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Continuous and discrete mechanics for the attitude dynamics of a rigid body on SO(3)

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CONTINUOUS AND DISCRETE MECHANICS FOR THE ATTITUDE DYNAMICS OF A RIGID BODY ON $SO(3)$

by Leonardo Colombo and Fernando Jiménez

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The present notes have been reviewed by the lecturer.

CONTINUOUS AND DISCRETE MECHANICS FOR THE ATTITUDE DYNAMICS OF A RIGID BODY ON $SO(3)$

LEONARDO COLOMBO AND FERNANDO JIMÉNEZ

Abstract. This review is devoted to the obtaining of the continuous and discrete Euler-Lagrange equations for the attitude dynamics of a rigid body in $SO(3)$. On that purpose, we introduce basics about Lagrangian mechanics, Hamilton's principle, Lie groups, Lie algebras, discrete Mechanics, geometric integration, Variational Integratos and some of its geometric properties. We present, quite exhaustively, the development of both continuous and discrete Euler-Lagrange equations for Lie group systems and, as particular cases, that ones concerning the rigid body and $SO(3)$.

Contents

1. Introduction

In this review we try to present a quite exhaustive approximation to *how to develop a Variational Integrator* for a Lagrangian system defined on a Lie group. Furthermore, we will focus in the case of the attitude dynamics of a rigid body, that is, a system whose configuration space is the special orthogonal Lie group $SO(3)$. Besides to the discrete setting, we describe also the continuous one, specifying the procedure to obtain the Euler-Lagrange equations from a Hamilton's principle.

Some of the important topics that come out naturally when we consider a variational approach to discrete mechanics are symplectic-energy-momentum methods, error analysis, constraints, etc.

In the last few years this area has grown to be very large and active area of research, with many points of view and many topics. As in standard continuous mechanics, some things are easier from a Hamiltonian perspective. We will focus nevertheless in the Lagrangian point of view as long as the course which these notes are based on did.

Geometric numerical integration deals with numerical integration methods that preserve geometric properties of the flow of a differential equation, such as invariants, symplecticity and the structure of a configuration manifold ([4], [5], [12], [17]).

Numerical methods that conserve energy, momentum or symplecticity of mechanical systems have been developed ([9], [10], [19], [21]). But the conservation property is often enforced by nonlinear constraints or by a projection onto the manifold defined by the constant conserved quantity.

For instance, there have been many works on symplectic integration, largely done from other points of view than the variational one. For an overview of symplectic integration, see [19] and [20].

Alternatively, a discrete-time mechanical system has been developed according to Hamilton's principle by [18] and [22]. The variational view of discrete-time mechanics is further developed in [6], [7], [24] and an intrinsic form of discrete-time variational principle is established in [16]. The resulting geometric numerical integrators, referred to as **Variational Integrators**, have desirable properties: they are symplectic, momentum preserving and they exhibit excellent energy conservation property. A step ahead, an interesting extension of these ideas to more involved geometries, such as that ones related to groupoids and algebroids, are presented in [23] and [13].

For differential equations that evolve on a Lie group, a group element can be updated by the corresponding group action so that the group structure is preserved naturally. This is referred as a **Lie group method** (see [4] and [5]). For mechanical methods evolving on a Lie group, a discrete time Euler-Poincaré equation has been introduced for a left-invariant Lagrangian system in [15], with application to the free attitude dynamics of a rigid body. A similar work is presented for the attitude dynamics of an axially symmetric rigid body acting under gravitational potential in [3].

The main idea beneath the discrete Mechanics and the obtaining a Variational Integrator is to replace the tangent space of a given manifold with two copies of that manifold. Namely, consider a mechanical system with configuration manifold Q. The velocity phase space is then TQ and the Lagrangian map $L : TQ \to \mathbb{R}$. As just mentioned, in discrete Mechanics the starting point is to replace TQ with $Q \times Q$ and we regard, intuitively, two nearby points as being the discrete analogue of a velocity vector. This defines the **discrete Lagrangian** $L_d : Q \times Q \to \mathbb{R}$ as an approximation of the continuous action integral $S = \int_0^T L\,dt$. Considering the **discrete action sum** as the sum of the discrete Lagrangian along N points (we consider a time grid of $N + 1$ points and time spacing h, such that $t_k = hk$), and applying the Hamilton's principle to this action sum we finally obtain the **discrete Euler-Lagrange equations**. Under some regularity hypotheses, these equations provide a discrete flow $\Upsilon_{L_d} : Q\times Q \to Q\times Q$ which has interesting symplectic-momentum conservation properties.

This technique is applicable to systems defined on Lie groups (actually a Lie group G can be considered as a differentiable manifold). If we also consider an updating choice for our integrator that preserves the group operation and preserves the structure of the configuration space we are dealing with **Lie Group Variational Integrators**. The main motivation for the development of integrators defined in a Lie group is the existence of several systems that evolve in that groups: planes, satellites, multibody systems, etc. In that sense, distinguised goups are $\mathbb{R}^n$, $SO(3)$, $SE(3)$, S^n, etc. Besides, there exist several features that distinguish the Lie Group Variational Integrators with respect to the usual symplectic integrators or Lie group methods. They are summarized in the following table:

As shown, Lie Group Variational Integrators have both geometric properties: symplecticity and group structure preservation. Furthermore, there are some computational reasons that make Lie Group Variational Integrators interesting, such as long time behavoir, accuracy, efficiency and computation time. For instance, a Lie Group Variational Integrator requires 16 less CPU time than the Lie Group Integrator

Methods	Symplecticity	Group Structure
Explicit Runge-Kutta	X	X
Symplectic Runge-Kutta	◯	X
Lie Group Method	X	◯
Lie Group Variational Integrator	◯	◯

and 98 less CPU time than the Symplectic Runge Kutta for similar total energy error.

The review is strutured as follows. In Section 2 we present some basics about Lagrangian mechanics, Hamilton's principle, Lie groups and Lie algebras in order to develop the Euler-Lagrange equations for a Lagrangian problem defined on a general Lie group. In Section 3 we introduce discrete Mechanics, Variational Integrators, its relation with the continuous Lagrangian flow and its geometric properties. We develop the discrete Euler-Lagrange equations for a Lagrangian problem defined on a general Lie group. In Section 4 de obtain the continuous Euler-Lagrange equations for the attitude dynamics of the rigid body on $SO(3)$ according to the Hamilton's principle. As a final and discrete conter part, in Section 5 we obtain the discrete Euler-Lagrange equations for the rigid body problem in $SO(3)$.

2. Continuous Euler-Lagrange equations for Lie groups

In this section we present some basic definitions about Lagrangian Mechanics and the Hamilton's principle. Besides, we show some basics regarding Lie groups and Lie algebras in order to develop the Euler-Lagrange equations for Lagrangian problems defined on Lie groups. Finally, we detail some important geometric properties of the Lagrangian flow.

2.1. **Basic definitions.** Consider a configuration manifold Q, with state space given by the tangent bundle TQ, and Lagrangian $L : TQ \to \mathbb{R}$.

Given an interval $[0, T]$, define the path space to be

$$C(Q) = \mathcal{C}([0, T], Q) = \{q : [0, T] \to Q \mid q \text{ is a } \mathcal{C}^2 - curve\}$$

and the action map $\mathcal{A} : C(Q) \to \mathbb{R}$ to be

$$\mathcal{A}(q) = \int_0^T L(q(t), \dot{q}(t))dt.$$

It can be proved that $C(Q)$ is a smooth manifold [1] and $\mathcal{A}$ as smooth as L. The tangent space $T_qC(Q)$ to $C(Q)$ at the point q is the set of $\mathcal{C}^2-$maps $v_q : [0,T] \to TQ$ such that $\pi_Q \circ v_q = q$, where $\pi_Q : TQ \to Q$ is the canonical projection.

We define the second-order submanifold of $T(TQ)$ to be

$$T^{(2)}Q \equiv \{w \in T(TQ) \mid T\pi_Q(w) = \pi_{TQ}(w)\} \subset T(TQ),$$

where $\pi_{TQ} : T(TQ) \to TQ$ and $\pi_Q : TQ \to Q$ are the canonical projections. $T^{(2)}Q$ is the set of derivatives $\dfrac{d^2q}{dt^2}(0)$ of curves $q : \mathbb{R} \to Q$ which are elements of the form $(q, \dot{q}), (\dot{q}, \ddot{q})$.

2.2. **Hamilton's principle and Euler-Lagrange equations.** The Lagrangian formulation of the mechanics is based on the Newton's law. One choose a configuration space Q, with coordinates $q^i, i = 1, \ldots, n$; that describes the configuration of the system under study. Then one introduce the Lagrangian $L(q^i, \dot{q}^i)$, which is shorthand notation for $L(q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n)$. Usually, L is kinetic minus potential energy of the system and one takes $\dot{q}^i = \frac{d}{dt}q^i$ to be the system velocity.

The Hamilton's principle states

$$\delta \int_0^T L(q^i, \dot{q}^i)dt = 0, \tag{1}$$

where, we choose curves $q^i(t)$ joining two fixed points in Q over a fixed interval $[a, b]$.

Hamilton's principle states that this function has a critical points at a solution within the space of curves. If we let δq^i be a variation, that is, the derivative of a family of curves with respect to a parameter, then (1) is equivalent to

$$\sum_{i=1}^{n} \int_a^b \left(\frac{\partial L}{\partial q^i}\delta q^i + \frac{\partial L}{\partial \dot{q}^i}\delta \dot{q}^i \right) dt = 0,$$

for all variations δq^i, where $\delta \dot{q}^i = \frac{d}{dt}\delta q^i$.

Using this, integrating by parts, employing the boundary conditions $\delta q^i = 0$ at $t = a$ and b and scince δq^i is arbitrary we can obtain the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0, \quad i = 1, \ldots, n.$$

2.3. **Preliminaries on Lie groups.** First, we give the basic definitions and properties of Lie groups. A Lie group is a differentiable manifold that has a group structure such that the group operation is smooth. A Lie algebra is the tangent space of the Lie group G at the identity of the group, $e \in G$, with the bracket $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that is bilinear, skew symmetric and satisfies the Jacobi identity. For $g, h \in G$, the *left-translation* map is defined as $L_h : G \to G$, by $L_h g = hg$. Similarly, the *right-translation* $R_h : G \to G$ is defined as $R_h g = gh$. Given $\xi \in \mathfrak{g}$ define a vector field $X_\xi : G \to TG$ such that $X_\xi(g) = T_e L_g \cdot \xi$, and let the corresponding unique integral curve passing through the identity e at $t = 0$ be denoted by $\gamma_\xi(t)$. The *exponential map* $\exp : \mathfrak{g} \to G$ is defined by $\exp \xi = \gamma_\xi(1)$. The application exp is a local diffeomorphism from a neighborhood of zero in $\mathfrak{g}$ onto a neighborhood of $e \in G$.

Define the *inner automorphism* $I_g : G \to G$ as $I_g(h) = ghg^{-1}$. The *adjoint operator* $Ad_g : \mathfrak{g} \to \mathfrak{g}$ is the differential of $I_g(h)$ with respect to h at $h = e$ along the direction $\eta \in \mathfrak{g}$, that is $Ad_g \eta = T_e I_g \cdot h$. The *ad* operator $ad_\xi : \mathfrak{g} \to \mathfrak{g}$ is obtained by differentiating $Ad_g \eta$ with respect to g at e in the direction ξ, that is $ad_\xi \eta = T_e(Ad_g \eta) \cdot \eta$. This corresponds to Lie bracket (i,e; $ad_\xi \eta = [\xi, \eta]$).

We define the *coadjoint operator* $Ad^* : G \times \mathfrak{g}^* \to \mathfrak{g}^*$ from the paring between vectors and covectors by $\langle Ad^*_g \alpha, \xi \rangle = \langle \alpha, Ad_g \xi \rangle$ for $\alpha \in \mathfrak{g}^*$. The *co-ad operator* $ad^* : \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}^*$ is defined by $\langle ad^*_\eta \alpha, \eta \rangle = \langle \alpha, ad_\eta \xi \rangle$ for $\alpha \in \mathfrak{g}^*$.

2.4. **The Euler-Lagrange equations on Lie Groups.** Consider a mechanical system evolving on a Lie group G. We derive the corresponding Euler-Lagrange equations from a variational principle.

The configuration space is a Lie group G. We trivialize (by a left trivialization) the tangent space TG as $G \times \mathfrak{g}$. A tangent vector $(g, \dot{g}) \in T_g G$ is expressed as

$$\dot{g} = T_e L_g \cdot \xi = g\xi.$$

We recall that the left-translation can be use to trivialize the tangent bundle TG and the cotangent bundle as follows

$$\begin{array}{rclc} TG & \to & G \times \mathfrak{g}\,, & (g, \dot{g}) \longmapsto (g, g^{-1}\dot{g}) = (g, T_g L_{g^{-1}} \dot{g}) = (g, \xi), \\ T^*G & \to & G \times \mathfrak{g}^*, & (g, \alpha_g) \longmapsto (g, T_e^* L_g(\alpha_g)) = (g, \alpha). \end{array}$$

In the same way, we have the following identifications: $TTG \equiv G \times 3\mathfrak{g}$, $T^*TG = G \times \mathfrak{g} \times 2\mathfrak{g}^*$, $TT^*G = G \times \mathfrak{g}^* \times \mathfrak{g} \times \mathfrak{g}^*$ and $T^*T^*G = G \times 3\mathfrak{g}^*$.

In the sequel, we assume that the Lagrangian of the mechanical system is given by $L(g, \xi) : G \times \mathfrak{g} \to \mathbb{R}$.

Define the action map as

$$\mathcal{A} = \int_{t_0}^{t_f} L(g, \xi) dt, \quad t_0, t_f \in [0, T] \subset \mathbb{R}.$$

As before, Hamilton's principle states that the variation of the action integral is equal to zero,

$$\delta \mathcal{A} = \delta \int_{t_0}^{t_f} L(g, \xi) dt = 0.$$

Now, let $g(t)$ be a differential curve in G defined for $t \in [t_0, t_f]$. The variation is a differentiable mapping $g_\epsilon(t) : (-c, c) \times [t_0, t_f] \to 0$ for $c >$

0 such that $g_0(t) = g(t), \forall t \in [t_0, t_f]$ and $g_\epsilon(t_0) = g(t_0), g_\epsilon(t_f) = g(t_f)$ $\forall \epsilon \in (-c, c)$. We express the variation using the exponential map (see [8], [14] for other approaches),

$$g_\epsilon(t) = g \exp \epsilon\eta(t),$$

for any arbitrary curve $\eta(t) \in \mathfrak{g}$. These variations are well defined for some constant c because the exponential map is a local diffeomorphism between $\mathfrak{g}$ and G, and it satisfies the properties of the fixed points $\eta(t_0) = \eta(t_f) = 0$. Since this is obtained by a group operation, it is also guaranteed that the variation lies on G for any $\eta(t)$.

The infinitesimal variation of g is given by,

$$\delta g(t) = \frac{d}{dt}\Big|_{\epsilon=0} g_\epsilon(t) = T_e L_{g(t)} \frac{d}{dt}\Big|_{\epsilon=0} \exp \epsilon\eta(t) = g(t)\eta(t). \tag{2}$$

for each $t \in [t_0, f_f]$, the infinitesimal variation $\delta g(t)$ lies in the tangent space $T_{g(t)}G$. Using this expression and $\dot{g} = g\xi$, the infinitesimal variation of $\xi(t)$ is obtained as follows (see [2], [11] for example).

$$\delta\xi(t) = \dot{\eta} + ad_{\xi(t)}\eta(t). \tag{3}$$

The equations (2) and (3) are infinitesimal variations of $(g(t), \xi(t)) : [t_0, t_f] \to G \times \mathfrak{g}$, respectively.

The variation of the Lagrangian is written as

$$\delta L(g, \xi) = \frac{\partial L}{\partial g}\delta g + \frac{\partial L}{\partial \xi}\delta\xi,$$

where $\frac{\partial L}{\partial g} \in T^*G$ denotes the derivative of L with respect to g, given by

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} L(g_\epsilon, \xi) = \frac{\partial L}{\partial g}\delta g,$$

and $\frac{\partial L}{\partial \xi}(g, \xi) \in \mathfrak{g}^*$ is defined similarly.

Therefore,

$$\begin{aligned}\delta L(g, \xi) &= \left\langle \frac{\partial L}{\partial g}(g, \xi), \delta g \right\rangle + \left\langle \frac{\partial L}{\partial \xi}(g, \xi), \delta\xi \right\rangle \\ &= \left\langle \frac{\partial L}{\partial g}(g, \xi), (T_e L_g \circ T_g L_{g^{-1}})\delta g \right\rangle + \left\langle \frac{\partial L}{\partial \xi}(g, \xi), \delta\xi \right\rangle,\end{aligned}$$

because $T(L_g \circ L_{g^{-1}}) = TL_g \circ TL_{g^{-1}}$ is equal to the identity on TG. Substituting (2) and (3) we have that

$$\begin{aligned}
\delta L(g,\xi) &= \left\langle \frac{\partial L}{\partial g}(g,\xi), T_eL_g\cdot\eta \right\rangle + \left\langle \frac{\partial L}{\partial \xi}(g,\xi), \dot{\eta} + ad_{\xi}\eta \right\rangle \\
&= \left\langle T_e^*L_g \cdot \frac{\partial L}{\partial g}(g,\xi) + ad_{\xi}^*\cdot\frac{\partial L}{\partial \xi}(g,\xi), \eta \right\rangle \\
&\quad + \left\langle \frac{\partial L}{\partial \xi}(g,\xi), \dot{\eta} \right\rangle.
\end{aligned} \tag{4}$$

Therefore, the variation of the action integral is given by

$$\delta\mathcal{A} = \int_{t_0}^{t_f} \delta L(g,\xi)dt.$$

Substituting (4) and using integration by parts, the variation of the action integral is given by

$$\begin{aligned}
\delta\mathcal{A} &= \int_{t_0}^{t_f} \left(\left\langle T_e^*L_g \cdot \frac{\partial L}{\partial g}(g,\xi) + ad_{\xi}^*\cdot\frac{\partial L}{\partial \xi}(g,\xi), \eta \right\rangle \right. \\
&\qquad \left. + \left\langle \frac{\partial L}{\partial \xi}(g,\xi), \dot{\eta} \right\rangle \right) dt \\
&= \left\langle \frac{\partial L}{\partial \xi}(g,\xi), \eta \right\rangle \Big|_{t_0}^{t_f} - \int_{t_0}^{t_f} \left\langle \frac{d}{dt}\frac{\partial L}{\partial \xi}(g,\xi), \eta \right\rangle dt \\
&\qquad - \int_{t_0}^{t_f} \left\langle T_e^*L_g \cdot \frac{\partial L}{\partial g}(g,\xi) + ad_{\xi}^*\cdot\frac{\partial L}{\partial \xi}(g,\xi), \eta \right\rangle dt.
\end{aligned}$$

Since $\eta(t) = 0$ at $t = t_0$ and $t = t_f$, the first term of the above equation vanishes thus, we obtain

$$\begin{aligned}
\delta\mathcal{A} &= \int_{t_0}^{t_f} \left(\left\langle T_e^*L_g \cdot \frac{\partial L}{\partial g}(g,\xi) + ad_{\xi}^*\cdot\frac{\partial L}{\partial \xi}(g,\xi), \eta \right\rangle \right. \\
&\qquad \left. - \left\langle \frac{d}{dt}\frac{\partial L}{\partial \xi}(g,\xi), \eta \right\rangle \right) dt.
\end{aligned} \tag{5}$$

From Hamilton's principle $\delta\mathcal{A} = 0$ $\forall\eta \in \mathfrak{g}$. Then, the corresponding Euler-Lagrange equations for $L : G \times \mathfrak{g} \to \mathbb{R}$ are given by

$$0 = \frac{d}{dt}\frac{\partial L}{\partial \xi}(g,\xi) - ad_{\xi}^*\frac{\partial L}{\partial \xi}(g,\xi) - (T_e^*L_g)\cdot\frac{\partial L}{\partial g}(g,\xi), \tag{6}$$

$$\dot{g} = g\xi. \tag{7}$$

If the Lagrangian is $G-$invariant the resulting equation is equivalent to the Euler-Poincaré eqs. and (7) is the reconstruction equation (see [14]). Therefore, both (6) and (7) can be considered as a generalization of the Euler-Poincaré equations.

Now, if we consider the identification of the tangent bundle TG with $G\times\mathfrak{g}$ by a right-trivialization, the corresponding Euler-Lagrange equations for $L : G\times\mathfrak{g}\to\mathbb{R}$ are given by

$$0 = \frac{d}{dt}\frac{\partial L}{\partial \xi}(g,\xi) + ad^*_\xi\frac{\partial L}{\partial \xi}(g,\xi) - (T^*_e R_g)\cdot\frac{\partial L}{\partial g}(g,\xi), \tag{8}$$

$$\dot{g} = \xi g. \tag{9}$$

2.5. **Legendre Transformation.** As before, we identify the tangent space TG with $G\times\mathfrak{g}$ using a left-trivialization. In the same way, we can identify the cotangent bundle T^*G with $G\times\mathfrak{g}^*$. For the given Lagrangian, the *Legendre transformation* $\mathbb{F}L : G\times\mathfrak{g}\to G\times\mathfrak{g}^*$ is defined as

$$\mathbf{F}L(g,\xi) = (g,\mu),$$

where $\mu\in\mathfrak{g}^*$ is given by $\mu = \frac{\partial L}{\partial \xi}(g,\xi)$.

If $\mathbb{F}L$ is a global diffeomorphism, the Lagrangian is called *hyperregular*, which induces a Hamiltonian system on $G\times\mathfrak{g}^*$, that is, the Legendre transformation yields Hamilton's equation that are equivalent to Euler-Lagrange equations.

$$\begin{aligned} 0 &= \frac{d}{dt}\mu - ad^*_\xi\mu - (T^*_e L_g)\cdot\frac{\partial L}{\partial g}(g,\xi), \\ \dot{g} &= g\xi. \end{aligned}$$

2.6. **Lagrangian Flow and Momentum preservation.** In this subsection we show two properties, the simplecticity and momentum preservation of the Lagrangian flow.

i) **Simplecticity**

Let Θ_L be the Lagrangian one-form

$$\Theta_L(g,\xi)\cdot(\delta g,\delta\xi) = \langle\frac{\partial L}{\partial \xi}(g,\xi), g^{-1}\delta g\rangle.$$

The Lagrangian symplectic 2-form is given by $\Omega_L = -d\Theta_L$ and the flow map $\mathcal{F}_L : (G\times\mathfrak{g})\times[0,t_f-t_0]\to G\times\mathfrak{g}$ as the flow of the Euler-Lagrange equations for $L : G\times\mathfrak{g}\to\mathbb{R}$

Proposition 2.1. *The lagrangian flow preserves the Lagrangian symplectic 2-form,*

$$(\mathcal{F}^T_L)^*\Omega_L = \Omega_L$$

for $T = t_f - t_0$

ii) **Noether's Theorem**

Suppose that a Lie group H with Lie algebra $\mathfrak{h}$ acts on G. We consider the left action $\Phi : H\times G\to G$ such that $\Phi(e,g) = g$ and $\Phi(h,\Phi(h',g)) = \Phi(hh',g)$ for any $g\in G$ and $h,h'\in H$. The left trivialization is given by $\phi_L : TG\to G\times\mathfrak{g}$ as $\phi_L(g,\dot{g}) =$

$(g, g^{-1}\dot{g})$. The infinitesimal generators $\zeta_G : G \to G \times \mathfrak{g}$ and $\zeta_{G\times\mathfrak{g}} : G \times \mathfrak{g} \to T(G \times \mathfrak{g}) \simeq G \times \mathfrak{g}$ for the action where $\zeta \in \mathfrak{h}$, are given by

$$\zeta_G(g) = \phi_L \circ \frac{d}{dt}\Big|_{\epsilon=0} \Phi_{\exp_H \epsilon\zeta}(g),$$

$$\zeta_{G\times\mathfrak{g}}(g, \xi) = \frac{d}{dt}\Big|_{\epsilon=0} \phi_L \circ T_g\Phi_{\exp_H \epsilon\zeta}(g) \cdot (\phi_L^{-1}(g, \xi)).$$

We define the momentum map $J_L : G \times \mathfrak{g} \to \mathfrak{h}^*$ as

$$J(g, \xi) \cdot \zeta = \Theta_L \cdot \zeta_{G\times\mathfrak{g}}(g, \xi).$$

Proposition 2.2. *Suppose that the Lagrangian is infinitesimal invariant under the lifted action for any $\zeta \in \mathfrak{h}$. Then, the Lagrangian flow preserves the momentum map*

$$J_L(\mathcal{F}_L^T(g, \xi)) = J_L(g, \xi).$$

3. Discrete Euler-Lagrange equations for Lie groups

In this section we introduce Discrete Mechanics and we define what a Variational Integrator is, which is its relation with the continuous Lagrangian flow and, besides, we enumerate some important geometric properties of the discrete flow. Moreover, we develop the discrete Euler-Lagrange equations for a Lagrangian problem defined in a Lie group and present its main geometric properties.

3.1. Discrete mechanics and Variational integrators.

In the following and along this section,we will summarize the main features of variational integrators [16]. A **discrete Lagrangian** is a map $L_d : Q \times Q \to \mathbb{R}$, which may be considered as an approximation of the integral action in a single time step h defined by a continuous Lagrangian $L\colon TQ \to \mathbb{R}$:

$$L_d(q_0, q_1) \approx \int_0^h L(q(t), \dot{q}(t))\, dt$$

where $q(t)$ is a solution of the Euler-Lagrange equations for L, where $q(0) = q_0$, $q(h) = q_1$ and $h > 0$ is small enough.

Define the **action sum** $\mathcal{A}_d\colon Q^{N+1} \to \mathbb{R}$, corresponding to the Lagrangian L_d by

$$\mathcal{A}_d = \sum_{k=1}^{N} L_d(q_{k-1}, q_k),$$

where $q_k \in Q$ for $0 \le k \le N$ and N is the number of steps. The discrete variational principle states that the solutions of the discrete system determined by L_d must extremize the action sum given fixed endpoints q_0 and q_N. By extremizing $\mathcal{A}_d$ over q_k, $1 \le k \le N-1$, we obtain the system of difference equations

$$D_1L_d(q_k, q_{k+1}) + D_2L_d(q_{k-1}, q_k) = 0. \tag{10}$$

or, in coordinates,

$$\frac{\partial L_d}{\partial x^i}(q_k, q_{k+1}) + \frac{\partial L_d}{\partial y^i}(q_{k-1}, q_k) = 0,$$

where $1 \leq i \leq n,\ 1 \leq k \leq N-1$ and x, y are respectively the n-first and n-second variables of the function L.

These equations are usually called the **discrete Euler–Lagrange equations**. Under some regularity hypotheses (the matrix $(D_{12}L_d(q_k, q_{k+1}))$ is regular), it is possible to define a (local) discrete flow $\Upsilon_{L_d}\colon Q \times Q \to Q \times Q$, by $\Upsilon_{L_d}(q_{k-1}, q_k) = (q_k, q_{k+1})$ from. Define the discrete Legendre transformations associated to L_d as

$$\begin{array}{rcl} \mathbb{F}^- L_d\colon Q \times Q & \to & T^*Q \\ (q_k, q_{k+1}) & \longmapsto & (q_k, -D_1 L_d(q_k, q_{k+1})), \\ \mathbb{F}^+ L_d\colon Q \times Q & \to & T^*Q \\ (q_k, q_{k+1}) & \longmapsto & (q_{k+1}, D_2 L_d(q_k, q_{k+1})). \end{array}$$

By means of these discrete Legendre transformation we can define the **discrete Hamilton's equations** as

$$\begin{aligned} p_k &= \mathbb{F}^- L_d(q_k, q_{k+1}) = -D_1 L_d(q_k, q_{k+1}), \\ p_{k+1} &= \mathbb{F}^+ L_d(q_k, q_{k+1}) = D_2 L_d(q_k, q_{k+1}), \end{aligned}$$

which implicitly define a discrete flow $\varphi_{L_d} : T^*Q \to T^*Q$, by $(q_k, p_k) \mapsto (q_{k+1}, p_{k+1})$.

Let define the discrete Poincaré–Cartan 2-form by $\omega_d = (\mathbb{F}^+ L_d)^*\omega_Q = (\mathbb{F}^- L_d)^*\omega_Q$, where ω_Q is the canonical symplectic form on T^*Q. The discrete algorithm determined by Υ_{L_d} preserves the symplectic form ω_d, i.e., $\Upsilon^*_{L_d}\omega_d = \omega_d$. Moreover, if the discrete Lagrangian is invariant under the diagonal action of a Lie group G, then the discrete momentum map $J_d\colon Q \times Q \to \mathfrak{g}^*$ defined by

$$\langle J_d(q_k, q_{k+1}), \xi \rangle = \langle D_2 L_d(q_k, q_{k+1}), \xi_Q(q_{k+1}) \rangle$$

is preserved by the discrete flow. Therefore, these integrators are symplectic-momentum preserving. Here, ξ_Q denotes the fundamental vector field determined by $\xi \in \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G. (See [16] for more details.)

Example 3.1. Let consider the usual mechanical Lagrangian $L(q, \dot{q}) = \frac{1}{2}\dot{q}^T M \dot{q} - V(q)$ and define the discrete Lagrangian by the usual Euler A discretization [4]

$$L_d(q_k, q_{k+1}) = hL\left(q_{k+\frac{1}{2}}, \frac{q_{k+1} - q_k}{h}\right),$$

where we have used the notation $q_{k+\frac{1}{2}} := \frac{q_{k+1}+q_k}{2}$. The resulting variational integrator, which defines the flow $\Upsilon_{L_d}(q_{k-1}, q_k) = (q_k, q_{k+1})$, is the following

$$M\left(\frac{q_{k+1}-2q_k+q_{k-1}}{h^2}\right) = -\frac{1}{2}\left(\nabla V(q_{k-\frac{1}{2}})+\nabla V(q_{k+\frac{1}{2}})\right).$$

This is clearly a discrete analog of the Newton's second law $M\ddot{q} = -\nabla V(q)$.

3.2. **Discrete-time Euler-Lagrange equations.** We are going to develop discrete-time Euler-Lagrange equations for a mechanical system evolving on an abstract Lie group G. Besides that one given in (2.3), a simple definition of Lie Group is the following: a Lie group is a group G which is a differentiable manifold, and for which the internal product $G\times G \to G$ is a differentiable mapping. Every Lie group G has associated a Lie algebra $\mathfrak{g}$, which can be defined as the tangent vector space of G at the identity element $e \in G$, that is $\mathfrak{g} = T_eG$. Of special interest in mathematics, physics and engeneering are the matrix Lie groups. We summarize some of them in TABLE 1 below.

TABLE 1. Matrix Lie Groups and Lie Algebras

G	$\mathfrak{g}$
$GL(n) = \left\{Y \middle\vert \det Y \neq 0\right\}$	$\mathfrak{gl}(n) = \left\{A \middle\vert \text{ arbitrary matrix}\right\}$
$SL(n) = \left\{Y \middle\vert \det Y = 1\right\}$	$\mathfrak{sl}(n) = \left\{A \middle\vert \ \mathrm{tr}A = 0\right\}$
$O(n) = \left\{Y \middle\vert Y^TY = I\right\}$	$\mathfrak{o}(n) = \left\{A \middle\vert \ A + A^T = 0\right\}$
$SO(n) = \left\{Y \in O(n) \middle\vert \det Y = 1\right\}$	$\mathfrak{so}(n) = \left\{A \middle\vert \ A + A^T = 0\right\}$
$Sp(n) = \left\{Y \middle\vert Y^TJY = J\right\}$	$\mathfrak{sp}(n) = \left\{A \middle\vert \ JA + JA^T = 0\right\}$

Nevertheless, we recall that the following development is based on an abstract group G. The Lie group method is explicitly adopted in the context of a variational integrator to construct a unified geometric integrator (**Lie group variational integrator**). As variational integrator, it preserves the geometric features of dynamics, such as symplecticity and any momentum map, as well as the geometry of the configuration manifold (G) by automatically remaining on the Lie group.

Consider a mechanical system evolving on G. The continuous problem is defined in the tangent bundle of the group, which, taking a left trivialization, is isomorphic to a copy of the group times a copy of its algebra. In other words $L : G \times \mathfrak{g} \to \mathbb{R}$. The procedure is pure *variational integration*-kind: the discrete time trajectory is derived such it minimizes the summation of the discrete Lagrangian, called, as we have seen in the previous subsection, the action sum. The discrete trajectory approximates the continuous one $g(t) \in G$ in the following way: $\{g_k\}_{k=0}^{N} \in G$, where $g_k \simeq g(t_k)$. Time is discretized in N steps of h size like $t_k = hk$, such that $Nh = T$. In this derivation, the discrete Legendre transformation provides an alternative description of mechanical systems referred to as discrete Hamiltonian mechanics. The point is to discretize Hamilton's principle, where the variations of the group elements are expressed in terms of the Lie algebra $\mathfrak{g}$ using for instance the exponential map, and updating group elements using group operations in order to remain in the group itself.

Consider the discrete space as $G \times G$. Define $f_k \in G$ such that

$$g_{k+1} = g_k f_k. \tag{11}$$

This is the kinematics or reconstruction equation, which provides g_{k+1} in terms of g_k and f_k. For sake of simplicity we will denote de group product $\cdot : G \times G \to G$, $(g, h) \mapsto g \cdot h$, just by gh. As mentioned before, this guarantees that the discrete flow lies on G without extra constrains or projections [5]. We choose $L_g : G \times G \to \mathbb{R}$ such that it approximates the action integral along the exact solution of the Euler-Lagrange equations over a single step (exact Lagrangian)

$$L_d^E(g_k, f_k) = \int_0^h L(\tilde{g}(t), \tilde{g}^{-1}(t)\dot{\tilde{g}}(t))\, dt,$$

where $\tilde{g} : [0, h] \to G$ satisfies the continuous Euler-Lagrange equations (see the continuous setting) over $[0, h]$ with boundary conditions $\tilde{g}(0) = g_k$ and $\tilde{g}(h) = g_k f_k$. The accuracy of the resulting variational integrator is equal to the accuracy of the discrete Lagrangian (see [16]).

Define the action sum as

$$\mathcal{A}_d = \sum_{k=0}^{N-1} L_d(g_k, f_k). \tag{12}$$

Discrete Hamilton's principle states that this action sum does not vary to the first order for all possible variations of a curve in G. In other words:

$$\delta\mathcal{A}_d \sum_{k=0}^{N-1} \delta L_d(g_k, f_k) = 0.$$

The variations of the sequence $\{g_k\}_{k=0}^N$ is expressed, similarly to the continuous case, as

$$g_k^\epsilon = g_k \exp(\epsilon\eta_k),$$

where exp is the usual exponential map, which is a diffeomorphism between $\mathfrak{g}$ and G. Here $\{\eta_k\}_{k=0}^N$ is a sequence in $\mathfrak{g}$ satisfying $\eta_0 = \eta_N = 0$. The infinitesimal variation is given by

$$\delta g_k = g_k \eta_k, \tag{13}$$

where $g_k\eta_k$ is a shorthand notation for $T_e l_{g_k} \eta_k$.

Taking into account equation (11), the infinitesimal variation of f_k is given by

$$\delta f_k = \frac{d}{d\epsilon}\Big|_{\epsilon=0} (g_k^\epsilon)^{-1} g_{k+1}^\epsilon = T_e l_{f_k} \left(-\mathrm{Ad}_{f_k^{-1}} \eta_k + \eta_{k+1}\right). \tag{14}$$

Both δg_k and δf_k belong to $\mathfrak{g}$. Following our program, now is necessary to calculate the variation of the discrete Lagrangian, which is given by

$$\delta L_d(g_k, f_k) = \langle D_{g_k} L_d(g_k, f_k)\,,\, \delta g_k\rangle + \langle D_{f_k} L_d(g_k, f_k)\,,\, \delta f_k\rangle.$$

Employing properties of the cotangent bundle and the expressions (13) and (14) we finally arrive to

$$\begin{aligned} \delta L_d(g_k, f_k) = {} & \langle T_e^* l_{g_k} D_{g_k} L_d(g_k, f_k)\,,\, \eta_k\rangle \\ & - \langle \mathrm{Ad}^*_{f_k^{-1}} \left(T_e^* l_{f_k} D_{f_k} L_d(g_k, f_k)\right)\,,\, \eta_k\rangle \\ & + \langle T_e^* l_{f_k} D_{f_k} L_d(g_k, f_k)\,,\, \eta_{k+1}\rangle. \end{aligned} \tag{15}$$

Introducing (15) in the summation and rearranging the sum index, the variation of the action sum can be written as

$$\begin{aligned} \delta\mathcal{A}_d = {} & \langle T_e^* l_{f_{N-1}} D_{f_{N-1}} L_d(g_{N-1}, f_{N-1})\,,\, \eta_{N-1}\rangle \\ & + \langle T_e^* l_{g_0} D_{g_0} L_d(g_0, f_0) - \mathrm{Ad}^*_{f_0^{-1}} \left(T_e^* l_{f_0} D_{f_0} L_d(g_0, f_0)\right)\,,\, \eta_0\rangle \\ & + \sum_{k=1}^{N-1} \langle T_e^* l_{g_k} D_{g_k} L_d(g_k, f_k) - \mathrm{Ad}^*_{f_k^{-1}} \left(T_e^* l_{f_k} D_{f_k} L_d(g_k, f_k)\right)\,,\, \eta_k\rangle \\ & + \sum_{k=1}^{N-1} \langle T_e^* l_{f_{k-1}} D_{f_{k-1}} L_d(g_{k-1}, f_{k-1})\,,\, \eta_k\rangle. \end{aligned}$$

Since $\eta_0 = \eta_N = 0$, the two first terms vanish. From the discrete Hamilton's principle $\delta\mathcal{A}_d = 0$ for all possible variations, which yields the discrete Euler-Lagrange equations on G:

$$\begin{aligned}&T_e^*l_{f_{k-1}}D_{f_{k-1}}L_d(g_{k-1},f_{k-1})+T_e^*l_{g_k}D_{g_k}L_d(g_k,f_k)\\&\qquad-\operatorname{Ad}^*_{f_k^{-1}}\left(T_e^*l_{f_k}D_{f_k}L_d(g_k,f_k)\right)=0,\\&g_{k+1}=g_kf_k.\end{aligned}\tag{16}$$

This set of equations (where we have added the reconstruction equation (11)), provides the discrete flow $\mathfrak{F}_{L_d}(g_{k-1},f_{k-1})=(g_k,f_k)$.

3.2.1. *Discrete Legendre transformation.* : Sometimes is more useful to express the discrete flow map in the cotangent bundle using the discrete Legendre transformation.

Define the discrete Legendre transforms $\mathbb{F}^+L_d,\mathbb{F}^-L_d:G\times G\to G\times\mathfrak{g}^*$ by:

$$\begin{aligned}\mathbb{F}^+L_d(g_k,f_k)&=(g_kf_k,\mu_{k+1}),\\\mathbb{F}^-L_d(g_k,f_k)&=(g_k,\mu_k),\end{aligned}$$

where $\mu_k,\mu_{k+1}\in\mathfrak{g}^*$ are given by

$$\begin{aligned}\mu_k&=-T_e^*l_{g_k}D_{g_k}L_d(g_k,f_k)+\operatorname{Ad}^*_{f_k^{-1}}\left(T_e^*l_{f_k}D_{f_k}L_d(g_k,f_k)\right),\\\mu_{k+1}&=T_e^*l_{f_k}D_{f_k}L_d(g_k,f_k).\end{aligned}$$

These transformations are well defined since, as is clear, the discrete Euler-Lagrange (16) equations can be expressed by means of the momentum matching equation

$$\mathbb{F}^+L_d(g_{k-1},f_{k-1})=\mathbb{F}^-L_d(g_k,f_k).$$

Combining both Legendre transforms we obtain the discrete Hamiltonian flow $\tilde{\mathfrak{F}}_{L_d}:G\times\mathfrak{g}^*\to G\times\mathfrak{g}^*$, $\tilde{\mathfrak{F}}_{L_d}(g_k,\mu_k)=(g_{k+1},\mu_{k+1})$

$$\tilde{\mathfrak{F}}_{L_d}=\mathbb{F}^+L_d\circ\left(\mathbb{F}^-L_d\right)^{-1}.$$

In consequence, the discrete Hamiltonian flow map can be alternatively written as

$$\tilde{\mathfrak{F}}_{L_d}=\mathbb{F}^{\pm}L_d\circ\mathfrak{F}_{L_d}\circ\left(\mathbb{F}^{\pm}L_d\right)^{-1}.$$

The discrete Hamiltonian equations can be explicitly written as follows:

$$\begin{aligned}&\operatorname{Ad}^*_{f_k^{-1}}\left(T_e^*l_{f_k}D_{f_k}L_d(g_k,f_k)\right)=\mu_k+T_e^*l_{g_k}D_{g_k}L_d(g_k,f_k),\\&g_{k+1}=g_kf_k,\\&\mu_{k+1}=\operatorname{Ad}^*_{f_k}\left(\mu_k+T_e^*l_{g_k}D_{g_k}L_d(g_k,f_k)\right).\end{aligned}$$

3.2.2. *Conservation properties of the discrete Lagrangian flow.* : The following developments can be considered as a special form of general properties of discrete Lagrangian flows, applied to a Lie group configuration manifold (see [16] for deeper understanding):

i) **Symplecticity**: Let $\Theta^+_{L_d}$, $\Theta^-_{L_d}$ be the discrete Lagrangian one-forms on $G \times G$ given by

$$\langle \Theta^+_{L_d}(g_k, f_k), (\delta g_k, \delta f_k)\rangle = \langle T^*_e l_{f_k} D_{f_k} L_{d(k)}\, ,\, f_k^{-1}\delta f_k + \mathrm{Ad}^{-1}_{f_k}\, g_k^{-1}\delta g_k\rangle,$$

$$\langle \Theta^-_{L_d}(g_k, f_k), (\delta g_k, \delta f_k)\rangle = -\langle T^*_e l_{g_k} D_{g_k} L_{d(k)} - \mathrm{Ad}^*_{f_k^{-1}}\left(T^*_e l_{f_k} D_{f_k} L_{d(k)}\right)\, ,\, g_k^{-1}\delta g_k\rangle.$$

From (14) we have that

$$\eta_{k+1} = f_k^{-1}\delta f_k + \mathrm{Ad}_{f_k^{-1}}\, g_k^{-1}\delta g_k.$$

Substituting in the definition of $\Theta^\pm_{L_d}$ above, and comparing with (15) it can be shown that $\mathrm{d}L_d = \Theta^+_{L_d} - \Theta^-_{L_d}$. Since $\mathrm{d}^2 = 0$, it follows that $\mathrm{d}\Theta^+_{L_d} = \mathrm{d}\Theta^-_{L_d}$, which is defined to be the discrete Lagrangian symplectic form Ω_{L_d} on $G \times G$:

$$\Omega_{L_d} = \mathrm{d}\Theta^+_{L_d} = \mathrm{d}\Theta^-_{L_d}.$$

Proposition 3.1. *The discrete Lagrangian flow preserves the discrete Lagrangian two-form as follows*

$$\left(\mathcal{F}^{N-1}_{L_d}\right)^*\Omega_{L_d} = \Omega_{L_d}. \tag{17}$$

Proof. Define the solution space $\mathcal{C}_{L_d}$ to be the set of solutions $\{g_k \in G\}_{k=0}^N$ of the equations (16). Since an element of $\mathcal{C}_{L_d}$ is uniquely determined by the initial conditions (g_0, f_0), we can identify $\mathcal{C}_{L_d}$ with the manifold of initial conditions on $G \times G$. Define the restricted action map $\tilde{\mathcal{A}}_d : G \times G \to \mathbb{R}$ by

$$\tilde{\mathcal{A}}_d(g_0, f_0) = \tilde{\mathcal{A}}_d(\{g_k\prime\}_k^N),$$

where $\{g_k\prime\}_k^N \in \mathcal{C}_{L_d}$ is the solution of the discrete Euler-Lagrange equations with the initial conditions $(g_0\prime, g_1\prime) = (g_0, g_0 f_0)$. Since this satisfies equations (16), the variation of the action sum reduces to

$$\langle \mathrm{d}\tilde{\mathcal{A}}_d\, ,\, w\rangle = \langle\left(\mathcal{F}^{N-1}_{L_d}\right)^*\Theta^+_{L_d} - \Theta^-_{L_d}\, ,\, w\rangle,$$

for any $w = (\delta g_k, \delta f_k) \in TG \times TG$. Taking the exterior derivative of the previous expression and taking into account that pullbacks and exterior derivative commute, we obtain

$$\mathrm{d}^2\tilde{\mathcal{A}}_d = \left(\left(\mathcal{F}^{N-1}_{L_d}\right)^*\mathrm{d}\Theta^+_{L_d} - \mathrm{d}\Theta^-_{L_d}\right).$$

Since $\mathrm{d}^2\tilde{\mathcal{A}}_d = 0$ we finally arrive to (17) q.e.d. □

ii) **Discrete Noether's Theorem**: Consider the action of a Lie group H on G, $\Phi : H \times G \to G$ and consider the infinitesimal generator $\zeta_G : G \to G \times \mathfrak{g}$ for $\zeta \in H$ defined by $\zeta_G(g) =$

$\frac{d}{d\epsilon}\Big|_{\epsilon=0}\Phi\exp_{H(\epsilon\zeta)}(g)$. Here we define the infinitesimal generator $\zeta_{G\times G} : G\times G \to TG\times TG$ as

$$\zeta_{G\times G}(g_k, f_k) = \Big(T_e l_{g_k}\zeta_G(g_k)\,,\, T_e l_{f_k}(-\mathrm{Ad}_{f_k^{-1}}\zeta_G(g_k) + \zeta_G(g_k f_k))\Big)\,.$$

We define two discrete Lagrangian momentum maps $J^+_{L_d}, J^-_{L_d} : G\times G \to \mathfrak{h}^*$:

$$\begin{aligned} \langle J^+_{L_d}(g_k, f_k), \zeta\rangle &= \langle \Theta^+_{L_d}\,,\, \zeta_{G\times G}(g_k, f_k)\rangle, \\ \langle J^-_{L_d}(g_k, f_k), \zeta\rangle &= \langle \Theta^-_{L_d}\,,\, \zeta_{G\times G}(g_k, f_k)\rangle. \end{aligned}$$

Proposition 3.2. *Suppose that the discrete Lagrangian is invariant under the lifted action over the group, i.e., $\langle dL_d\,,\, \zeta_{G\times G}\rangle = 0$ for any $\zeta\in\mathfrak{h}$. Then, the two Lagrangian momentum maps are the same, $J^+_{L_d} = J^-_{L_d}$ (which is denoted by $J_{L_d} : G\times G \to \mathfrak{h}^*$), and the discrete Lagrangian flow preserves the discrete Lagrangian momentum map:*

$$J_{L_d}\left(\mathcal{F}^{N-1}_{L_d}(g_0, f_0)\right) = J_{L_d}(g_0, f_0). \tag{18}$$

This is called the discrete Noether's theorem.

Proof. Since $\mathrm{d}L_d = \Theta^+_{L_d} - \Theta^-_{L_d}$, we have

$$\langle \mathrm{d}L_d\,,\, \zeta_{G\times G}\rangle = \langle \Theta^+_{L_d} - \Theta^-_{L_d}, \zeta_{G\times G}\rangle = \langle J^+_{L_d} - J^+_{L_d}\,,\, \zeta\rangle,$$

which is equal to 0 for any $\zeta\in\mathfrak{h}$ since the discrete Lagrangian is invariant under the lifted action over the group. Consequently $J^+_{L_d} = J^+_{L_d}$.

Since the action is the summation of the discrete Lagrangian, $\langle \mathrm{d}L_d\,,\, \zeta_{G\times G}\rangle$ implies that $\langle \mathcal{A}_d\,,\, \zeta_{G\times G}\rangle = 0$. We can restrict it to the solution space to obtain

$$\langle \mathrm{d}\tilde{\mathcal{A}}_d\,,\, \zeta_{G\times G}\rangle = 0.$$

Thus

$$\begin{aligned} \langle \mathrm{d}\tilde{\mathcal{A}}_d\,,\, \zeta_{G\times G}\rangle &= \langle (\mathcal{F}^{N-1}_{L_d})^*\Theta^+_{L_d} - \Theta^-_{L_d}\,,\, \zeta_{G\times G}\rangle \\ &= \langle J^+_{L_d}(\mathcal{F}^{N-1}_{L_d}(g_k, f_k)) - J^-_{L_d}(g_k, f_k)\,,\, \zeta\rangle \end{aligned}$$

for any $\zeta\in\mathfrak{h}$, which yields (18) q.e.d. □

4. Continuous Euler-Lagrange equations on $SO(3)$

In this section we develop the continuous Euler-Lagrange equations for the attitude dynamics of the rigid body on the special orthogonal group $SO(3)$ according the Hamilton's variational principle.

4.1. **Configuration manifold:** Consider a rigid body that can freely rotate a pivot point fixed in an element frame. The pivot point may not be located at the mass center of the rigid body, and it is assumed that there exists a potential field that depends on the attitude. We consider a body fixed frame whose origin is located at the pivot point.

The configuration manifold for the attitude dynamics of a rigid body is the special orthogonal group $SO(3)$, defined as

$$SO(3) = \{R \in \mathbb{R}^{3\times 3} \mid R^T R = I,\ \det R = 1\}.$$

A rotation matrix $R \in SO(3)$ is a linear transformation from a representation of a vector in the body fixed frame into a representation of the vector in the inertial frame.

The attitude kinematics equations are given by

$$\dot{R} = R\hat{\Omega}, \tag{19}$$

where the angular velocity represented in the body fixed frame is denoted by $\Omega \in \mathbb{R}^3$, and the hat map $\hat{\cdot} : \mathbb{R}^3 \to \mathfrak{so}(3)$ is an isomorphism between $\mathbb{R}^3$ and the set of skew-symmetric matrices, the Lie algebra $\mathfrak{so}(3)$, defined by

$$\hat{\Omega} = \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix}$$

for $\Omega = [\Omega_1, \Omega_2, \Omega_3] \in \mathbb{R}^3$. The Lie bracket on $\mathfrak{so}(3)$ corresponds to cross product on $\mathbb{R}^3$, that is, $[\hat{\Omega}, \hat{\Omega}'] = \Omega \times \Omega'$ for $\Omega, \Omega' \in \mathbb{R}^3$.

Using these kinematics equations, the tangent bundle $TSO(3)$ can be identified with $SO(3) \times \mathfrak{so}(3)$ after a left trivialization.

In the following, we give some properties of the hat map.

Proposition 4.1. *The hat map* $\hat{\cdot} : \mathbb{R}^3 \to \mathfrak{so}(3)$ *satisfies the following properties,*

- $\hat{x}y = x \times y = -y \times x = -\hat{y}x,$
- $\hat{x}^T\hat{x} = (x^T x)I - xx^T,$
- $\hat{x}\hat{y}\hat{x} = -(y^T x)\hat{x},$
- $-\frac{1}{2}tr(\hat{x}\hat{y}) = x^T y,$
- $\widehat{x \times y} = \hat{x}\hat{y} - \hat{y}\hat{x} = yx^T - xy^T,$
- $tr(\hat{x}A) = \frac{1}{2}tr\left(\hat{x}(A - A^T)\right),$
- $\widehat{Ax} = \hat{x}\left(\frac{1}{2}tr(A)I - A\right) + \left(\frac{1}{2}tr(A)I - A\right)^T \hat{x}.$
- $\hat{x}A + A^T\hat{x} = ((tr(A)I_{3\times 3} - A)\hat{x})$

for any $x, y \in \mathbb{R}^3$ *and* $A \in \mathbb{R}^{3\times 3}$.

The proof of this proposition is straightforward by using the definition of the hat isomorphism and some matrix properties.

4.2. **The Lagrangian function:** As we say before, the tangent bundle of the Lie group $SO(3)$ can be left-trivialized as $SO(3) \times \mathfrak{so}(3)$. Then we can define the Lagrangian function L over $SO(3) \times \mathfrak{so}(3)$.

The Lagrangian $L : SO(3) \times \mathfrak{so}(3) \to \mathbb{R}$ is the difference between the kinetic energy $T : SO(3) \times \mathfrak{so}(3) \to \mathbb{R}$ and the attitude dependent potential $U : SO(3) \to \mathbb{R}$.

$$L(R, \Omega) = T(R, \Omega) - U(R).$$

Let $\rho \in \mathbb{R}^3$ be the vector form the pivot to a mass element represented in the body fixed frame. The mass element has a velocity $\Omega \times \rho$. Thus, the kinematic energy is given by

$$\frac{1}{2}\int_{\mathcal{B}} \|\hat{\Omega}\rho\|^2 dm(\rho), \tag{20}$$

where the region of the body is denoted by $\mathcal{B}$. Since $\hat{\Omega}\rho = -\hat{\rho}\Omega$, the equation (20) can be written as

$$\begin{aligned} T(\Omega) &= \frac{1}{2}\int_{\mathcal{B}} \|\hat{\rho}\Omega\|^2 dm(\rho) = \int_{\mathcal{B}} (\hat{\rho}\Omega)^T(\hat{\rho}\Omega) dm(\rho) \\ &= \frac{1}{2}\int_{\mathcal{B}} \Omega^T\hat{\rho}^T\hat{\rho}\Omega dm(\rho) = \frac{1}{2}\Omega^T J\Omega, \end{aligned}$$

where the moment of inertia matrix $J \in \mathbb{R}^3$ is defined as $J = \int_{\mathcal{B}} \hat{\rho}^T\hat{\rho} dm$.

Alternatively, using the property $\|x\|^2 = x^T x = tr(xx^T)$ for any $x \in \mathbb{R}^3$, equation (20) can be written as

$$T(\Omega) = \frac{1}{2}\int_{\mathcal{B}} tr\left(\hat{\Omega}\rho\rho^T\hat{\Omega}^T\right) dm(\rho), \tag{21}$$

$$= \frac{1}{2} tr\left(\hat{\Omega}J_d\hat{\Omega}^T\right), \tag{22}$$

where a nonstandard moment of inertia matrix is defined as $J_d = \int_{\mathcal{B}} \rho\rho^T dm$.

In summary, the kinetic energy can be written in the standard form (21) or in a non-standard form (22). In (21), the kinetic energy is expressed as a function of the angular moment of inertia matrix, and in (22), it is expressed as a function of the Lie algebra with the non-standard momenta of inertia matrix. In this review we use the non-standard form. The Lagrangian function of the attitude dynamics of the rigid body is given by

$$L(R, \Omega) = \frac{1}{2} tr\left(\hat{\Omega}J_d\hat{\Omega}^T\right) - U(R). \tag{23}$$

Before proceeding to the next step, we are going to study the relationship between the moment of inertia matrix J and the non-standard moment of inertial matrix J_d. If we express ρ in coordinates as $\rho = [x, y, z]$, the inertia momenta are given by

$$J = \int_{\mathcal{B}} \begin{pmatrix} y^2 + z^2 & -xy & -zx \\ -xy & z^2 + x^2 & -yz \\ -zxx & -yz & x^2 + y^2 \end{pmatrix} dm,$$

$$J_d = \int_{\mathcal{B}} \begin{pmatrix} x^2 & xy & zx \\ xy & y^2 & yz \\ zx & yz & z^2 \end{pmatrix} dm.$$

Using the property $\hat{\rho}^T \hat{\rho} = (\rho^T \rho) I_{3\times 3} - \rho\rho^T$, it can be shown that

$$J_d = \frac{1}{2} tr(J) I_{3\times 3} - J. \tag{24}$$

Furthermore, the following equation is satisfied for any $\Omega \in \mathbb{R}^3$.

$$\widehat{J\Omega} = \hat{\Omega} J_d + J_d \hat{\Omega}. \tag{25}$$

4.3. **The Action integral:** Using the expression of the Lagrangian function, the action integral is defined as,

$$\begin{aligned} \mathcal{A} &= \int_{t_0}^{t_f} L(R, \Omega) dt \\ &= \int_{t_0}^{t_f} \left(\frac{1}{2} tr\left(\hat{\Omega} J_d \hat{\Omega}^T \right) - U(R) \right) dt. \end{aligned}$$

Hamilton's principle states that this action integral does not vary to the first order for all possible variations of a curve in $SO(3)$.

$$\delta \mathcal{A} = \delta \int_{t_0}^{t_f} \left(\frac{1}{2} tr\left(\hat{\Omega} J_d \hat{\Omega}^T \right) - U(R) \right) dt = 0. \tag{26}$$

4.4. **Variations.** Let $R(t)$ be a differentiable curve in $SO(3)$ defined for $t \in [t_0, t_f]$. The variation is a differentiable mapping $R_\epsilon(t) : (-c, c) \times [t_0, t_f] \to SO(3)$ for $c > 0$ such that $R_0(t) = R(t), R_\epsilon(t_0) = R(t_0), R_\epsilon(t_f) = R(t_f)$ for any $\epsilon \in (-c, c)$. The infinitesimal variation is given by

$$\delta R(t) = \left.\frac{d}{dt}\right|_{\epsilon=0} R_\epsilon(t) \in T_{R(t)} SO(3).$$

The variation determines a family of neighboring curves for $R(t)$ that have the same end points parameterized by a single variable ϵ. The infinitesimal variation of the rotation matrix using the exponential map as

$$R_\epsilon(t) = R(t)\exp\epsilon\hat{\eta}(t),$$

where $\eta(t)$ is defined as a differentiable curve in $\mathbb{R}^3$ so that $\hat{\eta}$ is a differentiable curve in $\mathfrak{so}(3)$. This is well defined since the exponential map ia a local diffeomorphism between $\mathfrak{so}(3)$ and $SO(3)$. Thus for any $\eta(t)$, there exists a constant $c > 0$ such that this variation is defined for any $\epsilon \in (-c, c)$. The corresponding infinitesimal variation is given by

$$\begin{aligned}\delta R(t) &= \frac{d}{dt}\Big|_{\epsilon=0} R_\epsilon(t) = R(t)\sum_{i=0}^{\infty}\frac{d}{dt}\frac{1}{i!}\epsilon^i\hat{\eta}^i\Big|_{\epsilon=0}\\ &= R(t)\hat{\eta}(t) \in T_{R(t)}SO(3).\end{aligned} \tag{27}$$

Since differentiation and the variation commute, we obtain

$$\delta\dot{R}(t) = \frac{d}{dt}(\delta R(t)) = \dot{R}(t)\hat{\eta}(t) + R(t)\dot{\hat{\eta}}(t).$$

The infinitesimal variation of the angular velocity can be obtained from the kinematic equation (19) as

$$\begin{aligned}\delta\hat{\Omega}(t) &= \delta(R^T(t)\dot{R}(t)) = \delta R^T(t)\dot{R}(t) + R^T(t)\delta\dot{R}(t)\\ &= -\hat{\eta}(t)\hat{\Omega}(t) + \hat{\Omega}(t)\hat{\eta}(t) + \dot{\hat{\eta}}(t).\end{aligned} \tag{28}$$

Since $\hat{x}\hat{y} - \hat{y}\hat{x} = \widehat{x \times y}$ for any $x, y \in \mathbb{R}^3$, this can be written as

$$\delta\Omega(t) = \dot{\eta}(t) + \Omega(t) \times \eta(t). \tag{29}$$

4.5. **Euler-Lagrange equation.** Now, we find the infinitesimal variation of the action integral using (27) and (28) as follows,

$$\begin{aligned}\delta\mathcal{A} &= \int_{t_0}^{t_f}\frac{1}{2}tr(\delta\hat{\Omega}J_d\hat{\Omega}^T) + \frac{1}{2}tr(\hat{\Omega}J_d\delta\hat{\Omega}^T) - \delta U(R)dt\\ &= \int_{t_0}^{t_f}\left(-\frac{1}{2}tr\left((\dot{\hat{\eta}} + \hat{\Omega}\hat{\eta} - \hat{\eta}\hat{\Omega})J_d\hat{\Omega}\right) + \frac{1}{2}tr\left(\hat{\Omega}J_d(-\dot{\hat{\eta}} + \hat{\eta}\hat{\Omega} - \hat{\Omega}\hat{\eta})\right) - \delta U(R)\right)dt\\ &= \int_{t_0}^{t_f}\left(-\frac{1}{2}tr\left(\dot{\hat{\eta}}(J_d\hat{\Omega} + \Omega\hat{J}_d)\right) + \frac{1}{2}tr\left(\hat{\eta}\hat{\Omega}(J_d\hat{\Omega} + \hat{\Omega}J_d) - \hat{\eta}(J_d\hat{\Omega} + \hat{\Omega}J_d)\hat{\Omega}\right)\right)dt\\ &- \int_{t_0}^{t_f}\delta U(R)\,dt,\end{aligned}$$

where we use the property $tr(AB) = tr(BA)$ for any matrices $A, B \in \mathbb{R}^{n\times n}$ repeatedly. Substituting (25), we obtain

$$\begin{aligned}\delta\mathcal{A} &= \int_{t_0}^{t_f}\left(-\frac{1}{2}tr\left(\dot{\hat{\eta}}\widehat{J\Omega}\right) + \frac{1}{2}tr\left(\hat{\eta}(\hat{\Omega}\widehat{J\Omega} - \widehat{J\Omega}\hat{\Omega})\right) - \delta U(R)\right)dt\\ &= \int_{t_0}^{t_f}\left(-\frac{1}{2}tr\left(\dot{\hat{\eta}}\widehat{J\Omega}\right) + \frac{1}{2}tr\left(\hat{\eta}\widehat{(\Omega \times J\Omega)}\right) - \delta U(R)\right)dt.\end{aligned} \tag{30}$$

The infinitesimal variation of the potential energy is given by

$$\delta U(R) = \frac{d}{d\epsilon}U(R_\epsilon)\Big|_{\epsilon=0} = \sum_{i,j=1}^{3} \frac{\partial U}{\partial [R]_{ij}} \frac{\partial [R \exp \epsilon\hat{\eta}]_{ij}}{\partial \epsilon}\Big|_{\epsilon=0}$$
$$(31) \qquad = \sum_{i,j=1}^{3} \frac{\partial U}{\partial [R]_{ij}} [R\hat{\eta}]_{ij} = -tr\left(\hat{\eta} R^T \frac{\partial U}{\partial R}\right)$$

where $[A]_{ij}$ denotes the (i,j)-th element of a matrix A, and $\frac{\partial U}{\partial R} \in \mathbb{R}^{3\times 3}$ is defined such that $\left(\frac{\partial U}{\partial R}\right)_{ij} = \frac{\partial U(R)}{\partial [R]_{ij}}$. Substituting (30) and (31) and using integration by parts, we obtain

$$(32) \qquad \delta\mathcal{A} = \int_{t_0}^{t_f} \frac{1}{2} tr\left[\hat{\eta}\left((J\dot{\Omega} + \Omega \times J\Omega)\hat{} + 2R^T\frac{\partial U}{\partial R}\right)\right] dt.$$

From Hamilton's principle, the above equation should be zero for all variations $\hat{\eta} \in \mathfrak{so}(3)$. Given that $\hat{\eta}$ is skew-symmetric, the expression in the braces should be symmetric. Thus, we obtain the Euler-Lagrange equation

$$(J\dot{\Omega} + \Omega \times J\Omega)\hat{} = \frac{\partial U^T}{\partial R} R - R^T \frac{\partial U}{\partial R},$$

or equivalently,

$$J\dot{\Omega} + \Omega \times J\Omega = M$$

where $M \in \mathbb{R}^3$ is determined by $S(M) = \frac{\partial U^T}{\partial R} R - R^T \frac{\partial U}{\partial R}$. More explicitly, it can be shown that the moment due to the attitude-dependent potential is given by

$$(33) \qquad M = r_1 \times u_1 + r_2 \times u_2 + r_3 \times u_3,$$

where $r_i, u_i \in \mathbb{R}^{1\times 3}$ are the i-th row vectors of R and $\frac{\partial U}{\partial R}$, respectively.

$$\begin{aligned}
\hat{M} &= \frac{\partial U^T}{\partial R} R - R^T \frac{\partial U}{\partial R} \\
&= \begin{pmatrix} u_1^T & u_2^T & u_3^T \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} - \begin{pmatrix} r_1^T & r_2^T & r_3^T \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \\
&= (u_1^T r_1 - r_1^T u_1) + (u_2^T r_2 - r_2^T u_2) + (u_3^T r_3 - r_3^T u_3).
\end{aligned}$$

Since $(u^T r - r^T u)\hat{} = \widehat{r \times u}$, we have

$$\hat{M} = (r_1 \times u_1 + r_2 \times u_2 + r_3 \times u_3)\hat{},$$

which is equivalent to (33).

The *Legendre transformation* $\mathbb{F}L : (SO(3) \times \mathfrak{so}(3)) \to (SO(3) \times \mathfrak{so}^*(3))$ is defined as

$$\begin{aligned}\mathbb{F}L(R, \hat{\Omega}) \cdot \hat{\eta} &= \frac{d}{dt}\Big|_{\epsilon=0} L(R, \hat{\Omega} + \epsilon\hat{\eta}) \\ &= \frac{d}{dt}\Big|_{\epsilon=0} \frac{1}{2}\text{tr}[(\hat{\Omega} + \epsilon\hat{\eta})^T J_d(\hat{\Omega} + \epsilon\hat{\eta})] \\ &= \frac{1}{2}\text{tr}[\widehat{J\Omega}^T \hat{\eta}] = \widehat{J\Omega} \cdot \hat{\eta}\end{aligned}$$

This gives the expression for the angular momenta expressed in the body fixed frame $\hat{\Pi} = \mathbb{F}L(R, \hat{\Omega}) = \widehat{J\Omega}$ and from $\Pi = \frac{\partial L}{\partial \Omega} = J\Omega \in \mathbb{R}^3$ we obtain the Hamilton's equations

$$\dot{\Pi} + J^{-1}\Pi \times \Pi = M.$$

Example 4.1. *A 3D pendulum is a rigid body supported by a frictionless pivot acting under uniform gravitational potential. Let $\rho_c \in \mathbb{R}^3$ be the vector from the pivot to the mass center represented in the body fixed frame, and let $e_3 = [0, 0, 1] \in \mathbb{R}^3$ be the gravity direction in the inertial frame. The gravitational potential energy is given by*

$$U(R) = -mge_3^T R\rho_c.$$

The derivative of the potential is

$$\frac{\partial U}{\partial R} = -mge_3\rho_c^T,$$

therefore the potential is $M = mg\rho_c \times R^T e_3$ becouse $u_1 = u_2 = 0, u_3 = -mge_3\rho_c^T$.

5. Discrete Euler-Lagrange equation on $SO(3)$

In this section, we are going to develop the discrete Euler-Lagrange equations of a rigid body on the special orthogonal group $SO(3)$. This is also referred to a Variational Integrator as it is obtained by the discrete Hamilton's principle applied to Lie groups. The procedure to derive the discrete Euler-Lagrange equations are presented in (3.1). However, we summarize them here. We will finish the section, as a kind of summary, with a comparative table between a $SO(3)$ and a **general** Variational Integrators.

In the continuous setting, a mechanical problem is usually defined in the tangent bundle of a configuration manifold Q which we consider n-dimensional (with n finite). Let $L : TQ \to \mathbb{R}$ be the Lagrangian function. Furthermore, we select $(q, \dot{q})$ as local coordinates for TQ. The continuous action integral is defined as the integral of the Lagrangian function between the initial and final time points $\mathcal{A} = \int_0^T L(q, \dot{q})\, dt$.

Applying the Hamilton's principle to the action integral we obtain the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0.$$

Again, if the discrete Lagrangian is regular, that is $D_{1,2}L_d$ is invertible, these equations define a discrete flow on Q, that is $\Upsilon_{L_d} : Q \times Q \to Q \times Q$ such that $\Upsilon_{L_d}(q_{k-1}, q_k) = (q_k, q_{k+1})$. Defining the Legendre transformation $\mathbb{F}L : TQ \to T^*Q$, where T^*Q is the cotangent bundle of the configuration space (phase space), $p = \mathbb{F}L(q, \dot{q}) \in T^*Q$ (which is a local diffeomorphism if L is regular, that is, the matrix $\frac{\partial^2 L}{\partial q \partial \dot{q}}$ is invertible), we arrive to the Hamilton's equations:

$$\dot{q} = \frac{\partial H(q,p)}{\partial p}, \quad \dot{p} = -\frac{\partial H(q,p)}{\partial q},$$

where $H : T^*Q \to \mathbb{R}$ is the Hamiltonian function.

In the discrete setting, we substitute the tangent bundle of the configuration manifold for two copies $Q \times Q$ with no loss of information. Now, we define the discrete Lagrangian $L_d : Q \times Q \to \mathbb{R}$ as an approximation of the action integral in a single time step, that is $L_d(q_k, q_{k+1}) \simeq \int_0^h L(q, \dot{q})\, dt$. Defining the discrete action sum as $\mathcal{A}_d = \sum_{k=0}^{N-1} L_d(q_k, q_{k+1})$ we can apply the discrete Hamilton's principle in order to obtain the discrete Euler-Lagrange equations:

$$D_1 L_d(q_k, q_{k+1}) + D_2 L_d(q_{k-1}, q_{k-1}) = 0.$$

By means of the discrete Legendre transforms $\mathbb{F}^{\pm} L_d : Q \times Q \to T^*Q$, we finally arrive to the discrete Hamilton's equations

$$\begin{aligned} p_k &= \mathbb{F}^- L_d(q_k, q_{k+1}) = -D_1 L_d(q_k, q_{k+1}), \\ p_{k+1} &= F^+ L_d(q_k, q_{k+1}) = D_2 L_d(q_k, q_{k+1}). \end{aligned}$$

We realize that both procedures are quite parallel.

We follow the *discrete program* to obtain the discrete Euler-Lagrange equation for the attitude dynamics of a rigid body on $SO(3)$. Let $h > 0$ be a fixed time stepsize. The label k denotes the value of a variable at $t_k = hk$. The integer N is defined such that $t_f = Nh$.

5.1. **Configuration Manifold.** The continuous-time attitude kinematics equation

$$\dot{R} = R\hat{\Omega},$$

ensures that its solution evolves on $SO(3)$, which is our Configuration Manifold, and given that the attitude matrix belongs to $SO(3)$, the equation just above can be expressed by the relation $\frac{d}{dt}\left(R^T R\right) = 0$ (we recall that $R^T R = I$, where I is the identity). A numerical integrator is a discrete approximation of the exact solution. In the following we define R_k as an approximation of the exact solution $R(t)$ at time hk, where h is the time step and k an integer, i.e. $R_k \simeq R(hk)$. The natural way to define an integrator would be

$$R_{k+1} = R_k + \Delta\, R_k,$$

but recalling that we deal with elements of $SO(3)$ the summation might not be well defined. Hence, we choose the discrete update map in such a way the approximation does not cause any deviation from the configuration manifold. More explicitly, define $F_k \in SO(3)$ as $F_k = R_k^T\, R_{k+1}$. Thus, we have

$$R_{k+1} = R_k\, F_k. \tag{34}$$

The rotation matrix F_k represents the relative attitude update between two integration steps, and by requiring that $F_k \in SO(3)$, we guarantee that the discrete flow R_k for $k \in \{0, ..., N\}$ evolves on $SO(3)$ automatically.

5.2. **Discrete Lagrangian.** The Lagrangian $L : SO(3) \times \mathfrak{so}(3) \to \mathbb{R}$ of the attitude dynamics of the rigid body is given by

$$L(R, \hat{\Omega}) = \frac{1}{2}\mathrm{tr}[\hat{\Omega}\, J_d\, \hat{\Omega}^T] - U(R),$$

for an attitude dependent potential $U : SO(3) \to \mathbb{R}$. The matrix $J_d \in \mathbb{R}^{3\times 3}$ is the nonstandard moment of inertia matrix defined as $J_d = \frac{1}{2}\mathrm{tr}[J]I - J$, for the standard moment of inertia matrix $J \in \mathbb{R}^{3\times 3}$.

Using the kinematics equation $\dot{R} = R\,\hat{\Omega}$ and (34), $\tilde{\Omega}_k$ can be approximated as

$$\hat{\Omega}_k = R_k^T \dot{R}_k \simeq \frac{1}{h} R_k^T \left(R_{k+1} - R_k\right) = \frac{1}{h}\left(F_k - I\right), \tag{35}$$

where we have chosen the easiest approximation for the first derivative of R_k, namely, $\dot{R}_k \simeq \frac{1}{h}\left(R_{k+1} - R_k\right)$ and the fact that $R_k \in SO(3)$.

As mentioned in the previous chapter, the continuous action integral is defined as de time integral of the Lagrangian function, i.e:

$$\mathcal{A} = \int_0^t L(R, \Omega)\, dt.$$

The discrete Lagrangian is just an arbitrary approximation of this action integral. Despite it is arbitrary, we can choose it in a suitable way, that is, the more accurate the discrete Lagrangian is as approximation of the action integral, the more accurate will be the numerical integrator at the end of the day.

From the trapezoidal rule (which is a second order approximation of the action integral), we choose the following form of the discrete Lagrangian $L_d : SO(3) \times SO(3) \to \mathbb{R}$:

$$L_d(R_k, \hat{\Omega}_k) = \frac{h}{2} L(R_k, \hat{\Omega}_k) + \frac{h}{2} L(R_{k+1}, \hat{\Omega}_k),$$

which in terms of F_k can be rewritten as:

$$\begin{aligned} L_d(R_k, F_k) &= \frac{h}{2}L\left(R_k, \frac{1}{h}(F_k - I)\right) + \frac{h}{2}L\left(R_{k+1}, \frac{1}{h}(F_k - I)\right) \\ &= \frac{1}{2h}\mathrm{tr}[F_k J_d F_k^T - F_k J_d - J_d F_k^T + J_d] - \frac{h}{2}U(R_k) - \frac{h}{2}U(R_{k+1}). \end{aligned}$$

Since the trace operation obeys the properties

$$\begin{aligned} \mathrm{tr}[ABC] &= \mathrm{tr}[CAB] = \mathrm{tr}[BCA], \\ \mathrm{tr}[A^T] &= \mathrm{tr}[A], \end{aligned}$$

and J_d is a symmetric matrix, we arrive to our final discrete Lagrangian:

$$L_d(R_k, F_k) = \frac{1}{h}\mathrm{tr}[(I - F_k)J_d] - \frac{h}{2}U(R_k) - \frac{h}{2}U(R_{k+1}). \tag{36}$$

5.3. **Action Sum.** Using the expression for the discrete Lagrangian (36), the action sum (that is, the discrete counter part of the action integral) is defined as:

$$\begin{aligned} \mathcal{A}_d &= \sum_{k=0}^{N-1} L_d(R_k, F_k) \\ &= \sum_{k=0}^{N-1} \frac{1}{h}\mathrm{tr}[(I - F_k)J_d] - \frac{h}{2}U(R_k) - \frac{h}{2}U(R_{k+1}). \end{aligned} \tag{37}$$

The discrete Hamilton's principle states that this action sum does not vary to the first order for all possible curves in $SO(3)$, that is:

$$\delta\mathcal{A}_d = \delta\sum_{k=0}^{N-1} L_d(R_k, F_k) = \sum_{k=0}^{N-1} \delta L_d(R_k, F_k) = 0.$$

The variation of the discrete Lagrangian will be calculated afterwards.

5.4. **Variation.** Similar to the continuous time case, a variation of a discrete curve $\{R_k\}_{k=0}^N$ is expressed as

$$R_k^\epsilon = R_k \exp(\epsilon\tilde{\eta}_k),$$

for $k \in \{0, ..., N\}$, where $\{\eta_k\}_{k=0}^N$ is a discrete curve on $\mathbb{R}^3$ satisfying $\eta_0 = \eta_N = 0$. As was mentioned before, $\hat{\cdot} : \mathbb{R}^3 \to \mathfrak{so}(3)$ is an isomorphism, where $\mathfrak{so}(3)$ is the Lie algebra corresponding to $SO(3)$, namely, if $(\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3$ then

$$\hat{\omega} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \in \mathfrak{so}(3).$$

The elements $\hat{\eta}_k$ of $\mathfrak{so}(3)$ are skew-symmetric 3×3 matrices, and, consequently, $\exp:\mathfrak{so}(3)\to SO(3)$. The corresponding infinitesimal variation is given by:

$$\delta R_k = \frac{d}{d\epsilon}\Big|_{\epsilon=0} R_k \exp(\epsilon\tilde{\eta}_k) = R_k\hat{\eta}_k \in T_{R_k}SO(3).$$

The infinitesimal variation of F_k is obtained using the chain rule and the above relation:

$$\delta F_k = \delta\left(R_k^T R_{k+1}\right) = -\hat{\eta}_k F_k + F_k\hat{\eta}_{k+1}. \tag{38}$$

5.5. **Discrete Euler-Lagrange equation.** The variation of the discrete Lagrangian is given by

$$\delta L_d(R_k, F_k) = -\frac{1}{h}\mathrm{tr}[\delta F_k\, J_d] - \frac{h}{2}\delta U(R_k) - \frac{h}{2}\delta U(R_{k+1}). \tag{39}$$

Recall that the variation of the potential can be written as

$$\delta U(R_k) = -\mathrm{tr}[\hat{\eta}_k R_k^T \frac{\partial U_k}{\partial R_k}], \tag{40}$$

where U_k is a shorthand notation for $U(R_k)$. Now, substituting (38) and (40) into (39), we obtain:

$$\begin{aligned}\delta L_d(R_k, F_k) = \mathrm{tr}[\hat{\eta}_k &\left\{\frac{1}{h}F_k J_d + \frac{h}{2}R_k^T\frac{\partial U_k}{\partial R_k}\right\}] + \\ &+ \mathrm{tr}[\hat{\eta}_{k+1}\left\{-\frac{1}{h}J_d F_k + \frac{h}{2}R_{k+1}^T\frac{\partial U_{k+1}}{\partial R_{k+1}}.\right\}]\end{aligned} \tag{41}$$

Therefore, the variation of the action sum is given by

$$\begin{aligned}\delta\mathcal{A}_d &= \sum_{k=0}^{N-1}\delta L_d(R_k, F_k) \\ &= \mathrm{tr}[\hat{\eta}_0\left\{\frac{1}{h}F_0 J_d + \frac{h}{2}R_0^T\frac{\partial U_0}{\partial R_0}\right\}] + \mathrm{tr}[\hat{\eta}_N\left\{-\frac{1}{h}J_d F_{N-1} + \frac{h}{2}R_N^T\frac{\partial U_N}{\partial R_N}\right\}] \\ &\quad + \sum_{k=1}^{N-1}\mathrm{tr}[\hat{\eta}_k\left\{\frac{1}{h}(F_k J_d - J_d F_{k-1}) + hR_k^T\frac{\partial U_k}{\partial R_k}\right\}].\end{aligned}$$

Since $\hat{\eta}_0 = \hat{\eta}_N = 0$, we arrive to

$$\delta\mathcal{A}_d = \sum_{k=1}^{N-1}\mathrm{tr}[\hat{\eta}_k\left\{\frac{1}{h}(F_k J_d - J_d F_{k-1}) + hR_k^T\frac{\partial U_k}{\partial R_k}\right\}]. \tag{42}$$

Since $\hat{\eta}_k$ belongs to $\mathfrak{so}(3)$, it is skew-symmetric $\delta\mathcal{A}_d$ vanishes if and only if the expression into braces is symmetric. Thus we obtain the discrete

Euler-Lagrange equation as

$$\begin{aligned}&\frac{1}{h}\left(F_{k+1}J_d + J_dF_{k+1}^T - J_dF_k - F_k^TJ_d\right)\\ &\quad = h\left(\frac{\partial U_{k+1}^T}{\partial R_{k+1}}R_{k+1} + R_{k+1}^T\frac{\partial U_{k+1}}{\partial R_{k+1}}\right).\end{aligned} \tag{43}$$

The expression in the parentheses in the right hand side is equal to the definition of the moment due to a potential $\hat{M}$ evaluated at $t = hk$. In summary, the discrete Euler-Lagrange equations on $SO(3)$ are given by:

$$\begin{aligned}\frac{1}{h}\left(F_{k+1}J_d + J_dF_{k+1}^T - J_dF_k - F_k^TJ_d\right) &= h\hat{M}_{k+1},\\ R_{k+1} &= R_k\,F_k.\end{aligned}$$

If the potential vanishes, i.e., $U = 0$ the equations simplify

$$\frac{1}{h}\left(F_{k+1}J_d + J_dF_{k+1}^T - J_dF_k - F_k^TJ_d\right) = 0, \tag{44}$$

$$R_{k+1} = R_k\,F_k. \tag{45}$$

For given (R_0, R_1) and $F_0 = R_0^TR_1$, we solve the implicit equation (44) to find F_1. Then R_2 is straightforward obtained by means of (45). This yields the discrete map $(R_0, R_1) \to (R_1, R_2)$. If our initial data is (R_k, F_k), we can obtain R_{k+1} from equation (45) and F_{k+1} implicitly from equation (44), which yields the discrete Lagrangian flow map $F_{DEL} : (R_k, F_k) \to (R_{k+1}, F_{k+1})$.

5.6. **Discrete Hamilton's equations.** The discrete Legrendre transformation yields a discrete Hamilton's equation. In that sense, we need to introduce the cotangent space. Let Q be a smooth manifold and let $q \in Q$. The cotangent space of Q at q, namely, T_q^*Q, is the dual space of the tangent space T_qQ. More explicitly, an element of the cotangent space p is a linear function on T_qQ, i.e. $p : T_qQ \to \mathbb{R}$. Similar to the tangent bundle, the cotangent bundle is the disjoint union of the cotangent space, $T^*Q = \bigcup_{q\in Q} T^*Q$.

Using the left-trivialization we identify $TSO(3)$ with $SO(3) \times \mathfrak{so}(3)$ (in general, we can identify by this trivization TG, where G is a Lie group, with the cartesian product $G \times \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra corresponding to G). Furthermore, we can make the identification $SO(3)\times\mathfrak{so}(3)$ with $SO(3)\times\mathbb{R}^3$ by means of the isomorphism $\hat{\cdot} : \mathbb{R}^3 \to \mathfrak{so}(3)$ described above. Therefore: $T^*SO(3) \simeq SO(3) \times \mathbb{R}^3$ since $(\mathbb{R}^3)^*$ is identified with $\mathbb{R}^3$ using the standar inner product on $(\mathbb{R})^3$, i.e. for $\Pi \in (\mathbb{R}^3)^*$ and $\Omega \in \mathbb{R}^3$, $\Pi\cdot\Omega = \Pi^T\Omega$. Taking this into account, the cotangent bundle $T^*SO(3)$ is further identified with $SO(3)\times\mathfrak{so}(3)$ using the identity $\Pi^T\Omega = -\frac{1}{2}\mathrm{tr}[\hat{\Pi}\hat{\Omega}]$. In summary, for given $(R, \hat{\Omega}) \in TSO(3) \simeq (SO(3) \times \mathfrak{so}(3))$ and $(R, \hat{\Pi}) \in T^*SO(3) \simeq (SO(3) \times \mathfrak{so}(3))$, the pairing between the tangent vector and cotangent vector is given by $\langle \hat{\Pi}\, ,\, \hat{\Omega}\rangle = -\frac{1}{2}\mathrm{tr}[\hat{\Pi}\hat{\Omega}]$.

Given the Lagrangian $L : (SO(3) \times \mathfrak{so}(3)) \to \mathbb{R}$, the Legendre transformation $\mathbb{F}L : (SO(3) \times \mathfrak{so}(3)) \to (SO(3) \times \mathfrak{so}(3))$ is defined as

$$\begin{aligned}\langle \mathbb{F}L(R, \hat{\Omega})\, ,\, \hat{\eta}\rangle &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} L(R, \hat{\Omega} + \epsilon\hat{\eta}) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \frac{1}{2}\mathrm{tr}[(\hat{\Omega} + \epsilon\hat{\eta})^T J_d(\hat{\Omega} + \epsilon\hat{\eta})] \\ &= \frac{1}{2}\mathrm{tr}[-(J_d\hat{\Omega} + \hat{\Omega}J_d)\hat{\eta}] = \frac{1}{2}\mathrm{tr}[\widehat{J\Omega}^T\hat{\eta}] = \langle \widehat{J\Omega}\, ,\, \hat{\eta}\rangle.\end{aligned}$$

This gives the expression for the momentum $\hat{\Pi} = \mathbb{F}L(R, \hat{\Omega}) = \widehat{J\Omega}$ in the continuous time as a cotangent vector, which, phisically, is a skew form of the angular momentum expressed in the body fixed frame.

The discrete Legendre transformation provides a relationship between the momentum evaluated at $t = hk$, namely, $\simeq\Pi_k\hat{\Pi}(hk)$, and the discrete variables (R_k, F_k), which is given by

$$\langle \hat{\Pi}_k\, ,\, \hat{\eta}_k\rangle = \frac{d}{d\epsilon}\Big|_{\epsilon=0} L_d(R_k^\epsilon, (R_k^\epsilon)^T R_{k+1}) = \mathrm{tr}\left[\hat{\eta}_k \left\{\frac{1}{h}F_k J_d + \frac{h}{2}R_k^T\frac{\partial U_k}{\partial R_k}\right\}\right].$$

Since $\mathrm{tr}[\hat{\eta}A] = \frac{1}{2}\mathrm{tr}[\hat{\eta}(A - A^T)]$ (due to trace properties and skew-symmetric matrices) this can be written as

$$\langle \hat{\Pi}_k, \hat{\eta}_k\rangle = -\mathrm{tr}\left[\hat{\eta}_k\left\{\frac{1}{h}(F_kJ_d - J_dF_k^T) - \frac{h}{2}M_k\right\}\right] = \langle \frac{1}{h}(F_kJ_d - J_dF_k^T), \hat{\eta}_k\rangle.$$

Thus, we obtain the first Hamilton's equation:

$$\hat{\Pi}_k = \frac{1}{h}(F_kJ_d - J_dF_k^T) - \frac{h}{2}\hat{M}_k. \tag{46}$$

Similarly, the second Hamilton's equation is given by

$$\begin{aligned}\langle \hat{\Pi}_{k+1}\, ,\, \hat{\eta}_{k+1}\rangle &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} L_d(R_k, R_k^T R_{k+1}^\epsilon) = \\ &= \frac{1}{2}\mathrm{tr}\left[\hat{\eta}_{k+1}\left\{-\frac{1}{h}(J_dF_k - F_k^TJ_d) - \frac{h}{2}\hat{M}_{k+1}\right\}\right] = \\ &= \langle \frac{1}{h}(J_dF_k - F_k^TJ_d) + \frac{h}{2}\hat{M}_{k+1}, \hat{\eta}_{k+1}\rangle.\end{aligned}$$

Therefore, we obtain

$$\hat{\Pi}_{k+1} = \frac{1}{h}(J_dF_k - F_k^TJ_d) + \frac{h}{2}\hat{M}_{k+1}. \tag{47}$$

Substituting (46) in (47) we have

$$\hat{\Pi}_{k+1} = F_k^T(\hat{\Pi}_k + \frac{h}{2}\hat{M}_k)F_k + \frac{h}{2}\hat{M}_{k+1}.$$

We can eliminate the hat map at every term to obtain a vector form of the equation, using the fact that $F^T\hat{\Pi}F = \widehat{F^T\Pi}$ for any $\Pi \in \mathbb{R}^3$

and $F \in SO(3)$. Assuming $U = 0$, consequently $\hat{M} = 0$, the discrete Hamilton's equations on $SO(3)$ are given by:

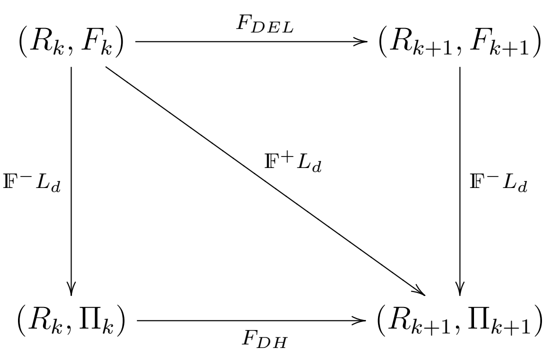
$$\Pi_k = \frac{1}{h}(F_k J_d - J_d F_k^T) \tag{48}$$

$$R_{k+1} = R_k F_k \tag{49}$$

$$\Pi_{k+1} = F_k^T \Pi_k \tag{50}$$

For given (R_0, Π_0), we solve the implicit equation (48) to find F_0. Then R_1 is obtained by (49) and finally Π_1 is straightforwardly obtained by (50). This procedure yields de discrete Hamiltonian flow $F_{DH} : (R_0, \Pi_0) \to (R_1, \Pi_1)$.

As was mentioned before, the continuous Legendre transformation relates the continuous velocity space $TSO(3) \simeq SO(3) \times \mathfrak{so}(3)$ with the phase space $T^*SO(3) \simeq SO(3) \times \mathfrak{so}(3)$ by $\hat{\Pi} = \mathbb{F}L(R, \hat{\Omega}) = \widehat{J\Omega}$. In the discrete case there exist two Legendre transformations which relate the discrete velocity and phase space in the following way: $\mathbb{F}^{\pm}L_d : SO(3) \times SO(3) \to SO(3) \times (\mathbb{R}^3)^* \simeq SO(3) \times (\mathbb{R}^3) \simeq SO(3) \times \mathfrak{so}(3)$. In addition, the Legendre transforms relate the discrete Lagrangian and Hamiltonian flows. Graphically it can be shown as:



5.7. **Computational approach.** The implicit equations given by (44) and (48) have the following structure: for given $g \in \mathbb{R}^3$ and $J_d \in \mathbb{R}^{3\times3}$ find $F \in SO(3)$ satisfying

$$\hat{g} = F\, J_d - J_d\, F^T. \tag{51}$$

5.7.1. *Cayley transformation.* : Among others, the Cayley transformation is a local diffeomorphism between the Lie group and the corresponding Lie algebra. In particular, it is defined for *quadratic Lie groups*, that is, groups defined by

$$G = \left\{ Y \in GL(n, \mathbb{R}) \;\middle|\; Y^T\, P\, Y = P \right\},$$

with $P \in GL(n,\mathbb{R})$ a given matrix (where $GL(n,\mathbb{R})$ is the group of $n \times n$ matrices with real entrances). The corresponding Lie algebra is

$$\mathfrak{g} = \left\{ \Omega \in \mathfrak{gl}(n,\mathbb{R}) \;\middle|\; P\Omega + \Omega P = 0 \right).$$

It is clear that the special ortogonal group $SO(n)$ is a particular case of quadratic Lie group with $P = Id$. Thus, the Cayley transformation is a local diffeomorphism between skew symmetric matrices (i.e. $\mathfrak{so}(3)$) and rotation matrices (i.e. $SO(3)$).

We can explicitly write down the equations for the Cayley transform in the case of $SO(3)$ matrices, namely, $\mathrm{cay} : \mathfrak{so}(3) \to SO(3)$:

$$\begin{aligned} F = \mathrm{cay}(f) &= (I + \hat{f})(I - \hat{f})^{-1} = (I - \hat{f})^{-1}(I + \hat{f}) = \\ &= \frac{1}{1 + f^T f}\left((1 - f^T f)I + 2\hat{f} + 2ff^T\right), \end{aligned}$$

where, as we have defined above, $f \in \mathbb{R}^3$, $\hat{f} \in \mathfrak{so}(3)$ and $F = \mathrm{cay}(f) \in SO(3)$. This represents a rotation of a rigid body along the direction $\frac{f}{\|f\|}$ with rotation angle θ determined by $\| f \| = \tan\left(\frac{\theta}{2}\right)$.

Another possibility to transform elements on the algebra to elements on the group is the exponential map. It is given by

$$F = \exp(v) = I + \frac{\sin \| v \|}{\| v \|}\hat{v} + \frac{1 - \cos \| v \|}{\| v \|^2}\hat{v}^2,$$

where again $v \in \mathbb{R}^2$, $\hat{v} \in \mathfrak{so}(3)$ and $F = \mathrm{cay}(v) \in SO(3)$. This represents a rotation along the direction $\frac{v}{\|v\|}$ with rotation angle θ determined by $\| v \|$. It can be shown that

$$\exp(v) = \mathrm{cay}\left(\tan\frac{\| v \|}{2}\,\frac{v}{\| v \|}\right).$$

Consequently, the Cayley transform can be considered as a different form of the exponential map, where the rotation angle is encoded in a different way. However, the Cayley transformation has a numerical advantage over the exponential map since it does not contain computationally-expensive *sin* and *cos* terms.

For further details regarding isomorphisms between the algebra and the group see [8]

5.7.2. *Implicit Equations:* Using the Cayley transformation, we transform the implicit equations (51) into an equivalent vector equation where the rotation angle is less than π. This is reasonable since the rotation matrix F represents a relative update betwen two consecutive integration steps. We expect the rotation angle of F to be small.

Since the two matrices $(I + \hat{f})$ and $(I - \hat{f})^{-1}$ commute, we can, for convenience and with some abusse of notation, in the following form

$$F = \mathrm{cay}(f) = \frac{I + \hat{f}}{I - \hat{f}}.$$

Substituting this into (51), we obtain

$$\hat{g} = \left(\frac{I+\hat{f}}{I-\hat{f}}\right) J_d - J_d \left(\frac{I-\hat{f}}{I+\hat{f}}\right),$$

where we recall here that $F \in SO(3)$ and consequently its inverse is equal to its transpose matrix. Multiplying the last expression by $(I-\hat{f})$ at the left side and by $(I+\hat{f})$ at the right side, we have

$$\begin{aligned} (I-\hat{f})\hat{g}(I+\hat{f}) &= (I+\hat{f})J_d(I+\hat{f}) - (I-\hat{f})J_d(I+\hat{f}), \\ \hat{g} + \widehat{g \times f} - \hat{f}\hat{g}\hat{f} &= 2\hat{f}J_d + 2J_d\hat{f}, \end{aligned}$$

where "$\times$" is the usual vectorial product between two elements belonging to R^3. Taking into account the following identities:

$$\widehat{f \times g} = \hat{f}\hat{g} - \hat{g}\hat{f}, \quad \widehat{Jf} = J_d\hat{f} + \hat{f}J_d, \quad \hat{f}\hat{g}\hat{f} = -(g^T f)\hat{f},$$

we arrive to the equation

$$(f + \widehat{g \times f + f}(g^T f)) = 2\widehat{Jf}.$$

Finally, for a given $g \in \mathbb{R}^3$, let define $G : \mathbb{R}^3 \to \mathbb{R}^3$ as

$$G(f) = f + g \times f + f(g^T f) - 2Jf. \tag{52}$$

We realize that the vector equation $G(f) = 0$ is equivalent to (51) near zero. The important point to remark is that we have transformed a matricial implicit equation (51) into a vector equation.

5.7.3. *Computatinal approach:* Despite we have arrived to a vector equation (less computationally expensive than a matricial one) we have to find out an efficient way to solve it. This is reached by using either a fixed point iteration. A fixed point iteration gives the following equation

$$f^{k+1} = \frac{1}{2}J^{-1}(f^k + g \times f^k + f^k(g^T f^k)),$$

where k is the iteration label. Similary, the Newton iteration gives

$$f^{k+1} = f^k - \nabla G(f^k)^{-1}G(f^k),$$

where the Jacobian is defined by

$$\nabla G(f^k) = \hat{g} + (g^T f^k)I + f^k g^T - 2J.$$

To run these numerical methods, an initial guess is iterated untill $\| f^{k+1} - f^k \| < \epsilon$ for a given tolerance ϵ.

5.8. **General Setting.** In the last part of these notes we have developed numerical methods to approximate the solutions for the dynamics of a rigid body on the special orthogonal group $SO(3)$. More concretely, we have obtained a variational integrator referred to a Lie group by means of the discrete Hamilton's principle. The scenario is very specific, i.e., the problem is defined in a concrete group. Nevertheless, as was detailed (3.1), the procedure can be *extrapolated* to a general group G. We summarize the paralell process in the table below (the left column is referred to the $SO(3)$ setting and the right one to the G setting).

In general scenario, the dynamical problem would be defined in the tangent bundle of a Lie group (configuratin manifold) G, that is, $L : TG \to \mathbb{R}$. Taking a left trivialization, we can rewrite the Lagrangian as a function of the group and the algebra associated to that group $L : G \times \mathfrak{g} \to \mathbb{R}$. In the i)-row we select the discrete update map in such a way the approximation does not cause any deviation from the configuration manifold G. The update map is f_k, which represents the relative update between two integration steps, is also an element of the group G. Since f_k is composed on the left with g_k, the discrete flow $g_k \in G$ for $0 \leq k \leq M$ evolves on the group automatically.

In the ii)-row we define a discrete Lagrangian which depends on the configuration manifold and the update map as an approximation of the continuous integral action. Once this discrete Lagrangian is defined, we fix de **discrete action sum** as

$$\mathcal{A}_d = \sum_{k=0}^{N-1} \mathcal{L}_d(g_k, f_k).$$

A difference between the introduction and the last equation shows clearly up: we select $\mathcal{L}_d$ for the discrete Lagragian in the G setting because the notation L_d has been fixed in the previous section for the $SO(3)$ problem. The update map (F_k in the $SO(3)$ case, f_k in the G case) will depend on the elements of the algebra ($\mathfrak{so}(3)$ and $\mathfrak{g}$ respectively). We have seen that the Cayley map or the exponential map are suitable choices in the case of the rigid body. In the iii) and iv)-rows we show the dependence of the variations of both group elements and update map on the algebra elements η_k.

Once we have all these ingrediets we are ready to apply the Hamilton's principle to the discrete action sum $\mathcal{A}_d$, obtaining consequently the **discrete Euler-Poincaré equations** for a general group G. We present again these equations as a final corollary of these notes ($T_e^* l : \mathfrak{g}^* \to \mathfrak{g}^*$ represents the cotangent map of the usual left action of the group $l_g : G \to G$, $h \mapsto gh$, and D is the derivative of the discrete Lagrangian function with respect of one of its arguments):

Table 2. General Group

	$SO(3)$	G
$i)$	$R_{k+1} = R_k F_k$	$g_{k+1} \circ f_k \quad f_k \in G$
$ii)$	$L_d(R_k, F_k)$	$\mathcal{L}_d(g_k, f_k)$
$iii)$	$\delta R_k = R_k \hat{\eta}_k$	$\delta g_k = g_k \eta_k$
$iv)$	$\delta F_k = -\hat{\eta}_k F_k + F_k \hat{\eta}_{k+1}$	$\delta f_k = f_k \left(\eta_k - \mathrm{Ad}_{f_k} \eta_k\right)$

$$
\begin{aligned}
&T_e^* l_{f_{k-1}} D_{f_{k-1}} \mathcal{L}_d(g_{k-1}, f_{k-1}) - \mathrm{Ad}^*_{f_k^{-1}} \left(T_e^* l_{f_k} D_{f_k} \mathcal{L}_d(g_k, f_k)\right) + \\
&+ T_e^* l_{g_k} D_{g_k} \mathcal{L}_d(g_k, f_k) = 0, \\
&g_{k+1} = g_k \circ f_k,
\end{aligned}
$$

which again are equations (16). This setting provides the discrete flow $(g_k, f_k) \Rightarrow (g_{k+1}, f_{k+1})$ in $G \times G$. On the other hand we have the Hamiltonian version in $G \times \mathfrak{g}^*$

$$
\begin{aligned}
&\mathrm{Ad}^*_{f_k^{-1}} \left(T_e^* l_{f_k} D_{f_k} \mathcal{L}_d(g_k, f_k)\right) = \mu_k + T_e^* l_{g_k} D_{g_k} \mathcal{L}_d(g_k, f_k), \\
&g_{k+1} = g_k \circ f_k, \\
&\mu_{k+1} = \mathrm{Ad}^*_{f_k} \left(\mu_k + T_e^* l_{g_k} D_{g_k} \mathcal{L}_d(g_k, f_k)\right).
\end{aligned}
$$

This setting provides the discrete flow $(g_k, \mu_k) \Rightarrow (g_{k+1}, \mu_{k+1})$.

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