INTRODUCTION TO GRADED BUNDLES V: HIGHER ORDER LAGRANGIAN MECHANICS

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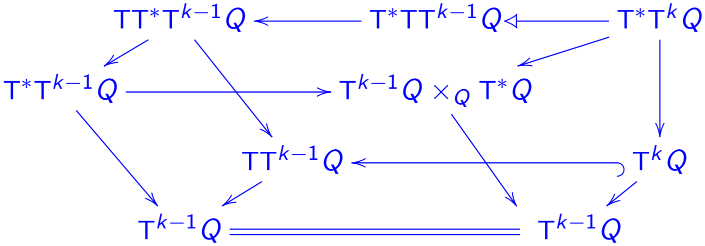
Higher order Lagrangians

The mechanics with a higher order Lagrangian $L : \mathsf{T}^k Q \to \mathbb{R}$ is traditionally constructed as a vakonomic mechanics, thanks to the canonical embedding of of the higher tangent bundle $\mathsf{T}^k Q$ into the tangent bundle $\mathsf{T}\mathsf{T}^{k-1}Q$ as an affine subbundle of holonomic vectors:

$$\left(q, \dot{q}, \ddot{q}, \dots, \overset{(k-1)}{q}, \overset{(k)}{q}\right) \mapsto \left(q, \dot{q}, \ddot{q}, \dots, \overset{(k-1)}{q}, \dot{q}, \ddot{q}, \dots, \overset{(k-1)}{q}, \overset{(k)}{q}\right) .$$

Thus we work with the standard Tulczyjew triple for $\mathsf{T}M$, where $M = \mathsf{T}^{k-1}Q$, with the presence of vakonomic constraint $\mathsf{T}^k Q \subset \mathsf{T}\mathsf{T}^{k-1}Q$:



Higher order Euler-Lagrange equations

The Lagrangian function $L = L(q, \dot{q}, \dots, \overset{(k)}{q})$ generates the phase dynamics

$$\mathcal{D} = \left\{ (v, p, \dot{v}, \dot{p}) : \; \dot{v}_{i-1} = v_i, \;\; \dot{p}_i + p_{i-1} = \frac{\partial L}{\partial \overset{(i)}{q}}, \dot{p}_0 = \frac{\partial L}{\partial q}, p_{k-1} = \frac{\partial L}{\partial \overset{(k)}{q}} \right\}.$$

This leads to the higher Euler-Lagrange equations in the traditional form:

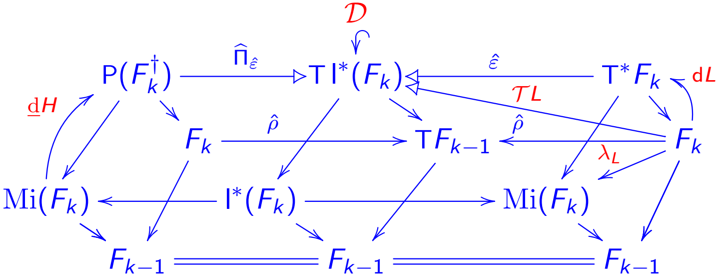
$$\overset{(i)}{q} = \frac{\mathrm{d}^i q}{\mathrm{d}t^i}, \; i = 1, \dots, k,$$

$$0 = \frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) + \cdots + (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(\frac{\partial L}{\partial \overset{(k)}{q}}\right).$$

These equations can be viewed as a system of ordinary differential equations of order k on $\mathsf{T}^k Q$ or, which is the standard point of view, as an ordinary differential equation of order $2k$ on Q.

Lagrangian framework for graded bundles

A weighted Lie algebroid on $\mathsf{I}(F_k)$ gives the Tulczyjew triple



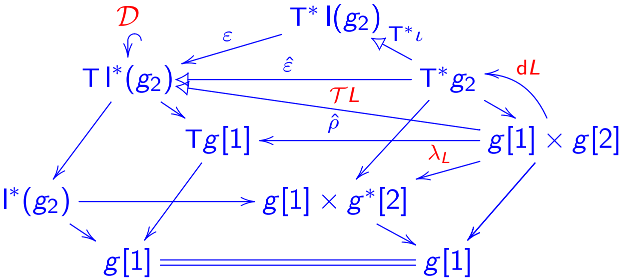
Here, the diagram consists of relations, $\hat{\varepsilon} : \mathsf{T}^* F_k \longrightarrow\!\triangleright \mathsf{T}^*\,\mathsf{I}(F_k) \to \mathsf{T}\,\mathsf{I}^*(F_k)$, and $\mathrm{Mi}(F_k) = F_{k-1} \times_M \bar{F}_k$ is the so called Mironian of F_k. In the classical case, $\mathrm{Mi}(\mathsf{T}^k M) = \mathsf{T}^{k-1} M \times_M \mathsf{T}^* M$.

$\mathcal{T}L$ is the Tulczyjew differential and λ_L the Legendre relation.

The fact that we obtain the Euler-Lagrange equations of higher order comes from the vakonomic constraint and the additional gradation.

Example

Let g be a Lie algebra and put $F_2 = g_2 = g[1] \times g[2]$, with coordinates (x^i, z^j) on g_2 and coordinates (x^i, y^j, z^k) on $\mathsf{l}(g_2) = g[1] \times g[1] \times g[2]$. The vector bundle projection is $\tau(x, y, z) = x$ and the corresponding diagram looks like



The embedding $\iota : g_2 \hookrightarrow \mathsf{l}(g_2)$ takes the form $\iota(x, z) = (x, x, z)$. In coordinates $(x, y, z, \alpha, \beta, \gamma)$ on $\mathsf{T}^*\,\mathsf{l}(g_2)$, the phase relation $\mathsf{T}^*\iota : \mathsf{T}^*g_2 \longrightarrow\!\!\triangleright\, \mathsf{T}^*\,\mathsf{l}(g_2)$ relates $(x, z, \alpha + \beta, \gamma)$ with $(x, x, z, \alpha, \beta, \gamma)$.

Example continued

The Lie algebroid structure $\varepsilon : \mathsf{T}^* \mathsf{l}(g_2) \to \mathsf{T}\,\mathsf{l}^*(g_2)$ reads

$$(x, y, z, \alpha, \beta, \gamma) \mapsto (x, \beta, \gamma, z, \mathrm{ad}^*_y\beta, \alpha)\,,$$

so $\hat{\varepsilon}$ relates $(x, z, \alpha + \beta, \gamma)$ with $(x, \beta, \gamma, z, \mathrm{ad}^*_x\beta, \alpha)$.
Given a Lagrangian $L : g_2 \to \mathbb{R}$, the Tulczyjew differential relation $\mathcal{T}L : g_2 \to \mathsf{T}\,\mathsf{l}^*(g_2)$ therefore reads

$$\mathcal{T}L(x, z) = \left\{ \left(x, \beta, \frac{\partial L}{\partial z}(x, z), z, \mathrm{ad}^*_x\beta, \alpha \right) : \alpha + \beta = \frac{\partial L}{\partial x}(x, z) \right\}\,.$$

Hence, for the phase dynamics,

$$z = \dot{x}\,, \quad \mathrm{ad}^*_x\beta = \dot{\beta}\,, \quad \alpha = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z}(x, z)\right)\,,$$

and

$$\beta = \frac{\partial L}{\partial x}(x, z) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z}(x, z)\right)\,.$$

Higher Euler equations

This leads to the Euler-Lagrange equations on g_2:

$$\begin{aligned} \dot{x} &= z\,, \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial x}(x,z)-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z}(x,z)\right)\right) &= \mathrm{ad}^*_x\left(\frac{\partial L}{\partial x}(x,z)-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z}(x,z)\right)\right)\,. \end{aligned}$$

These equations are second order and induce the Euler-Lagrange equations on g which are of order 3:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial x}(x,\dot{x})-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z}(x,\dot{x})\right)\right)=\mathrm{ad}^*_x\left(\frac{\partial L}{\partial x}(x,\dot{x})-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z}(x,\dot{x})\right)\right)\,.$$

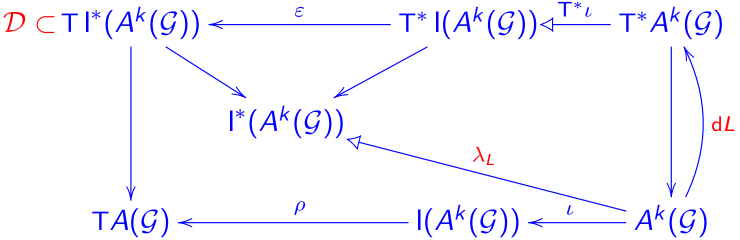
For instance, the 'free' Lagrangian $L(x,z)=\frac{1}{2}\sum_i I_i(z^i)^2$ induces the equations on g (c_{ij}^k are structure constants, no summation convention):

$$I_j\,\dddot{x}^j=\sum_{i,k}c_{ij}^k I_k x^i\ddot{x}^k\,.$$

The latter can be viewed as 'higher Euler equations'.

Higher order Lagrangian mechanics on Lie algebroids

Let us consider a general Lie groupoid $\mathcal{G}$ and a Lagrangian $L : A^k \to \mathbb{R}$ on $A^k = A^k(\mathcal{G})$. We will refer to such systems as a k-th order Lagrangian system on the Lie algebroid $A(\mathcal{G})$. The relevant diagram here is



Here, $\mathsf{I}(A^k(\mathcal{G}))$ is the corresponding Lie algebroid prolongation, $\mathcal{D} = \varepsilon \circ r \circ \mathrm{d}L(A^k(\mathcal{G}))$, and λ_L is the Legendre relation.

Note that we deal with reductions: in the case $\mathcal{G}$ is a Lie group,

$$A^k(\mathcal{G}) = \mathsf{T}^k(\mathcal{G})/\mathcal{G} \quad \text{and} \quad \mathsf{I}(A^k(\mathcal{G})) = \mathsf{T}\mathsf{T}^{k-1}(\mathcal{G})/\mathcal{G}\,.$$

For instance, using x^A as base coordinates, and y^a_i as fibre coordinates of degree $i = 1, \ldots, k$ in A^k, extended by the appropriate momenta π^j_b of degree $j = 1, \ldots, k$ in $\mathsf{l}^*(A^k)$, we get the equations for the Legendre relation in the form (no Lie algebroid structure appears!):

$$k\pi^1_a = \frac{\partial L}{\partial y^a_k},$$

$$(k-1)\pi^2_b = \frac{\partial L}{\partial y^b_{k-1}} - \frac{1}{k}\frac{d}{dt}\left(\frac{\partial L}{\partial y^b_k}\right),$$

$$\vdots$$

$$\pi^k_d = \frac{\partial L}{\partial y^d_1} - \frac{1}{2!}\frac{d}{dt}\left(\frac{\partial L}{\partial y^d_2}\right) + \frac{1}{3!}\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial y^d_3}\right) - \cdots$$
$$+(-1)^k\frac{1}{(k-1)!}\frac{d^{k-2}}{dt^{k-2}}\left(\frac{\partial L}{\partial y^d_{k-1}}\right) - (-1)^k\frac{1}{k!}\frac{d^{k-1}}{dt^{k-1}}\left(\frac{\partial L}{\partial y^d_k}\right),$$

which we recognise as the Jacobi–Ostrogradski momenta.

Higher order Lagrangian mechanics on Lie algebroids

The remaining equation for the dynamics is

$$\frac{d}{dt}\pi^k_a = \rho^A_a(x)\frac{\partial L}{\partial x^A} + y^b_1 C^c_{ba}(x)\pi^k_c\,,$$

where ρ^A_a and C^c_{ba} are structure functions of the Lie algebroid $A = A(\mathcal{G})$. The above equation can then be rewritten as

$$\rho^A_a(x)\frac{\partial L}{\partial x^A} = \left(\delta^c_a \frac{d}{dt} - y^b_1 C^c_{ba}(x)\right)\left(\frac{\partial L}{\partial y^c_1} - \frac{1}{2!}\frac{d}{dt}\left(\frac{\partial L}{\partial y^c_2}\right)\cdots - (-1)^k \frac{1}{k!}\frac{d^{k-1}}{dt^{k-1}}\left(\frac{\partial L}{\partial y^c_k}\right)\right)$$

which we define to be the k-th order Euler–Lagrange equations on $A(\mathcal{G})$.

The above higher order algebroid Euler-Lagrange equations are in complete agrement with the ones obtained by Jóźwikowski & Rotkiewicz, Colombo & de Diego, as well as Martínez. We clearly recover the standard higher Euler–Lagrange equations on $\mathsf{T}^k M$ as a particular example.

The tip of a javelin

For instance, let L be the Lagrangian, governing the motion of the tip of a javelin defined on $\mathsf{T}^2\mathbb{R}^3$,

$$L(x,y,z)=\frac{1}{2}\left(\sum_{i=1}^{3}(y^i)^2-(z^i)^2\right).$$

We can understand $G=\mathbb{R}^3$ here as a commutative Lie group, and since L is G-invariant, we get immediately the reduction to the graded bundle $\mathbb{R}^3[1]\times\mathbb{R}^3[2]$. The Euler-Lagrange equations on $\mathsf{T}^2\mathbb{R}^3$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial y^i}-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z^i}\right)\right)=0\,,$$

give in this case

$$\frac{\mathrm{d}y^i}{\mathrm{d}t}=\frac{1}{2}\frac{\mathrm{d}^2z^i}{\mathrm{d}t^2}\,,$$

so the Euler-Lagrange equation on $\mathbb{R}^3$ ($y=\dot{x}$, $z=\ddot{x}$) reads

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2}=\frac{1}{2}\frac{\mathrm{d}^4x^i}{\mathrm{d}t^4}\,.$$

Dynamics with the configuration space $\wedge^n \mathsf{T} M$

- We want to built a similar framework for higher dimensional objects, being motivated by the study of dynamics of one-dimensional non-parametrized objects (strings).
- The motion of a system is given by an n-dimensional submanifold in the manifold M ("space-time"). An infinitesimal piece of the motion is the first jet of the submanifold. However, this model leads to essential complications even in one-dimensional case (relativistic particle). For instance, the infinitesimal action (Lagrangian) is not a function on first jets, but a section of certain line bundle over the first-jet manifold, a 'dual' of the bundle of "first jets with volumes".
- Compromise: take for the space of infinitesimal pieces of motions the space of simple n-vectors, which represent first jets of n-dimensional submanifolds together with an infinitesimal volume.It is technically convenient to extend this space to all n-vectors, i.e. to the vector bundle $\wedge^n \mathsf{T} M$ of n-vectors on M.

Dynamics with the configuration space $\wedge^n \mathsf{T} M$

- A Lagrangian L is a function

$$L : \wedge^n \mathsf{T} M \to \mathbb{R}\,.$$

If L is positive homogeneous, the action functional does not depend on the parametrization of the submanifold and the corresponding Hamiltonian (if it exists) is a function on the dual vector bundle $\wedge^n \mathsf{T}^* M$ (the phase space).

- The dynamics should be an equation (in general, implicit) for n-dimensional submanifolds in the phase space, i.e.

$$\mathcal{D} \subset \wedge^n \mathsf{T} \wedge^n \mathsf{T}^* M\,.$$

- A submanifold S in the phase space $\wedge^n \mathsf{T}^* M$ is a solution of $\mathcal{D}$ if and only if its tangent space $\mathsf{T}_\alpha S$ at $\alpha \in \wedge^n \mathsf{T}^* M$ is represented by a bivector from $\mathcal{D}_\alpha$.
If we use a parametrization, then the tangent bivectors associated with this parametrization must belong to $\mathcal{D}$.

The Hamiltonian side for multivector bundles

Recall that $\wedge^2\mathsf{T}\wedge^2\mathsf{T}^*M$ is a double graded bundle (actually a GrL-bundle)

$$\begin{array}{ccccc} & & \wedge^2\mathsf{T}\wedge^2\mathsf{T}^*M & & \\ & \swarrow & & \searrow & \\ \wedge^2\mathsf{T}^*M & & & & \wedge^2\mathsf{T}M \\ & \searrow & & \swarrow & \\ & & M & & \end{array} .$$

We have:

- the canonical Liouville 2-form on $\wedge^2\mathsf{T}^*M$:
$$\theta^2_M = \frac{1}{2}p_{\mu\nu}\,\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\,;$$
- the canonical multisymplectic form
$$\omega^2_M = \mathrm{d}\theta^2_M = \frac{1}{2}\,\mathrm{d}p_{\mu\nu}\wedge\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\,;$$
- the vector bundle morphism
$$\beta^2_M\colon\ \wedge^2\mathsf{T}\wedge^2\mathsf{T}^*M\to\mathsf{T}^*\wedge^2\mathsf{T}^*M\,,\quad :u\mapsto i_u\omega^2_M\,.$$

The Lagrangian side for multivector bundles

In local coordinates,

$$\beta^2_M(x^\mu, p_{\lambda\kappa}, \dot{x}^{\nu\sigma}, y^\eta_{\theta\rho}, \dot{p}_{\gamma,\delta,\epsilon,\zeta}) = (x^\mu, p_{\lambda\kappa}, -y^\eta_{\eta\rho}, \dot{x}^{\nu\sigma})\,.$$

Using the canonical isomorphism of double vector bundles

$$\mathcal{R} : \mathsf{T}^* \wedge^2 \mathsf{T}^* M \to \mathsf{T}^* \wedge^2 \mathsf{T} M\,,$$

we can define $\alpha^2_M = \mathcal{R} \circ \beta^2_M$, which is another double graded bundle morphism,

$$\alpha^2_M \colon \wedge^2 \mathsf{T} \wedge^2 \mathsf{T}^* M \to \mathsf{T}^* \wedge^2 \mathsf{T} M\,,$$

(of double graded bundles over $\wedge^2 \mathsf{T} M$ and $\wedge^2 \mathsf{T}^* M$).

In local coordinates,

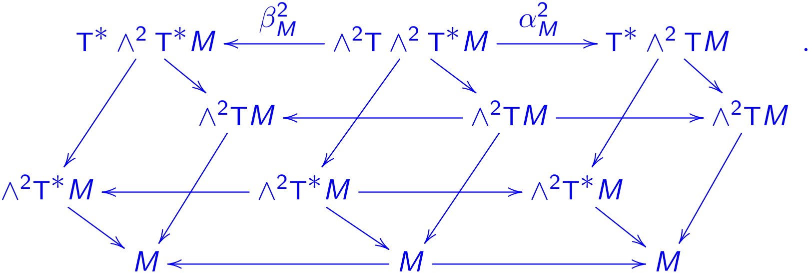
$$\alpha^2_M(x^\mu, p_{\lambda\kappa}, \dot{x}^{\nu\sigma}, y^\eta_{\theta\rho}, \dot{p}_{\gamma\delta\epsilon\zeta}) = (x^\mu, \dot{x}^{\nu\sigma}, y^\eta_{\eta\rho}, p_{\lambda\kappa})\,.$$

The map α^2_M can also be obtained as a certain 'dual' of the canonical isomorphism

$$\kappa^2_M : \mathsf{T} \wedge^2 \mathsf{T} M \to \wedge^2 \mathsf{T}\mathsf{T} M\,.$$

The Tulczyjew triple and dynamics

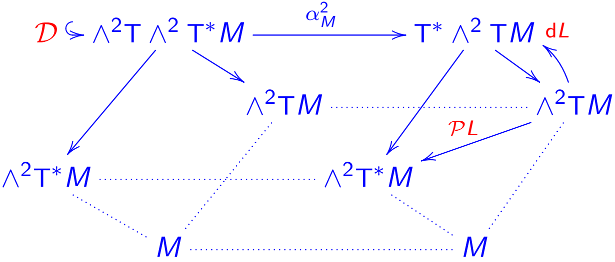
Combining the maps β^2_M and α^2_M, we get the following Tulczyjew triple for multivector bundles, consisting of double graded bundle morphisms:



The way of obtaining the implicit phase dynamics D, as a submanifold of $\wedge^2\mathsf{T}\wedge^2\mathsf{T}^*M$, from a Lagrangian $L:\wedge^2\mathsf{T}M\to\mathbb{R}$ or from a Hamiltonian $H:\wedge^2\mathsf{T}^*M\to\mathbb{R}$ is now standard.

The phase dynamics - Lagrangian side

$\wedge^2 \mathsf{T}M$ - (kinematic) configurations, $L : \wedge^2 \mathsf{T}M \to \mathbb{R}$ - Lagrangian



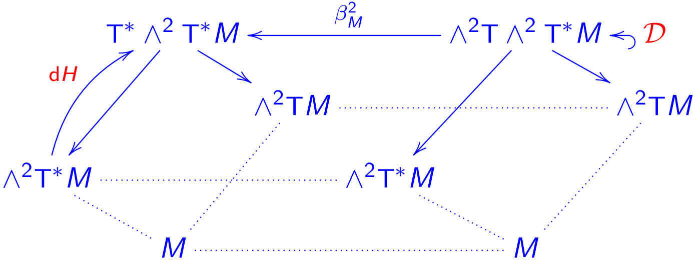
$$\mathcal{D} = (\alpha^2_M)^{-1}(\mathrm{d}L(\wedge^2\mathsf{T}M))$$

$$\mathcal{D} = \left\{ (x^\mu, p_{\lambda\kappa}, \dot{x}^{\nu\sigma}, y^\eta_{\theta\rho}, \dot{p}_{\gamma\delta\epsilon\zeta}) : \quad y^\eta_{\eta\rho} = \frac{\partial L}{\partial x^\rho}, \quad p_{\lambda\kappa} = \frac{\partial L}{\partial \dot{x}^{\lambda\kappa}} \right\} .$$

Thus we get Lagrange (phase) equations.

The phase dynamics - Hamiltonian side

$H : \wedge^2 \mathsf{T}^* M \to \mathbb{R}$



$$\mathcal{D} = (\beta^2_M)^{-1}(\mathrm{d}H(\wedge^2 \mathsf{T}^* M))$$

$$\mathcal{D} = \left\{ (x^\mu, p_{\lambda\kappa}, \dot{x}^{\nu\sigma}, y^\eta_{\theta\rho}, \dot{p}_{\gamma\delta\epsilon\zeta}) : \quad y^\eta_{\eta\rho} = -\frac{\partial H}{\partial x^\rho}, \quad \dot{x}^{\nu\sigma} = \frac{\partial H}{\partial p_{\nu\sigma}} \right\} .$$

Thus we get Hamilton equations.

The EL and Hamilton equations

For a surface in the phase space $\wedge^2 \mathsf{T}^* M$,

$$(t,s) \mapsto \left(x^\mu(t,s), p_{\kappa\lambda}(s,t)\right),$$

the Euler-Lagrange equations read

$$\begin{aligned} \dot{x}^{\mu\nu} &= \frac{\partial x^\mu}{\partial t}\frac{\partial x^\nu}{\partial s} - \frac{\partial x^\mu}{\partial s}\frac{\partial x^\nu}{\partial t}, \\ \frac{\partial L}{\partial x^\sigma} &= \frac{\partial x^\mu}{\partial t}\frac{\partial}{\partial s}\left(\frac{\partial L}{\partial \dot{x}^{\mu\sigma}}(t,s)\right) - \frac{\partial x^\mu}{\partial s}\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}^{\mu\sigma}}(t,s)\right). \end{aligned}$$

As for the Hamilton equations, we have

$$\begin{aligned} \frac{\partial H}{\partial p_{\mu\nu}} &= \frac{\partial x^\mu}{\partial t}\frac{\partial x^\nu}{\partial s} - \frac{\partial x^\mu}{\partial s}\frac{\partial x^\nu}{\partial t}, \\ -\frac{\partial H}{\partial x^\sigma} &= \frac{\partial x^\mu}{\partial t}\frac{\partial p_{\mu\sigma}}{\partial s} - \frac{\partial x^\mu}{\partial s}\frac{\partial p_{\mu\sigma}}{\partial t}. \end{aligned}$$

An example

In the dynamics of strings, the manifold of infinitesimal configurations is $\wedge^2 \mathsf{T}M$, where M is the space time with the Lorentz metric g. This metric induces a scalar product h in fibers of $\wedge^2 \mathsf{T}M$: for

$$w = \frac{1}{2}\dot{x}^{\mu\nu}\frac{\partial}{\partial x^\mu} \wedge \frac{\partial}{\partial x^\nu}, \quad u = \frac{1}{2}\dot{x}'^{\mu\nu}\frac{\partial}{\partial x^\mu} \wedge \frac{\partial}{\partial x^\nu}$$

we have

$$(u|w) = h_{\mu\nu\kappa\lambda}\dot{x}^{\mu\nu}\dot{x}'^{\kappa\lambda}\,,$$

where

$$h_{\mu\nu\kappa\lambda} = g_{\mu\kappa}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\kappa}\,.$$

The Lagrangian is a function of the volume with respect to this metric, the so called Nambu-Goto Lagrangian,

$$L(w) = \sqrt{(w|w)} = \sqrt{h_{\mu\nu\kappa\lambda}\dot{x}^{\mu\nu}\dot{x}^{\kappa\lambda}}\,,$$

which is defined on the open submanifold of positive bivectors.

An example

The dynamics $\mathcal{D} \subset \wedge^2 \mathsf{T} \wedge^2 \mathsf{T}^* M$ is the inverse image by α^2_M of the image $\mathrm{d}L(\wedge^2 \mathsf{T} M)$ and it is described by the Lagrange (phase) equations

$$\begin{aligned} y^{\alpha}_{\alpha\nu} &= \frac{1}{2\rho}\frac{\partial h_{\mu\kappa\lambda\sigma}}{\partial x^{\nu}}\dot{x}^{\mu\kappa}\dot{x}^{\lambda\sigma}, \\ p_{\mu\nu} &= \frac{1}{\rho}h_{\mu\nu\lambda\kappa}\dot{x}^{\lambda\kappa}, \end{aligned}$$

where

$$\rho = \sqrt{h_{\mu\nu\lambda\kappa}\dot{x}^{\mu\nu}\dot{x}^{\lambda\kappa}}\,.$$

The dynamics $\mathcal{D}$ is also the inverse image by β^2_M of the lagrangian submanifold in $\mathsf{T}^* \wedge^2 \mathsf{T}^* M$, generated by the Morse family

$$\begin{aligned} H \quad &: \wedge^2 \mathsf{T}^* M \times \mathbb{R}_+ \to \mathbb{R}\,, \\ &: (p, r) \mapsto r\left(\sqrt{(p|p)} - 1\right). \end{aligned}$$

In the case of minimal surface, i.e. the Plateau problem, we replace the Lorentz metric with a positively defined one.

Plateau problem

In particular, if $M = \mathbb{R}^3 = \{(x^1 = x, x^2 = y, x^3 = z)\}$ with the Euclidean metric, the Lagrangian reads

$$L(x^\mu, \dot{x}^{\kappa\lambda}) = \sqrt{\sum_{\kappa,\lambda} (\dot{x}^{\kappa\lambda})^2}\,.$$

The Euler-Lagrange equation for surfaces, being graphs $(x, y) \mapsto (x, y, z(x, y))$, provides the well-known equation for minimal surfaces, found already by Lagrange :

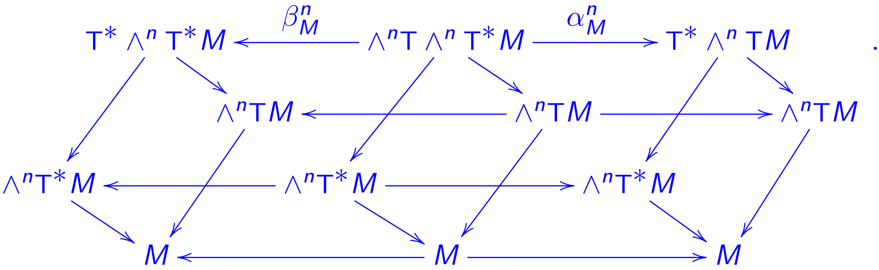
$$\frac{\partial}{\partial x}\left(\frac{z_x}{\sqrt{1+z_x^2+z_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{z_y}{\sqrt{1+z_x^2+z_y^2}}\right) = 0\,.$$

In another form:

$$(1+z_x^2)z_{yy} - 2z_x z_y z_{xy} + (1+z_y^2)z_{xx} = 0\,.$$

A generalization

We have a straightforward generalization for all integer $n \geq 1$ replacing 2:



The map β^n_M comes from the canonical multisymplectic $(n+1)$-form ω^n_M on $\wedge^n \mathsf{T}^* M$, being the differential of the canonical Liouville n-form θ^n_M:

$$\begin{aligned} \beta^n_M \quad &: \wedge^n \mathsf{T} \wedge^n \mathsf{T}^* M \to \mathsf{T}^* \wedge^n \mathsf{T}^* M \\ &: u \mapsto \iota_u \omega^n_M . \end{aligned}$$

The map α^n_M is just the composition of β^n_M with the canonical isomorphism of double vector bundles $\mathsf{T}^* \wedge^n \mathsf{T}^* M$ and $\mathsf{T}^* \wedge^n \mathsf{T} M$.