.2 Lagrangian calculus in Poisson geometry

Let (M, ω) be a symplectic manifold and $\flat_\omega : TM \to T^*M$ be the vector bundle isomorphism induced by the symplectic structure ω. Denote by w the Poisson 2-vector associated with w and by $\#_w : T^*M \to TM$ the corresponding vector bundle morphism. Then, as we know,

$$\#_w = -\flat_\omega^{-1}.$$

On the other hand, if N is a submanifold of M then, using Proposition 2.3, it follows that

$$T_q^\perp N = \#_w((T_qN)^0).$$

This result suggest us to introduce, in a natural way, the definition of the Poisson orthogonal of a submanifold N of a Poisson manifold as an extension of the notion of the symplectic orthogonal.

Coisotropic submanifolds of a Poisson manifold

In this section we recall the definition of a coisotropic submanifold of a Poisson manifold. For this purpose, we use the notion of the Poisson orthogonal of a submanifold of a Poisson manifold (for more details see [21]).

Let M be a Poisson manifold and N be a submanifold of M. Denote by $\#_w : T^*M \to TM$ the vector bundle morphism induced by the Poisson 2-vector w of M.

If $q \in N$ is a point, then, the *Poisson orthogonal* of N at the point q is the vector space of T_qM defined by

$$\#_w((T_qN)^0).$$

Thus, the submanifold N is said to be *coisotropic* if

$$\#_w((T_qN)^0) \subset T_qN \text{ for all } q \in N.$$

Coisotropic manifolds play the same role in the Poisson setting than Lagrangian submanifolds in the symplectic setting (see [21]).

We have the following useful result that gives different characterizations of the coisotropic submanifolds.

Proposition 2.10. [21] *Let C be a submanifold of a Poisson manifold M. The following statements are equivalent:*

i. *C is coisotropic*

ii. *$w(\alpha, \beta) = 0$ for every $\alpha, \beta \in T^0C$.*

iii. *For every couple of functions $f, g \in \mathcal{C}^\infty(M)$ such that $f_{|C}$, and $g_{|C}$ are constant, then $\{f, g\}_{|C} = 0$.*

iv. *For every $q \in N$, $T_qN \cap \#_w(T_q^*M)$ is a coisotropic subspace of the symplectic vector space $\#_w(T_q^*M)$.*

Finally, let us see that if N is a coisotropic submanifold of a Poisson manifold M, the the characteristic distribution

$$q \in N \to \mathcal{D}_N(q) = \#_w((T_qN)^0) \subset T_qN$$

is completely integrable. In fact, suppose that $\{f_i\}_{i=1,\cdots n}$ is a set of local $\mathcal{C}^\infty$-functions on an open subset U of M such that

$$N \cap U = \{q \in M \mid f_i(q) = 0, \ i = 1, \cdots, n\}$$

and $\{df_i\}_{i=1,\cdots n}$ are linearly independent 1-forms at q, for every $q \in U$. Then,

$$T_q^0N = \langle df_i(q) \rangle \qquad\qquad \mathcal{D}_N(q) = \langle X_{f_i}(q) \rangle$$

Thus, $\mathcal{D}_N$ is locally finitely generated. Moreover, for any f_i, f_j,

$$\{f_i, f_j\} = w(df_i, df_j) = X_{f_j}(f_i) = 0.$$

Hence, from (1.4),

$$[X_{f_i}, X_{f_j}] = -X_{\{f_i,f_j\}} = 0, \ \ \forall f_i, f_j \in \mathcal{C}^\infty(M)$$

and the characteristic distribution is completely integrable.

2.2.1 Lagrangian Lie subalgebroids in the symplectic Lie algebroid $\mathcal{T}^A A^*$

We finish this chapter giving a first result on Lagrangian subbundles of $\mathcal{T}^A A^*$ which will provide us with some ideas about the local structure of the Lagrangian Lie subalgebroids of $\mathcal{T}^A A^*$.

We remark that a vector subbundle L of $\mathcal{T}^A A^*$ over a submanifold C of A^* is said to be Lagrangian if L_α is a Lagrangian subspace of the symplectic vector space $(\mathcal{T}^A_\alpha A^*, \Omega(\alpha))$, for all $\alpha \in C$.

Proposition 2.11. *Let $(\tau_A, [\![\, , \,]\!], \rho)$ be a Lie algebroid and L be a Lagrangian subbundle of $\mathcal{T}^A A^*$ over C, such that $\rho^1(L) \subset TC$. Then, C is coisotropic on A^*.*

Proof. It is enough to prove that the Hamiltonian vector field X_f on TA^* associated with every smooth function f such that $df \in T^0C$ belongs to TC.

Given a function f such that $df \in T^0C$, consider its Hamiltonian section $\mathcal{H}_f$ on $\mathcal{T}^A A^*$. For every $X \in \Gamma(L)$ we have

$$\Omega(\mathcal{H}_f, X) = d^{\mathcal{T}^A A^*} f(X) = \rho^1(X)(f) = 0$$

because $\rho^1(L) \subset TC$. Thus, $\mathcal{H}_f \in \Gamma(L^\perp) = \Gamma(L)$ and $\rho^1(\mathcal{H}_f) = X_f \in TC$. □

Remark 2.12. Notice that in the particular case of L being not a subbundle, but a Lie subalgebroid over C the condition $\rho^1(L) \subset TC$ is trivially satisfied and the previous result is valid.

In view of the previous result, the first step to give a local description of the Lagrangian Lie subalgebroids is to study the local structure of the coisotropic submanifolds of the dual bundle of a Lie algebroid. As well, given that Theorem 2.8 proved that every Lagrangian submanifold is locally affine, and we want to generalize such result, we will focus on the study of the coisotropic affine subbundles of linear Poisson manifolds. This is the purpose of the next chapter.

Chapter 3

Coisotropic affine subbundles of linear Poisson structures

In the previous chaper we have proved that the base space of a Lagrangian Lie subalgebroid of the A-tangent bundle to A^* is a coisotropic submanifold of A^*. That is why this chapter focus on the study of coisotropic submanifolds on the dual bundle of a Lie algebroid. As we justified on the previous chapter, in this project we focus only on the study of the affine coisotropic subbundles of A^* (see Theorem 2.8). The general case in which the submanifold is not an affine subbundle is left for a further work.

We begin the chapter by describing a particular model of a coisotropic affine subbundle of A^*. The aim of the chapter is to prove that every affine coisotropic subbundle of A^* is given locally by the same model.

First we study them in the particular case when the Lie algebroid A is the tangent bundle of a manifold. In such case, as it is already known, we replace the Poisson structure of the dual bundle by a symplectic structure. This fact provides us a first simplified aproximation by assuming no degenerations on the Poisson structure.

Finally, taking advantatge of the obtained results on the cotangent bundle, we give a general description for the dual bundle A^* of an arbitrary Lie algebroid A, emphasizing the most important differences that arises when the Poisson structure admits degenerations.

We begin this section by giving a version of Proposition 2.7 adapted to the context of Lie algebroids. In the same way as Proposition 2.7 gave the first example of a Lagrangian submanifold of the cotangent bundle, the next result gives the first example of a coisotropic submanifold of the dual bundle to a Lie algebroid.

Proposition 3.1. *Let* $(\tau_A, [\![\ ,\ ]\!], \rho)$ *be a Lie algebroid over* M *and* $\phi \in \Gamma(A^*)$*. Then,*

$$\phi(M) \text{ is coisotropic submanifold of } A^* \;\Leftrightarrow\; d^A\phi = 0$$

Proof. First of all notice that

$$\phi(M) = \left\{ \alpha \in A^* \mid \hat{X}(\alpha) - \phi(X)\left(\tau_{A^*}(\alpha)\right) = 0 \ \forall X \in \Gamma(A) \right\}$$

which implies that

$$T^0\left(\phi(M)\right) = \left\langle d(\hat{X} - \phi(X) \circ \tau_{A^*}) \mid X \in \Gamma(A) \right\rangle$$

Then, using (1.24), the following equality proves the result

$$\begin{aligned}
\#(d(\hat{X} - \phi(X)\circ\tau_{A^*}), d(\hat{Y} - \phi(Y)\circ\tau_{A^*})) &= \left\{\hat{X}, \hat{Y}\right\} - \left\{\phi(X)\circ\tau_{A^*}, \hat{Y}\right\} \\
&+ \left\{\phi(Y)\circ\tau_{A^*}, \hat{X}\right\} + \{\phi(X)\circ\tau_{A^*}, \phi(Y)\circ\tau_{A^*}\} \\
&= -\widehat{[\![X,Y]\!]} - \rho(Y)\phi(X)\circ\tau_{A^*} + \rho(X)\phi(Y)\circ\tau_{A^*} \\
&= -\widehat{[\![X,Y]\!]} + d^A\phi(X,Y) + \phi([\![X,Y]\!])\circ\tau_{A^*} \\
&= d^A\phi(X,Y).
\end{aligned}$$

□

Next we extend the previous result with the following theorem.

Theorem 3.2. *Let $(\tau_A, [\![\ ,\ ]\!], \rho)$ be a Lie algebroid over M and B be a Lie subalgebroid over a submanifold N of M. Suppose that $\phi : N \to B^*$ is a 1-cocycle in the Lie subalgebroid $B \to N$. Then,*

$$C(B,\phi) = \left\{\alpha \in A^* \mid \alpha_{|B(\tau_{A^*}(\alpha))} = \phi(\tau_{A^*}(\alpha)),\ \tau_{A^*}(\alpha) \in N\right\}.$$

is a coisotropic affine subbundle of the Linear Poisson manifold A^*.

Proof. Let (q^i, q^a) be local coordinates on an open subset U of M such that

$$N \cap U = \left\{(q^i, q^a) \in U \mid q^a = 0\ \forall a\right\}.$$

Now suppose that $\{e_\alpha\}$ is a local basis of $\Gamma(B)$ on the open subset $N \cap U$ and that $e_\gamma = \{\tilde{e}_\alpha, \tilde{e}_\theta\}$ is a local basis of $\Gamma(A)$ such that

$$\tilde{e_{\alpha|N\cap U}} = e_\alpha, \quad \text{for all } \alpha.$$

Denote by $(q^A, y_\gamma) = (q^i, q^a, y_\alpha, y_\theta)$ the induced coordinates in A^* on $\tau_{A^*}^{-1}(U)$ and by $(\rho^A_\gamma, C^{\gamma''}_{\gamma\gamma'})$ the corresponding local structure functions of A. Then, using that B is a Lie subalgebroid of A we deduce that,

$$\rho^a_\alpha(q^i, 0) = 0, \qquad\qquad C^\theta_{\alpha,\alpha'}(q^i, 0) = 0.$$

On the other hand, if $\left\{\tilde{e}^\alpha, \tilde{e}^\theta\right\}$ is the dual basis of $\{\tilde{e}_\alpha, \tilde{e}_\theta\}$, and

$$\phi(q^i, 0) = \phi_\alpha(q^i, 0)\tilde{e}^\alpha(q^i, 0)$$

then, since ϕ is a 1-cocycle, it follows that

$$\rho^i_\alpha \frac{\partial\phi_{\alpha'}}{\partial q^i} + \rho^i_{\alpha'}\frac{\partial\phi_\alpha}{\partial q^i} - C^{\alpha''}_{\alpha\alpha'}\phi_{\alpha''} = 0.$$

Now, from the definition of $C(B,\phi)$, we have that

$$C(B,\phi) \cap \tau_{A^*}^{-1}(U) = \left\{(q^i, q^a, y_\alpha, y_\theta) \in A^* \mid q^a = 0,\ \text{and } y_\alpha = \phi(q^i, 0)\right\}.$$

This implies that

$$T^0C(B,\phi) = \left\langle dq^a, dy_\alpha - \frac{\partial\phi_\alpha}{\partial q^i}dq^i\right\rangle.$$

Thus, from (1.18) we conclude that

$$\#_w(T^0C(B,\phi)) \subset TC(B,\phi)$$

and, therefore, $C(B,\phi)$ is coisotropic submanifold of A^*. Finally, it is clear that $C(B,\phi)$ is an affine subbundle of A^*. $\square$

From Theorem 3.2, we deduce the following result.

Corollary 3.3. [22] *Let $(\tau_A, [\![\ ,\ ]\!], \rho)$ be a Lie algebroid over M and B be a Lie subalgebroid over a submanifold N of M. Then, the annihilator B^0 of B is a coisotropic vector subbundle of the linear Poisson manifold A^*.*

3.1 Coisotropic affine submanifolds on cotangent bundles

Let M be a smooth manifold, $\mathcal{F}_N$ be a foliation on a submanifold N of M and $\phi : N \to \mathcal{F}_N^*$ be a 1-cocycle on $\mathcal{F}_N$. Note that $\mathcal{F}_N$ is a Lie subalgebroid of the standard Lie algebroid $TM \to M$ and, thus, one may consider the coisotropic submanifold of T^*M given by

$$C(\mathcal{F}_N,\phi) = \left\{\alpha \in T^*M \mid \pi_M(\alpha) \in N, \text{ and } \alpha_{|\mathcal{F}_N(\pi_M(\alpha))} = \phi(\pi_M(\alpha))\right\}$$

where $\pi_M : T^*M \to M$ is the canonical projection. In this section, our aim is to prove that any affine coisotropic submanifold C of the cotangent bundle T^*M can be described locally by this procedure. To do so, we will use the following result.

Proposition 3.4. *Let $(\tau_A, [\![\ ,\ ]\!], \rho)$ be a Lie algebroid over M and C be an affine coisotropic subbundle of A^* over a submanifold N. Then, the characteristic distribution $\mathcal{D}$ of C is τ_{A^*}-projectable.*

Proof. We recall that the characteristic distribution $\mathcal{D}_C$ of C is given by

$$\mathcal{D}_C = \#(T^0C) = \left\langle X_{f_{|C}} \mid f \in \mathcal{C}^\infty(A^*) \text{ and } df_{|C} \in T^0C\right\rangle.$$

Thus, it is enough to prove that $\#(df_i)$ are τ_{A^*}-projectable for some f_i such that df_i span T^0C. Equivalently, we have to show that for every basic function $\tilde{g} \in \mathcal{C}^\infty(A^*)$, $X_{f_i}(\tilde{g})$ is again a basic function.
Now assume that C is modelled over a vector subbundle V over N. Then, it is easy to see that,

$$T^0C = \left\langle d\hat{X}, d\tilde{h} \mid X \in \Gamma(V^0) \text{ and } h \in \mathcal{C}^\infty(M),\ h_{|N} = 0\right\rangle.$$

From (1.16), we have that $X_{\hat{X}}(\tilde{g})$ and $X_{\tilde{h}}(\tilde{g})$ are basic functions. $\square$

Suppose that C is a coisotropic submanifold of the cotangent bundle of a manifold M. Assume that $\dim M = m$, $\dim C = m+s$, with $s \leq m$ and that $\mathcal{D}_C = T^\perp C$ is the characteristic distribution of C. Then,

$$\operatorname{rank} \mathcal{D}_C(\alpha) = \dim T^\perp_\alpha C = 2m - (m+s) = m - s, \quad \text{for } \alpha \in C. \tag{3.1}$$

If C is affine and fibered over M, from Proposition 3.4 there exists a distribution $\mathcal{F}$ over M given by,

$$\mathcal{F}(q) = (T_\alpha\pi_M)(\mathcal{D}_C(\alpha)), \quad \text{for } q \in M \text{ with } \pi_M(\alpha) = q.$$

As C is fibered over M, $\mathcal{F}$ is a regular distribution and its rank coincides with the rank of $\mathcal{D}_C$. Indeed, since

$$\text{rank}\mathcal{F} = \dim \mathcal{F}(q) = \dim \mathcal{D}_C(\alpha) - \dim(\ker(T_\alpha \pi_{M|\mathcal{D}_C})),$$

it is enough to check that $\mathcal{D}_C(\alpha) \cap \ker T_\alpha \pi_M = \{0\}$. As $C \overset{\pi_C}{\to} M$ is a fibration, then $T_\alpha(T^*M) = \ker T_\alpha \pi_M + T_\alpha C$. Therefore, if $v \in \mathcal{D}_C(\alpha) \cap \ker T_\alpha \pi_M$, it follows that $i_v \omega = \flat_\omega(v) = 0$, so that, $v = 0$ because ω is symplectic.

Moreover, taking into account that $\mathcal{D}_C$ is completely integrable (recall section 2.2), from the $T\pi_M$-projectability of the Lie bracket one has that $\mathcal{F}$ is a foliation. Thus, from section 1.2 we have that $\mathcal{F}$ is a Lie subalgebroid of TM. $\mathcal{F}$ is said to be the foliation on M associated with the characteristic distribution $\mathcal{D}_C$.

Proposition 3.5. *Let C be an affine coisotropic submanifold of $T^*M \overset{\pi_M}{\to} M$ fibered over M and $\mathcal{F}$ be the foliation on M associated with $\mathcal{D}_C$. If $j : T^*M \to \mathcal{F}^*$ denotes the canonical projection, then for every $\alpha \in C$,*

i. $\ker T_\alpha j = \left\langle \mathcal{F}^0(\pi_M(\alpha)) \right\rangle_\alpha^v$, *where* $\,^v_\alpha : T^*_{\pi_M(\alpha)} M \to T_\alpha(T^*_{\pi_M(\alpha)} M)$ *is the canonical isomorphism.*

ii. $\ker T_\alpha j \subset T_\alpha C$

Proof. i. Remark that,

$$\dim \ker T_\alpha j = \dim T_\alpha(T^*M) - \dim T_\alpha \mathcal{F} = s$$
$$\dim(\mathcal{F}^0(\pi_M(\alpha)))^v_\alpha = \dim \mathcal{F}^0(\pi_M(\alpha)) = s.$$

Moreover, for every $\beta \in \mathcal{F}^0(\pi_M(\alpha))$,

$$T_\alpha j(\beta^v_\alpha) = T_\alpha j \left(\frac{d}{dt}_{|t=0} (\alpha + t\beta) \right)$$
$$= \frac{d}{dt}_{|t=0} (j(\alpha) + tj(\beta)) = \frac{d}{dt}_{|t=0} j(\alpha) = 0.$$

Thus, $(\mathcal{F}^0(\pi_M(\alpha)))^v_\alpha \subset \ker T_\alpha j$ and as the dimensions coincide $\ker T_\alpha j = \left\langle \mathcal{F}^0(\pi_M(\alpha)) \right\rangle_\alpha^v$.

ii. From the previous item it is enough to see $(\mathcal{F}^0(\pi_M(\alpha)))^v_\alpha \subset T_\alpha C$. First, as (T^*M, ω) is a symplectic manifold, it follows that $\flat_\omega(T^\perp C) = T^0 C$. For every $\gamma \in T^0_\alpha C$, we denote by X_γ its inverse image by $\flat_\omega$. For every $\beta \in \mathcal{F}^0(\pi_M(\alpha))$ we have,

$$\gamma(\beta^v_\alpha) = \flat_\omega(X_\gamma)(\beta^v_\alpha) = T^*_\alpha \pi_M(\beta)(X_\gamma) = \beta(T_\alpha \pi_M(X_\gamma))$$

where we used that $\flat_\omega(\beta^v_\alpha)(v) = -T^*_\alpha \pi_M(\beta)(v)$ for any $\beta \in T^*M$ and any $v \in T_\alpha(T^*M)$. Finally, since $X_\gamma \in \mathcal{D}_{|C}$ it follows that $(T_\gamma \pi_M)(X_\gamma) \in \mathcal{F}(\pi_M(\gamma))$ and we conclude,

$$\gamma(\beta^v_\alpha) = 0$$

and $\ker T_\alpha j \subset T_\alpha C$.

□

Now, we have all the requierd tools to give the local description of the affine coisotropic submanifolds of the cotangent bundle. We give the result in two parts, first the case where C is fibered over the base manifold and then the general case.

Proposition 3.6. *Let C be an affine coisotropic submanifold of T^*M fibered over the base manifold M and $\mathcal{F}$ be the projection of its characteristic distribution. If we denote by $j : T^*M \to \mathcal{F}^*$ the canonical projection, then for every $\xi \in C$ there exists an open neighbourhood V and a 1-cocycle $\phi \in \Gamma(\mathcal{F}^*)$, $d^{\mathcal{F}}\phi = 0$, such that V is an open set of $j^{-1}(\phi(M))$.*

$$\begin{array}{ccc} T^*M & \xrightarrow{\quad j \quad} & \mathcal{F}^* \\ \uparrow & \overset{j_C}{\nearrow} & \phi \uparrow \downarrow \pi_M \\ C & \xrightarrow[\pi_{M|C}]{\quad\quad} & M \end{array} \tag{3.2}$$

Proof. If $\alpha \in C$ then, from Proposition 3.5, we have that

$$\operatorname{rank} j_C = \dim T_\alpha C - \dim \ker T_\alpha j = m.$$

Thus, we can choose an open subset V of C such that $\xi \in V$ and $j_C(V)$ is a submanifold of $\mathcal{F}^*$. Now, since $\mathcal{F}$ is a Lie subalgebroid of TM, then $\mathcal{F}^*$ admits a linear Poisson structure. We will see hat $j_C(V)$ is a coisotropic submanifold of $\mathcal{F}^*$.
Given $\alpha \in V$, for every $\beta \in T^0_{j(\alpha)} j_C(V)$ and every $v \in T_\alpha C$ we have

$$(T^*_\alpha j(\beta))\,(v) = \beta(T_\alpha j(v)) = 0,$$

so that, $(T^*_\alpha j)\,(T^0_{j(\alpha)} j_C(V)) \subset T^0_\alpha C$. As well, as $\mathcal{F}$ is a Lie subalgebroid of TM, then from Proposition 1.10 it follows that j is a Poisson morphism. Therefore, if $w_{\mathcal{F}^*}$ and w_{T^*M} are the Poisson 2-vectors of $\mathcal{F}^*$ and T^*M respectively, then for every $\beta, \eta \in T^0_{j_C(\alpha)} j(V)$,

$$w_{\mathcal{F}^*}(j(\alpha))(\beta, \eta) = w_{T^*M}(\alpha)(T^*_\alpha j(\beta), T^*_\alpha j(\eta)) = 0$$

where we have used (1.5) and the fact that C is coisotropic on T^*M. Thus, $j_C(V)$ is coisotropic in $\mathcal{F}$.
From the commutativity of the diagram

$$\begin{array}{ccc} T^*M & \xrightarrow{\quad j \quad} & \mathcal{F}^* \\ & {}_{\pi_M}\searrow \quad \swarrow_{\tau_{\mathcal{F}^*}} & \\ & M & \end{array}$$

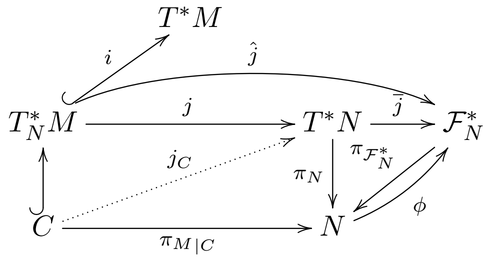
it follows that $j_C(V)$ is fibered over M. Thus, taking into account that $\operatorname{rank} j_C = m$, there exists a section $\phi \in \Gamma(\mathcal{F}^*)$ such that $\phi(M) = j_C(V)$, i.e.,

$$V \subset j^{-1}(\phi(M))$$

Finally, by Proposition 3.1, $d^{\mathcal{F}}\phi = 0$. □

Next, we discuss the general case when the coisotropic submanifold is fibered over a submanifold of M.

Theorem 3.7. [Local structure of fibered coisotropic submanifolds with projectable characteristic distribution] *Let C be a coisotropic submanifold of T^*M fibered over a submanifold N of M such that its characteristic distribution is π_M-projectable over a foliation $\mathcal{F}_N$ on N. Then, if $\hat{j} : T^*_N M \to \mathcal{F}^*_N$ denotes the canonical projection, for every $\xi \in C$ there exists* *an open neighbourhood V and a 1-cocycle $\phi \in \Gamma(\mathcal{F}_N^*)$, $d^{\mathcal{F}_N}\phi = 0$, such that V is an open set of $\hat{j}^{-1}(\phi(N))$:*



Proof. Proceding as in the proof of the Theorem 3.6, one may prove that

$$\ker T_\alpha j \subset T_\alpha^\perp C, \text{ for } \alpha \in C.$$

Then, as C is coisotropic, $T_\alpha^\perp C \subset T_\alpha C$ and we may choose an open subset V of C such $\xi \in V$ and $j_C(V)$ is a submanifold of T^*N.
Now, let us see that

$$T_\alpha j(T_\alpha^\perp C) = ((T_\alpha j)(T_\alpha C))^\perp = T_{j(\alpha)}^\perp j(C), \text{ for } \alpha \in V. \tag{3.3}$$

First, if we set $\dim N = n$, $\dim M = m = n + k$ and $\dim C = n + s$ with $k \leq s$,

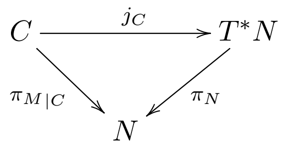
$$\begin{aligned}\dim\left(T_\alpha j\right)\left(T_\alpha^\perp C\right) &= \dim T_\alpha^\perp C - \dim\ker(T_\alpha j_{|T_\alpha^\perp C}) \\ &= 2(n+k) - (n+s) - (2n+k-2n) = n+k-s = m-s\end{aligned}$$

$$\begin{aligned}\dim((T_\alpha j)(T_\alpha C))^\perp &= \dim T_{j(\alpha)}(T^*N) - \dim T_\alpha j(T_\alpha C) \\ &= 2n - (n+s-(2n+k-2n)) = n+k-s = m-s.\end{aligned}$$

Furthermore, as we saw in the proof of Theorem 2.8, $i^*\omega_M = j^*\omega_N$ where $i : T_N^*M \to T^*M$. Thus, for any given $u \in T_\alpha^\perp C$ and every $v \in T_\alpha C$,

$$\omega_N(j(\alpha))(T_\alpha j(u), T_\alpha j(v)) = j^*\omega_N(\alpha)(u,v) = i^*\omega_M(\alpha)(u,v) = 0.$$

Therefore, $T_\alpha j(T_\alpha^\perp C) = ((T_\alpha j)(T_\alpha C))^\perp = T_{j(\alpha)}^\perp j(C)$. Consequently, since $T_\alpha^\perp C \subset T_\alpha C$, it follows that $j(C)$ is coisotropic.
As in the Lagrangian case, from the commutativity of the diagram below it follows that $j_C(V)$ is fibered over N



So far, we have proved that $j_C(V)$ is coisotropic in T^*N and fibered over the whole base manifold N. Hence, if we show that the characteristic distribution $\mathcal{D}_{j_C(V)}$ of $j_C(V)$ projects to $\mathcal{F}_N$, then, from Proposition 3.6 we will have that there exists an open neighbourhood U of $j(\xi)$ such that

$$U \subset \bar{j}^{-1}(\phi(N))$$

with $\bar{j} : T^*N \to \mathcal{F}_N^*$ the canonical projection, $\phi \in \Gamma(\mathcal{F}_N^*)$ and $d^{\mathcal{F}_N}\phi = 0$. But, using again the commutativity of the previous diagram and Proposition 3.3 we obtain,

$$\mathcal{F}_N(\pi_M(\alpha)) = (T_\alpha \pi_M)\left(T_\alpha^\perp C\right) = \left(T_{j(\alpha)}\pi_N\right)\left(T_\alpha j(T_\alpha^\perp C)\right) = \left(T_{j(\alpha)}\pi_M\right)\left(D_{j(C)}(j(\alpha))\right).$$

for $\alpha \in C$. Eventually, restricting V more if necessary, we conclude that

$$V \subset j^{-1}(\phi(N))$$

□

Remark 3.8. If C is an affine coisotropic subbundle of T^*M, the characteristic distribution $\mathcal{D}_C$ of C is τ_{A^*}-projectable (see Proposition 3.4). Conversely, using Theorem 3.7 we deduce that if C is a fibered coisotropic submanifold with projectable characteristic distribution, then C is locally an affine subbundle.

3.2 Coisotropic affine subbundles on Lie Algebroids

The idea now is to generalize the above results to the dual bundle of any Lie algebroid, which in general, is not symplectic but Poisson. The most natural way of doing it is trying to adjust the proofs we have to the context of Lie algebroids and Poisson manifolds.

If we try to follow the same steps as in the cotangent bundle case, we remark that Proposition 3.4 is still valid. Furthermore, in the cotangent bundle case we have seen that the converse is also true, i.e., if the characteristic distribution is projectable, then the submanifold is locally an affine subbundle. However, this last result is not, in general, true as shows the following example.

Example 3.9. Let $(\tau_A, [\![\ ,\ ]\!], \rho)$ be a Lie algebroid and $C = \{\alpha \in A^* | f_i(\alpha) = 0,\ i = 1, \cdots, n\}$ be an affine coisotropic subbundle of A^*. Let φ_j, $j = 1, \cdots, m$ be Casimir functions and define $\overline{C} = \{\alpha \in A^* \mid f_i(\alpha) = 0,\ \varphi_j(\alpha) = 0 \text{ for } i = 1, \cdots, n,\ j = 1, \cdots, m\}$. We have,

$$T^\perp \overline{C} = T^\perp C \subset T\overline{C},$$

that is, $\overline{C}$ is coisotropic submanifold and its characteristic distribution is τ_{A^*}-projectable. However, $\overline{C}$ is not in general, an affine subbundle.

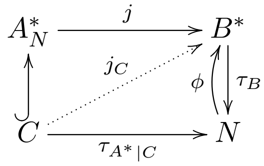
Example 3.10. Let us see a particular example of the previous situation. Consider the Lie algebra $\mathfrak{so}(3)$ of the special orthogonal group $SO(3)$ that may be identified with $(\mathbb{R}^3, \times)$. The Poisson 2-vector of the Lie-Poisson structure on $\mathfrak{so}^*(3) \cong \mathbb{R}^3$ is given by

$$w = x\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + y\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} + z\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$$

Hence it is easy to prove that the functions $n_r(x, y, z) = x^2 + y^2 + z^2 - r^2$ are Casimir functions for such a Poisson structure (for further details on these topics see [14]).

Note that, in this case, the vector bundle projection $\tau_{A^*} = \tau_{\mathbb{R}^3}$ is the zero map. Thus, the characteristic distribution of an arbitrary coisotropic submanifold is $\tau_{\mathbb{R}^3}$-projectable. Now, consider the function $f(x, y, z) = x$ and the manifold $C = \left\{v \in \mathbb{R}^3 \mid f(v) = 0\right\}$. It is coisotropic since $TC = \left\langle \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$ and $\#_w(df) = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}$. Furthermore, the manifold $\bar{C} = \left\{v \in \mathbb{R}^3 \mid f(v) = 0 = n_r(v)\right\}$ is also coisotropic since $TC = \left\langle -z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z} \right\rangle$ and its characteristic distribution coincides with that of C. However, $\overline{C}$ is not an affine subspace of $\mathbb{R}^3$.

In view of the previous example, we focus on the case in which C is an affine subbundle which is the strongest condition. Now, in order to generalize the above results we construct a diagram analogous to (3.2) to proceed in a similar way



The first problem that we encounter is defining an object B where before we had in a natural way a foliation over N in TN. What one could expect, is B being a Lie subalgebroid of A such that the properties i. and ii. of Proposition 3.5 hold. These properties imply that the rank of j_C is constant. As we will see, this will lead us to define B through its anihilator and to guarantee that B is a Lie subalgebroid of A we will need the following results.

Proposition 3.11. [22] *Let $(\tau_A, [\![\ ,\ ]\!], \rho)$ be a Lie algebroid over M and B be a vector subbundle of A over a submanifold N of M. Then, B is a Lie subalgebroid of A if, and only if, the annihilator B^0 of B is a coisotropic submanifold of A^*.*

Proposition 3.12. *Let $(\tau_A, [\![\ ,\ ]\!], \rho)$ be a Lie algebroid over M and B be a vector subbundle of A over a submanifold N of M. If C is a coisotropic affine subbundle of A^* which is modelled over the vector subbundle B^0 of A^*, then B is a Lie subalgebroid of A.*

Proof. It is easy to prove that

$$T^0B^0 = \left\langle \left\{ d\hat{X}, d\tilde{f} \mid X \in \Gamma(A) \text{ and } f \in \mathcal{C}^\infty(M) \text{ with } X_{|N} \in \Gamma(B) \text{ and } f_{|N} = 0 \right\} \right\rangle.$$

Moreover, for every $X \in \Gamma(B)$ there exists $\phi_X \in \mathcal{C}^\infty(M)$ such that

$$T^0C = \left\langle \left\{ d(\hat{X} + \tilde{\phi}_X), d\tilde{f} \mid X \in \Gamma(A) \text{ and } f \in \mathcal{C}^\infty(M) \text{ with } X_{|N} \in \Gamma(B) \text{ and } f_{|N} = 0 \right\} \right\rangle$$

Now, if $f \in \mathcal{C}^\infty(M)$ and $f_{|N} = 0$, we have that

$$\#_{w_{A^*}}(d\tilde{f})_{|C} \in \mathcal{X}(C)$$

where $\#_{w_{A^*}}$ is the linear Poisson 2-vector on A^*.
Thus, since, $\#_{w_{A^*}}(d\tilde{f})(\tilde{\phi}_X) = 0$ note that from (1.28), $\#_{w_{A^*}}(d\tilde{f})$ is a τ_{A^*}-vertical vector field and it follows that

$$\#_{w_{A^*}}(d\tilde{f})_{|B^0} \in \mathcal{X}(B^0) \tag{3.4}$$

On the other hand, if $X \in \Gamma(A)$ and $X_{|N} \in \Gamma(B)$, we have that

$$\#_{w_{A^*}}(d(\hat{X} + \tilde{\phi}_X))_{|C} \in \mathcal{X}(C)$$

which implies that

$$0 = \left(\#_{w_{A^*}}(d(\hat{X} + \tilde{\phi}_X))\right)(\hat{Y} + \tilde{\phi}_Y)_{|C} = \left(\#_{w_{A^*}}(d\hat{X})(\hat{Y}) + \#_{w_{A^*}}(d\hat{X})(\tilde{\phi}_Y) + \#_{w_{A^*}}(d\tilde{\phi}_X)(\hat{Y})\right)_{|C}$$

Thus using that $\#_{w_{A^*}}(d\hat{X})(\hat{Y})$ is a linear function and that $\#_{w_{A^*}}(d\hat{X})(\tilde{\phi}_Y)$ and $\#_{w_{A^*}}(d\hat{Y})(\tilde{\phi}_X)$ are basic functions we conlude that

$$\#_{w_{A^*}}(d\hat{X})(\hat{Y}) = 0$$

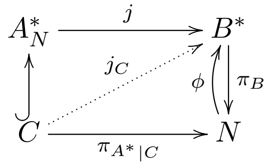
and therefore,

$$\#_{w_{A^*}}(d\hat{X})_{|B^0} \in \mathcal{X}(B^0). \tag{3.5}$$

Now, form (3.4) and (3.5) we deduce that B^0 is a coisotropic submanifold of A^*. Consequently, using Proposition 3.11 it follows that B is a Lie subalgebroid of A. □

Next, we present the local description of the coisotropic affine subbundles of the dual bundle to a Lie algebroid.

Theorem 3.13. [Local structure of coisotropic affine subbundles of the dual bundle to a Lie Algebroid]. *Let $(\tau_A, [\![\ ,\ ]\!], \rho)$ be a Lie algebroid over M, N be a submanifold of M and B be a vector subbundle over N. If $C \hookrightarrow A^*$ is an coisotropic affine subbundle of A^* modelled over B^0 and $j : A^*_N \to B^*$ is the canonical projection, then for every $\xi \in C$ there exists an open neighbourhood V and a 1-cocycle $\phi \in \Gamma(B^*)$, $d^B\phi = 0$, such that V is an open set of $j^{-1}(\phi(N))$*



Proof. Thanks to Proposition 3.12 we know that B is a Lie subalgebroid of A, and the previous diagram makes sense.
Proceeding as in the proof of Proposition 3.5 one proves that $\ker T_\alpha j = \left\langle B^0(\tau(\alpha))\right\rangle^v_\alpha \subset T_\alpha C$, for every $\alpha \in C$. It follows that j_C has constant rank equals to the dimension of N and we may choose an open subset V of C so that $j_C(V)$ is a submanifold of B^*.
As B is a Lie subalgebroid of A, B^* is a linear Poisson manifold and proceeding as in the proof of Proposition 3.6, we deduce that the submanifold $j_C(V)$ is coisotropic on B^* and fibered over N. Thus, there exists $\phi \in \Gamma(B^*)$ such that, restricting V if necessary, $V \subset j^{-1}(\phi(N))$ and, from Proposition 3.1 $d^B\phi = 0$. □

This result gives us the local structure of the coisotropic affine subbundles of A^*. For our immediate objective this description suffices. Nevertheless, the problem of describing the coisotropic fibered non affine submanifolds whose characteristic distribution is projectable is still open. One might conjecture that a fibered coisotropic submanifold with projectable characteristic distribution is under certain conditions of regularity, as in Example 3.9.

Finally we study the coisotropic submanifolds of A^* in the particular case when A is an action Lie algebroid or an Atiyah algebroid associated with a principal bundle. We remark that the example of the cotangent bundle is already studied in the previous section and the Lie algebra case is just a particular case of both examples.

Example 3.14. i. **Action Lie algebroids** Let $\tau_A : M \times \mathfrak{g} \to M$ be an action Lie algebroid associated with the left infinitesimal action $\Phi : \mathfrak{g} \to \mathcal{X}(M)$. In order to describe the coisotropic affine subbundles of $A^* = M \times \mathfrak{g}^*$ we only need a Lie subalgebroid and a 1-cocycle on it. In Example 1.11 we saw that $N \times \mathfrak{h}$, with N a submanifold of M and $\mathfrak{h}$ a Lie subalgebra of $\mathfrak{g}$ acting on $\mathcal{X}(N)$, is an action Lie subalgebroid of $A = M \times \mathfrak{g}$. Furthermore, if $\alpha \in \mathfrak{h}^*$ is a 1-cocycle for the Lie subalgebra $\mathfrak{h}$, then α induces, in a natural way, a 1-cocycle of the action Lie subalgebroid $\tau_B : B = N \times \mathfrak{h} \to N$. Thus,

$$C = \left\{(q, \beta) \in N \times \mathfrak{g}^* \mid \beta_{|\mathfrak{h}} = \alpha\right\}$$

is a coisotropic affine subbundle of the linear Poisson manifold $A^* = M \times \mathfrak{g}^*$. Note that if $\tilde{\alpha} \in \mathfrak{g}^*$ is an extension of $\alpha \in \mathfrak{h}^*$, then

$$C = \left\{(q, \tilde{\alpha} + \gamma) \in N \times \mathfrak{g}^* \mid \gamma \in \mathfrak{h}^0\right\} \subset A^* = M \times \mathfrak{g}^*.$$

ii. **Atiyah algebroid** Let $G \times M$ be the total space of a trivial principal G-bundle over M. As we know, the Atiyah algebroid associated with the principal G-bundle is $\tau_A : A = \mathfrak{g} \times TM \to M$, where $\mathfrak{g}$ is the Lie algebra of G.

Now, suppose that $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, N is a submanifold of M and $\mathcal{F}_N$ is a foliation on N. Then, the trivial vector bundle $\tau_B : B = \mathfrak{h} \times \mathcal{F}_N \to N$ is a Lie subalgebroid of the Atiyah algebroid $\tau_A : A = \mathfrak{g} \times TM \to M$.

Next, let $\alpha : \mathfrak{h} \to \mathbb{R}$ be a 1-cocycle of the Lie subalgebra $\mathfrak{h}$ and $\phi : N \to \mathcal{F}_N^*$ be a cocycle for the foliation $\mathcal{F}_N$. Then, α and ϕ induce, in a natural way, a 1-cocycle (α, ϕ) of the Lie subalgebroid $\tau_B : B = \mathfrak{h} \times \mathcal{F}_N \to N$. In fact,

$$(\alpha, \phi)(q) = (\alpha, \phi(q)), \text{ for } q \in N$$

Thus,

$$C = \left\{(\mu, \beta) \in A^* = \mathfrak{g}^* \times T_N^* M \mid \mu_{|\mathfrak{h}} = \alpha, \beta_{|\mathcal{F}_N(q)} = \phi(q), \text{ for } q \in N\right\}$$

is a coisotropic affine subbundle of the Atiyah algebroid $\tau_A : A = \mathfrak{g} \times TM \to M$.

Finally, if $\tilde{\alpha} \in \mathfrak{g}^*$ and $\tilde{\phi} : N \to T_N^* M$ are extensions of α and ϕ respectively, it follows that

$$C = \bigcup_{q \in N} \left\{(\tilde{\alpha} + \gamma, \tilde{\phi} + \beta) \in A_q^* = \mathfrak{g}^* \times T_q^* M \mid \gamma \in \mathfrak{h}^0 \text{ and } \beta \in T_q^0 N\right\}$$

Chapter 4

Lagrangian Lie subalgebroids of the A-tangent bundle to A^*

In Chapter 2 (see Proposition 2.11), we proved that the base space of every Lagrangian Lie subalgebroid of the A-tangent bundle to A^* is a coisotropic submanifold on A^*. In the previous chapter we discussed the local structure of such submanifolds in the particular case when they are affine. Now, we discuss the local description of the Lagrangian Lie subalgebroids of A-tangent bundle to A^* fibered over an affine subbundle of A^*. Following the same pattern as in the other chapters we give first a model of Lagrangian Lie subalgebroids of $\mathcal{T}^A A^*$ fibered over a coisotropic affine subbundle of A^* and then we show that locally, under certain conditions, every Lagrangian Lie subalgebroid fibered over a coisotropic affine subbundle can be described by the same procedure.

4.1 The local structure of some Lagrangian Lie subalgebroids of the A-tangent bundle to A^*.

Proposition 4.1. *Let $(\tau_A, [\![\ ,\ ]\!]_A, \rho_A)$ be a Lie algebroid over M and $(\tau_B, [\![\ ,\ ]\!]_B, \rho_B)$ be a Lie subalgebroid over a submanifold N of M. Let $\phi \in \Gamma(B^*)$ be a cocycle and $C = C(B, \phi)$ be the affine coisotropic submanifold associated to B and ϕ. Then,*

$$L = \mathcal{T}^B C = \bigcup_{\alpha_q \in C} \left\{(b_q, X_{\alpha_q}) \in B \times T_{\alpha_q} C \mid \rho_B(b_q) = \left(T_{\alpha_q} \tau_{A^*|C}\right)(X_{\alpha_q})\right\} \tag{4.1}$$

is a Lagrangian Lie subalgebroid of the symplectic Lie algebroid $(\mathcal{T}^A A^*, \Omega)$

$$\begin{array}{ccc} L & \xrightarrow{\quad I \quad} & \mathcal{T}^A A^* \\ {\scriptstyle \tau_L = (\tau_{\mathcal{T}^A A^*})_{|L}} \big\downarrow & & \big\downarrow {\scriptstyle \tau_{\mathcal{T}^A A^*}} \\ C & \xrightarrow[\quad i \quad]{} & A^* \end{array}$$

Proof. First of all let us check that L has constant rank equals to the half of the rank of $\mathcal{T}^A A^*$. If $\dim N = n$, $\dim M = m$, $\operatorname{rank} B = b$ and $\operatorname{rank} A = a$, it follows that $\dim C = a + n - (b + n - n) = a + n - b$. Then,

$$\operatorname{rank} L = \operatorname{rank} B + \dim C - \dim N = a = \frac{1}{2} \operatorname{rank} \mathcal{T}^A A^*,$$

and L is a vector subbundle of $\mathcal{T}^A A^*$.
Now, let us prove that L is a Lie subalgebroid of $\mathcal{T}^A A^*$ by checking the conditions of Definition 1.11:

i. $\rho^1_{|L} : L \to TC$ is well defined (we recall that ρ^1 is the anchor map of the Lie algebroid $\tau_{\mathcal{T}^A A^*} : \mathcal{T}^A A^* \to A^*$).

ii. Given two projectable sections $(X, X'), (Y, Y') \in \Gamma(\mathcal{T}^A A^*)$ such that $(X, X')_{|C}, (Y, Y')_{|C} \in \Gamma(L)$, we have that $X_{|N}, Y_{|N} \in \Gamma(B)$ and $X'_{|C}, Y'_{|C}$ are vector fields on C $(\tau_{A^*})_{|C}$-projectables on $\rho_A(X)_{|N}$ and $\rho_A(Y)_{|N}$ respectively. On the other hand,

$$[\![(X, X'), (Y, Y')]\!]_{\mathcal{T}^A A^*} = \left([\![X, Y]\!]_A, [X', Y']\right). \tag{4.2}$$

Hence, since B is a Lie subalgebroid and C is a submanifold it follows that

$$\left([\![(X, X'), (Y, Y')]\!]_{\mathcal{T}^A A^*}\right)_{|C} = \left([\![X, Y]\!]_B, [X', Y']_{|C}\right) \in \Gamma(L).$$

The above facts imply that L is a Lie subalgebroid.
Let us prove that L is Lagrangian. To do so, first we need to set a local basis of $\Gamma(L)$. Recall that $\left\langle \left\{ (df_X)_{|C} = d(\hat{X} - \tilde{\phi}(X) \circ \tau_{A^*})_{|C} \mid X \in \Gamma(A) \text{ and } X_N \in \Gamma(B) \right\} \right\rangle \subset T^0 C$. Here, $\tilde{\phi} : M \to A^*$ is an exension of ϕ. We can consider the Hamiltonian sections associated with f_X,

$$\mathcal{H}_{f_X} = \mathcal{H}_{\hat{X}} - \mathcal{H}_{\tilde{\phi}(X) \circ \tau_{A^*}} = X^{*\mathbf{c}} + (d^A(\tilde{\phi}(X) \circ \tau_{A^*}))^{\mathbf{v}} = (X, X^{*c} + (d^A(\tilde{\phi}(X) \circ \tau_B)))^v).$$

They are sections of L since $X_{|N} \in \Gamma(B)$ and

$$(\rho^1(\mathcal{H}_{f_X}))_{|C} = (X_{f_X})_{|C} = (\#_{w_{A^*}}(df_X))_{|C} \in TC$$

because C is coisotropic. Since $\left\{ (df_X)_{|C} \mid X \in \Gamma(A) \text{ and } X_{|N} \in \Gamma(B) \right\}$ span a vector subbundle of $T^*_C A^*$ of rank b, it follows that $\left\langle \mathcal{H}_{f_X} \mid X \in \Gamma(A) \text{ and } X_{|N} \in \Gamma(B) \right\rangle$ spans a vector subbundle of rank b.
Consider,

$$\mathcal{V}_C = \left\{ \beta \in \Gamma(A^*) \mid (\beta^v)_{|C} \in \mathcal{X}(C) \right\}.$$

For every $\beta \in \mathcal{V}_C$ we have that $\beta^{\mathbf{v}} = (0, \beta^v)$ is a section of $\Gamma(L)$. Moreover, if $\alpha \in C$ we have that $\{\beta^v(\alpha) \mid \beta \in \mathcal{V}_C\}$ is the fiber of a vector subbunlde of $\mathcal{T}^A A^*$ over C of rank $a - b$. In addition, for every $\beta \in \mathcal{V}_C$ we have that $(\beta^{\mathbf{v}})_{|C} = (0, (\beta^v)_{|C})$ is independent of $\mathcal{H}_{f_X|C}$ for every $X \in \Gamma(A)$ such that $X_{|N} \in \Gamma(B)$. Thus,

$$\Gamma(L) = \left\langle \mathcal{H}_{f_X|C}, \beta^{\mathbf{v}}{}_{|C} \mid X \in \Gamma(A), X_{|N} \in \Gamma(B) \text{ and } \beta \in \mathcal{V}_C \right\rangle.$$

Finally, let us prove that $(I, i)^*\Omega = \Omega_L = 0$. First, using that C is coisotropic, i.e, $\#(T^0 C) \subset TC$ we have

$$\Omega(\mathcal{H}_{f_X}, \mathcal{H}_{f_Y})_{|C} = \{f_X, f_Y\}_{|C} = 0.$$

Furthermore, from (1.35), for every $\beta, \gamma \in \mathcal{V}_C$,

$$\Omega(\beta^{\mathbf{v}}, \gamma^{\mathbf{v}}) = 0$$

Eventually, also from (1.35), we have that

$$\Omega(\mathcal{H}_{f_X}, \beta^{\mathbf{v}}) = \Omega(X^{*\mathbf{c}} + \left(d^{\mathcal{T}^A A^*}(\tilde{\phi}(X) \circ \tau_{A^*}))\right)^{\mathbf{v}}, \beta^{\mathbf{v}}) = \Omega(X^{*\mathbf{c}}, \beta^{\mathbf{v}}) = \widetilde{\beta(X)}$$

and, since $X_{|N} \in \Gamma(B)$ and $\beta_{|N} \in \Gamma(B^0)$ we conclude that $\Omega(\mathcal{H}_{f_X}, \beta^{\mathbf{v}}) = 0$. This proves that L is Lagrangian. $\square$

Remark that the model of Lagrangian Lie subalgebroids that we presented verfies $\tau^1(L) \subset B$. Denote by $\mathcal{V}_C^{\mathbf{v}}$ the vector subundle of $\mathcal{T}^A A^*$ over C whose fiber at the point $\alpha \in C$ is

$$\{\beta^v(\alpha) \mid \beta \in \mathcal{V}_C(\alpha)\}$$

Then, we have that

$$\tau^1(L) \subset B \;\Leftrightarrow\; \mathcal{V}_C^{\mathbf{v}} \subset L \;\Leftrightarrow\; L^\perp = L \subset (\mathcal{V}_C^{\mathbf{v}})^\perp$$

In fact, if $\alpha \in C$ we have that

$$B^0(\tau_{A^*}(\alpha)) = \left\{\beta \in A^*_{\tau_{A^*}(\alpha)} \mid \beta^v_\alpha \in T_\alpha C\right\} = \mathcal{V}_C(\alpha)$$

and thus, using that

$$\Omega(\beta^{\mathbf{v}}, X) = \beta(\widetilde{\tau^1(X)}), \text{ for } X \in L_\alpha \text{ and } \beta \in \mathcal{V}_C(\alpha)$$

we deduce the result.

Now, we will prove that every Lagrangian subbundle of $(\mathcal{T}^A A^*, \Omega)$ verifying the previous property can be described locally by the same procedure.

Theorem 4.2. *Let $(\tau_A, [\![\ ,\ ]\!]_A, \rho_A)$ be a Lie algebroid over M. Let L be a lagrangian subbundle of $(\mathcal{T}^A A^*, \Omega)$ with base manifold an affine subbundle C of A^* over a submanifold N of M. Then,*

i. C is coisotropic on A^* and there exists a Lie subalgebroid B of A over N and a 1-cocycle $\phi \in \Gamma(B^*)$ such that $C = C(B, \phi)$.

ii. If $\rho^1(L) \subset TC$ and $\tau^1(L) \subset B$, locally $L = \mathcal{T}^B C$.

Proof. i. It is already proved on Proposition 2.11 and Theorem 3.13. Thus, locally $C = j^{-1}(\phi(N))$.

ii. In a sufficiently small neighbourhood of each point we have

$$\left\langle (df_X)_{|C} = (d(\hat{X} - \phi(X) \circ \tau_B))_{|C} \mid X \in \Gamma(A),\ X_{|N} \in \Gamma(B) \right\rangle \subset T^0 C.$$

First we prove that the Hamiltonian sections $(\mathcal{H}_{f_X})_{|C}$ associated with f_X form a set of b independent sections of $\Gamma(L)$ by checking that $(\mathcal{H}_{f_X})_{|C} \in \Gamma(L^\perp)$. For every $Y \in \Gamma(L)$,

$$(\Omega(\mathcal{H}_{f_X}, Y))_{|C} = (d^{\mathcal{T}^A A^*} f_X(Y))_{|C} = (\rho^1(Y) f_X)_{|C} = 0$$

because $f_{X|C} = 0$. Likewise, we have that for every $\beta \in \mathcal{V}_C$, $\beta^{\mathbf{v}} \in \Gamma(L)$. Indeed, for every $Y \in \Gamma(L)$,

$$\Omega(\beta^{\mathbf{v}}, Y) = \beta(\widetilde{\tau^1(Y)})$$

and therefore, using that $\beta \in \Gamma(B^0)$ and the fact that $\tau^1(L) \subset B$ we conclude that

$$\Omega(\beta^{\mathbf{v}}, Y) = 0$$

which implies that $\beta^v_{|C} \in \Gamma(L^\perp) = \Gamma(L)$.

On the other hand, it is clear that if $\alpha \in C$, it follows that $\langle\{\mathcal{H}_{f_X}(\alpha), \beta^{\mathbf{v}}(\alpha) \mid X \in \Gamma(A),\ X_{|N} \in \Gamma(B) \text{ and } \beta \in \mathcal{V}_C\}\rangle$ is a real vector space of dimension a. This implies that

$$L(\alpha) = \left\langle\{\mathcal{H}_{f_X}(\alpha), \beta^{\mathbf{v}}(\alpha) \mid X \in \Gamma(A),\ X_{|N} \in \Gamma(B) \text{ and } \beta \in \mathcal{V}_C\}\right\rangle$$

and thus, L is (locally) $\mathcal{T}^B C$.

□

Remark that if in stead of L being a Lagrangian subbundle we require L being a Lagrangian Lie subalgebroid the condition $\rho^1(L) \subset TC$ could be avoid. Nontheless, the second condition $\tau^1(L) \subset B$ is still necessary as shows the following example.

Example 4.3. Let $(\tau_A, [\![\ ,\ ]\!]_A, \rho_A)$ be a Lie algebroid over M, B be a Lie subalgebroid over N, $\phi \in \Gamma(B^*)$ be a 1-cocycle and C be the affine coisotropic submanifold associated with B and ϕ. Suppose that e_0 is a section of $\tau_A : A \to M$ which belongs to the center of A and such that $e_0(q) \notin B_q$ for all $q \in N$. Assume also that $e^0 \in \Gamma(A^*)$ satisfies $e^0(e_0) = 1$ and $(e^0)_{|N} \in \Gamma(B^0)$. Then, one may prove that $\rho(e_0) = 0$ and, thus, $e_0^{*\mathbf{c}} = (e_0, 0) \in \Gamma(\mathcal{T}^A A^*)$.

Now consider the vector subbundle L of $\mathcal{T}^A A^*$ over C whose fiber at the point $\alpha \in C$ is

$$L(\alpha) = \left(\mathcal{T}^B_\alpha C \cap \left\langle\{(e^0)^{\mathbf{v}}(\alpha)\}\right\rangle^0\right) \oplus \langle (e_0)^{*\mathbf{c}}(\alpha)\rangle$$

It follows that L is a Lagrangian subalgebroid over C. However, $\tau^1((e_0)^{*\mathbf{c}}(\alpha)) = e_0(\tau_C(\alpha)) \notin B(\tau_C(\alpha))$, for all $\alpha \in C$.

4.2 Some examples of Lagrangian Lie subalgebroids of the A-tangent bundle to A^*.

Next, we obtain some particular examples of Lagrangian Lie subalgebroids of the A-tangent bundle to A^*, when A is an action Lie algebroid or A is an Atiyah algebroid.

Example 4.4. i. **Action Lie algebroid** Let $\tau_A : A = M \times \mathfrak{g} \to M$ be an action Lie algebroid induced by a left infinitesimal action $\Phi : \mathfrak{g} \to \mathcal{X}(M)$ and $B = N \times \mathfrak{h}$ be the total space of an action Lie subalgebroid as in Example 1.12. Suppose that $\alpha \in \mathfrak{h}^*$ is a 1-cocycle for the Lie subalgebra $\mathfrak{h}$. Then, if $\tilde{\alpha} \in \mathfrak{g}^*$ is an extension of α we have that

$$C = \{(q, \tilde{\alpha} + \gamma) \in N \times \mathfrak{g}^* \mid \gamma \in \mathfrak{h}^0\}$$

is a coisotropic affine subbundle of $A^* = M \times \mathfrak{g}^*$ (see Example 3.14).

On the other hand, as we know (see Example 1.13), the A-tangent bundle to A^* may be identified with the trivial vector bundle $(M \times \mathfrak{g}^*) \times (\mathfrak{g} \times \mathfrak{g}^*) \to M \times \mathfrak{g}$. Under this identification,

$$\begin{aligned} \mathcal{T}^B C) = \{ & ((q, \tilde{\alpha} + \gamma), (\xi, \gamma')) \in (M \times \mathfrak{g}^*) \times (\mathfrak{g} \times \mathfrak{g}^*) \mid \\ & q \in N,\ \gamma \in \mathfrak{h}^0,\ \xi \in \mathfrak{h} \text{ and } \gamma' \in \mathfrak{h}^0\} \end{aligned}$$

is a Lagrangian Lie subalgebroid of $\mathcal{T}^A A^* \cong (M \times \mathfrak{g}^*) \times (\mathfrak{g} \times \mathfrak{g}^*)$ over the submanifold $C(B, \phi)$ of $A^* = M \times \mathfrak{g}^*$.

ii. **Atiyah algebroid** Let $\tau_A : A = \mathfrak{g} \times TM \to M$ be an Atiyah algebroid associated with a trivial principal G-bundle $G \times M \to M$ and $B = \mathfrak{h} \times \mathcal{F}_N$ be the total space of a Lie subalgebroid of $\tau_A : A = \mathfrak{g} \times TM \to M$ (over a submanifold N of M) as in Example 1.12.

Now, suppose that $\alpha : \mathfrak{h}^*$ is a 1-cocycle of the Lie subalgebra $\mathfrak{h}$ and that $\phi : N \to \mathcal{F}_N^*$ is a 1-cocycle of the foliation $\mathcal{F}_N$ on N. Denote by (α, ϕ) the corresponding 1-cocycle of the Lie subalgebroid $\tau_B : B = \mathfrak{h} \times \mathcal{F}_N \to N$ (see Example 1.12). If $\tilde{\alpha} \in \mathfrak{g}^*$ and $\tilde{\phi} : N \to T_N^* M$ are extensions of α and ϕ respectively, it follows that

$$C = \bigcup_{q \in N} \left\{ \left(\tilde{\alpha} + \gamma, \tilde{\phi}(q) + \beta \right) \in A_q^* = \mathfrak{g}^* \times T_q^* M \mid \gamma \in \mathfrak{h}^0 \text{ and } \beta \in T_q^0 N \right\}$$

is an affine coisotropic submanifold of the linear Poisson manifold $A^* = \mathfrak{g}^* \times T^*M$.

On the other hand, the A-tangent bundle to A^* may be identified with the trivial vector bundle $\mathfrak{g}^* \times (\mathfrak{g} \times \mathfrak{g}^*) \times T(T^*M) \to \mathfrak{g}^* \times T^*M = A^*$ (see Example 3.14), and, under this indetification,

$$L = \left\{ \left(\tilde{\alpha} + \gamma, (\xi, \gamma'), T_q\tilde{\phi}(X_q) \right) \mid \gamma, \gamma' \in \mathfrak{h}^0,\ \xi \in \mathfrak{h},\ q \in N \text{ and } X_q \in T_q N \right\}$$

is a Lagrangian Lie subalgebroid of $\mathcal{T}^A A^* = \mathfrak{g}^* \times (\mathfrak{g} \times \mathfrak{g}^*) \times T(T^*M)$ over the submanifold $C(B, (\alpha, \phi))$.

Conclusions and Future Work

In this master thesis we have discussed the local structure of a Lagrangian Lie subalgebroid L of the A-tangent bundle to A^*, $\mathcal{T}^A A^*$ (A being a Lie algebroid), which satisfies certain conditions. First of all, we proved that the base manifold C of L turns out to be coisotropic on A^*. Then, we disccussed the local structure of a fibered coisotropic manifold of A^*. Since a Lagrangian submanifold of the cotangent bundle T^*Q which is fibered over a submanifold of Q is affine, we focused our attention to the particular case when C is an affine subbundle of A^*. In fact, we proved that such a manifold is completely determined by a Lie subalgebroid B of A and a 1-cocycle on B (see Theorem 3.13). In particular, we deduce that when A is the standard Lie algebroid TM then C is (locally) an affine subbundle if, and only if, its characteristic distribution is projectable. However, this result is not true if A is an arbitrary Lie algebroid (see Example 3.9).

Finally, a local description of a Lagrangian Lie subalgebroid L of $\mathcal{T}^A A^*$ over a coisotropic affine subbundle C of A^* is given. In fact, if B is the Lie subalgebroid of A associated with C and $\tau^1(L) \subset B$ we deduce that L is locally the prolongation of B over C.

Thus, this master thesis gives a first approach to the local structure of the Lagrangian Lie subalgebroids of $\mathcal{T}^A A^*$ but more work must be done and some new problems, that are summarized below, arise:

- Local description of the fibered cosiotropic submanifolds of A^* whose characteristic distribution is projectable.
- Local description of an arbitrary fibered coisotropic submanifold of A^*.
- Local description of coisotropic submanifols of A^* (i.e., C is no longer fibered over a submanifold of the base manifold).
- Local desription of Lagrangian Lie subalgebroids of $\mathcal{T}^A A^*$ over a coisotropic submanifold of A^*.

Furthermore, it is noteworthy that as we mentioned on the introduction, our final aim is to apply all these results to develop a Hamilton-Jacobi theory for Hamiltonian systems on linear Poisson manifolds.

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