INTRODUCTION TO GRADED BUNDLES IV: MECHANICS VIA TULCZYJEW TRIPLES

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Weighted Lie algebroids out of reductions

For a Lie groupoid $G \rightrightarrows M$, consider the subbundle $\mathsf{T}^k G^{\underline{s}} \subset \mathsf{T}^k G$ consisting of all higher order velocities tangent to source-leaves. The bundle

$$F_k = A^k(G) := \mathsf{T}^k G^{\underline{s}}\Big|_M ,$$

inherits graded bundle structure of degree k as a graded subbundle of $\mathsf{T}^k G$. Of course, $A = A^1(G)$ can be identified with the Lie algebroid of G.

Theorem

The linearisation of $A^k(G)$ is given as

$$\mathsf{l}(A^k(G)) \simeq \{(Y, Z) \in A(G) \times \mathsf{T}A^{k-1}(G) | \quad \rho(Y) = \mathsf{T}\tau(Z)\},$$

viewed as a vector bundle over $A^{k-1}(G)$ with respect to the obvious projection of part Z onto $A^{k-1}(G)$, where $\rho : A(G) \to \mathsf{T}M$ is the standard anchor of the Lie algebroid and $\tau : A^{k-1}(G) \to M$ is the obvious projection. Moreover, the above bundle is canonically a weighted Lie algebroid, a Lie algebroid prolongation in the sense of Popescu and Martínez.

Total linerization of $A^k(G)$

Continuing description of $\mathsf{l}(A^k(G))$, we note that the linearisation functor as a subfunctor of the tangent functor respects products and commutes with the tangent functor. In particular, we have

$$\mathsf{l}^{(2)}(A^3(G)) \subset \mathsf{l}(A(G) \times \mathsf{T}A^2(G)) = A(G) \times \mathsf{T}\mathsf{l}(A^2(G)),$$

Thus, proceeding by induction, we get:

$$\mathsf{L}(A^3(G)) \subset A(G) \times \mathsf{T}A(G) \times \mathsf{T}^{(2)}A(G).$$

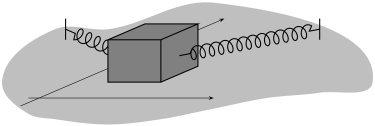
Theorem

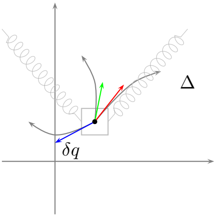
The full linearisation of $A^k(G)$ is given as

$$\mathsf{L}(A^k(G)) = \Big\{(X_1, \cdots, X_k) \in A(G) \times \mathsf{T}A(G) \cdots \times \mathsf{T}^{(k-1)}A(G) |$$
$$\rho(X_1) = \mathsf{T}\pi(X_2), \ \cdots, \ \mathsf{T}^{(k-2)}\rho(X_{k-1}) = \mathsf{T}^{(k-1)}\pi(X_k)\Big\},$$

where $\mathsf{T}^{(l)} = \mathsf{T}\mathsf{T}\cdots\mathsf{T}$ (l-times), $\pi : A(G) \to M$ is the standard projection, and $\rho : A(G) \to \mathsf{T}M$ is the anchor of the Lie algebroid.

Variational calculus in statics





- Q - manifold of configurations
- Γ - admissible processes, i.e., one-dimensional oriented submanifolds with boundary (sometimes, however, we use a parametrization)
- $\mathcal{W} : \Gamma \to \mathbb{R}$ - the cost function

$$\mathcal{W}(\gamma) = \int_\gamma W\,,$$

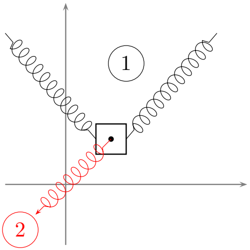
for W being a positively homogeneous function on the set $\Delta \subset \mathsf{T}Q$ of vectors δq tangent to admissible processes.

Variational calculus in statics

Definition

Point $q \in Q$ is an equilibrium point of the system if for all processes starting in q the cost function is non-negative, at least initially.
First-order condition: $W(q) \geq 0$.

Interactions between systems are described by composite systems



- system (1) and (2) have the same configurations Q
- $\Delta = \Delta_1 \cap \Delta_2$
- $W = W_1 + W_2$

The interaction with an 'external' system is usually described in terms of forces $\varphi \in \mathsf{T}^*Q$: $W_2(\delta q) = -\langle \varphi, \delta q \rangle$.

Variational Calculus

The subset $\mathcal{C} \subset \mathsf{T}^*Q$ of all external forces in equilibrium with our system is called the constitutive set.

We will consider only 'potential systems' without constraints, where $\Delta = \mathsf{T}Q$ and $W(\delta q) = \langle \mathrm{d}U, \delta q \rangle$ for a function $U : Q \to \mathbb{R}$, so that the constitutive set is $\mathcal{C} = \mathrm{d}U(Q)$.

In general, also for other theories, e.g. statics of an elastic rod, mechanics, different field theories, etc., we need

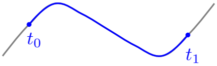
- Configurations Q,
- Processes (or at least infinitesimal processes),
- Functions on Q (to define regular systems),
- Covectors T^*Q (to define constitutive sets).

Mechanics for finite time interval

Let M be a manifold of positions of mechanical system. We will use smooth paths in M and first-order Lagrangians $L : \mathsf{T}M \to \mathbb{R}$.

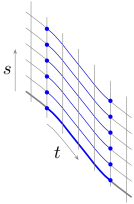
- Configurations:

$$Q = \{q : [t_0, t_1] \to M\} .$$



- Functions: $S(q) = \int_{t_0}^{t_1} L(\dot{q})dt.$
- Processes in Q come from homotopies $q_s(t) = \chi(s, t)$,

  $$\chi : \mathbb{R}^2 \supset [0, 1] \times [t_0, t_1] \to M .$$



- Tangent vectors are equivalence classes of curves.
- Cotangent vectors are equivalence classes of functions.

Mechanics for finite time interval

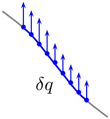
We need convenient representations of vectors and covectors:

$$\frac{d}{ds}\bigg|_{s=0} S\circ q_s = \int_{t_0}^{t_1} \langle \mathcal{E}L(\ddot{q}), \delta q\rangle dt + \left. \langle \mathcal{P}L(\dot{q}), \delta q\,\rangle\right|_{t_0}^{t_1},$$

where $\mathcal{E}L : \mathsf{T}^2 M \to \mathsf{T}^* M$ and $\mathcal{P}L = \mathrm{d}^{\mathsf{v}} L : \mathsf{T}M \to \mathsf{T}^* M$ are bundle maps.

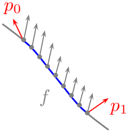
- Tangent vectors are in one-to-one correspondence

  with paths δq in $\mathsf{T}M$



- Covectors are in one-to-one correspondence with triples (f, p_0, p_1)

  $f : [t_0, t_1] \to \mathsf{T}^* M$, $p_i \in \mathsf{T}^*_{q(t_i)} M$.



Mechanics for finite time interval

We have found another representation of covectors (Liouville structure):

$$\alpha : \mathbb{P}Q = \{(f, p_0, p_1)\} \longrightarrow \mathsf{T}^*Q$$

Definition

The (phase) dynamics is a subset $\mathcal{D}$ of $\mathbb{P}Q = \{(f, p_0, p_1)\}$ given by

$$\mathcal{D} = \alpha^{-1}(dS(Q)),$$

i.e.,

$$\mathcal{D} = \{(f, p_0, p_1) : \quad f(t) = \mathcal{E}L(\ddot{q}(t)), \quad p_a = \mathcal{P}L(\dot{q}(t_a)), \; a = 0, 1\} \; .$$

Explicitly, writing $q = (x^i(t))$, $\dot{q} = (x^i(t), \dot{x}^j(t))$,

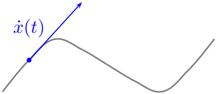
$$f(t) = \frac{\partial L}{\partial x^i}(\dot{q}(t)) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}^i}(\dot{q}(t))\right) , \quad p_a = \frac{\partial L}{\partial \dot{x}^i}(\dot{q}(t_a)) , \; a = 0, 1 \, .$$

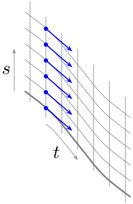
Mechanics: infinitesimal version

Let M be a manifold of positions of mechanical system. We will use smooth curves in M and first-order Lagrangians

- Configurations: $Q = \mathsf{T}M$, $q = (x, \dot{x})$
- Functions: $S(q) = L(x, \dot{x})$
- Curves in Q come from homotopies: $\chi : \mathbb{R}^2 \to M$
- Tangent vectors: $\mathsf{T}Q = \mathsf{T}\mathsf{T}M$, i.e, equivalence classes of curves in $\mathsf{T}M$, $\delta q = \delta \dot{x}$.

  Additionally, $\kappa_M : \mathsf{T}\mathsf{T}M \to \mathsf{T}\mathsf{T}M$, $\kappa(\chi)(s,t) = \chi(t,s)$.
- Covectors: $\mathsf{T}^*Q = \mathsf{T}^*\mathsf{T}M$





Dynamics

By (usually implicit) first-order dynamics on a manifold N we will understand a submanifold D in $\mathsf{T}N$.

A curve $\gamma : \mathbb{R} \to N$ satisfies this dynamics (is a solution), if its tangent prolongation belongs to D, $\widehat{\gamma} : \mathbb{R} \to D \subset \mathsf{T}N$.

Example

A vector field X on N, i.e. a section of the tangent bundle $X : N \to \mathsf{T}N$, defines the dynamics $D = X(N) \subset \mathsf{T}N$.

In local coordinates, for the vector field $X = f_a(q)\frac{\partial}{\partial q^a}$, we have

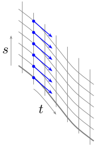
$$D = \{(q^a, \dot{q}^b) \in \mathsf{T}N : \dot{q}^b = f_b(q)\}$$

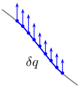
and the explicit dynamical equations $\frac{\mathrm{d}q^a}{\mathrm{d}t}(t) = f_a(q(t))$ are the equations for trajectories of this vector field.

Canonical isomorphisms

- Tangent vectors $\delta\dot{x}$ are in one-to-one correspondence with vectors tangent to curves $t \mapsto \delta x(t)$ in $\mathsf{T}M$

$$\kappa_M : \mathsf{TT}M \ni \delta\dot{x} \mapsto (\delta x)^{\cdot} \in \mathsf{TT}M$$





- We get also the tangent evaluation between TT^*M and $\mathsf{TT}M$ defined on elements $\dot{p}$ and $(\delta x)^{\cdot}$ with the same tangent projection δx on $\mathsf{T}M$:

$$\langle\langle \dot{p}, (\delta x)^{\cdot} \rangle\rangle = \left.\frac{d}{dt}\right|_{t=0} \langle p(t), \delta x(t) \rangle .$$

- The map dual to κ,

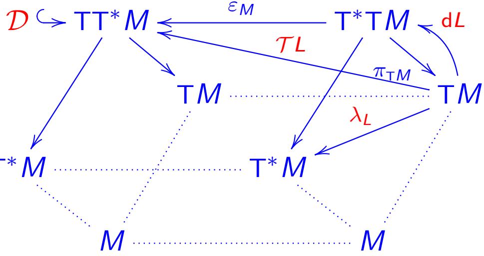
$$\alpha_M : \mathsf{TT}^*M \longrightarrow \mathsf{T}^*\mathsf{T}M$$

gives us an identification of covectors from $\mathsf{T}^*\mathsf{T}M$ with elements of TT^*M.

The Tulczyjew triple - Lagrangian side

Any $\mathcal{D} \subset \mathsf{T}N$ can be viewed as implicit dynamics whose solutions are curves $\gamma : \mathbb{R} \to N$ s.t. $\dot{\gamma} \in \mathcal{D}$. For the lagrangian phase equations:

M - positions,
$\mathsf{T}M$ - (kinematic) configurations,
$L : \mathsf{T}M \to \mathbb{R}$ - Lagrangian
T^*M - phase space



$$\mathcal{D} = \varepsilon_M(\mathrm{d}L(\mathsf{T}M)) = \mathcal{T}L(\mathsf{T}M)\,,$$

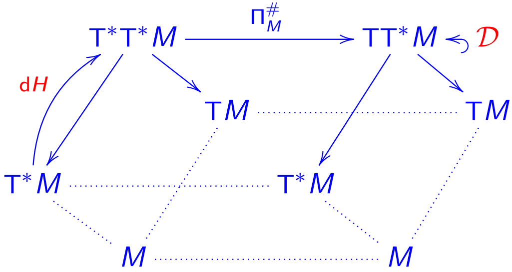
the image of the Tulczyjew differential $\mathcal{T}L$, is the phase dynamics,

$$\mathcal{D} = \left\{ (x, p, \dot{x}, \dot{p}) : \quad p = \frac{\partial L}{\partial \dot{x}}, \quad \dot{p} = \frac{\partial L}{\partial x} \right\}\,,$$

whence the Euler-Lagrange equation: $\frac{\partial L}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right)$. Note that L can be as well singular for the price that $\mathcal{D}$ is an implicit equation.

The Tulczyjew triple - Hamiltonian side

$H : \mathsf{T}^*M \to \mathbb{R}$

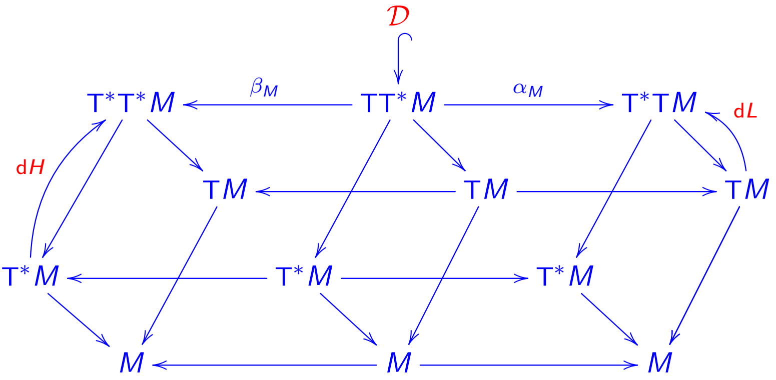


$$\mathcal{D} = \Pi^\#_M(\mathrm{d}H(\mathsf{T}^*M))$$

$$\mathcal{D} = \left\{ (x, p, \dot{x}, \dot{p}) : \quad \dot{p} = -\frac{\partial H}{\partial x}, \quad \dot{x} = \frac{\partial H}{\partial p} \right\},$$

whence the Hamilton equations.

Tulczyjew triple in mechanics



The dynamics is in the middle, the right-hand side is Lagrangian, the left-hand side – Hamiltonian.

The Legendre transform

The Legendre transform is a pass from the Lagrange to the Hamilton description of the dynamics:

we try to describe the Lagrangian phase dynamics as a Hamiltonian phase dynamics.

It is easy in the case of hyperregular Lagrangians (the Legendre map $(q,p)\mapsto\lambda_L(q,\dot q)=(q,p)$ is a diffeomorhism).

In this case the Lagrangian phase dynamics D_L is simultaneously Hamiltonian with the Hamiltonian function

$$\begin{aligned} H(q,p) &= \dot q^a p_a - L(q,\dot q)\,, \\ (q,\dot q) &= \lambda_L^{-1}(q,p)\,. \end{aligned}$$

In other words, the Lagrangian submanifolds $\mathrm{d}L(\mathsf{T}M)\subset\mathsf{T}^*\mathsf{T}M$ and $\mathrm{d}H(\mathsf{T}^*M)\subset\mathsf{T}^*\mathsf{T}^*M$ are related by the canonical isomorphism $\mathcal{R}_{\tau_M}$.

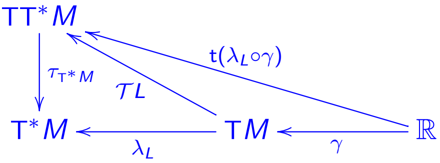
Euler-Lagrange equations

The Euler-Lagrange equation for a curve $\underline{\gamma} : \mathbb{R} \to M$ takes in this model the form

$$\mathrm{t}(\lambda_L \circ \gamma) = \mathcal{T}L \circ \gamma\,,$$

where $\mathcal{T}L = \varepsilon \circ \mathrm{d}L$ and $\gamma = \mathrm{t}(\underline{\gamma})$ is the tangent prolongation of $\underline{\gamma}$.

In this sense, the Euler-Lagrange equation can be viewed as a first-order differential equation on curves γ in $\mathsf{T}M$:



The equation just tells that the curve $\mathcal{T}L \circ \gamma$ is admissible, i.e. that it is a tangent prolongation of a curve (it must be $\lambda_L \circ \gamma$) on the phase space, $\mathcal{T}L \circ \gamma = \mathrm{t}(\lambda_L \circ \gamma)$.

Euler-Lagrange equations (continued)

In local coordinates,

$$\mathcal{T}L(q,\dot{q}) = (q, \frac{\partial L}{\partial \dot{q}}(q,\dot{q}), \dot{q}, \frac{\partial L}{\partial q}(q,\dot{q}))\,.$$

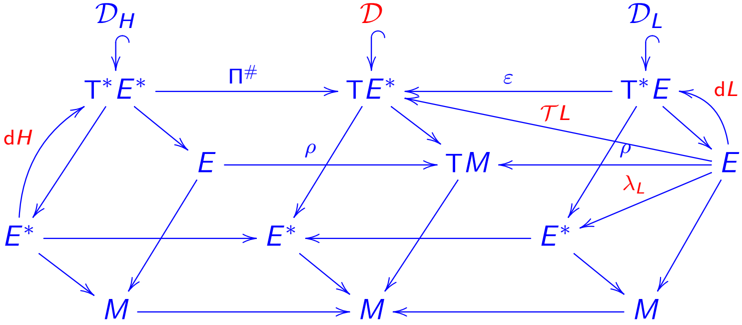
For $\gamma(t) = (q(t), \dot{q}(t))$ this implies the equations

$$\dot{q}(t) = \frac{\mathrm{d}q}{\mathrm{d}t}(t)\,, \quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}}(q(t),\dot{q}(t)) = \frac{\partial L}{\partial q}(q(t),\dot{q}(t))\,.$$

These equations are second-order equations for curves $q = q(t)$ in M.

Regularity of the Lagrangian is completely irrelevant for this formalism. Irregular Lagrangians just produce complicated and implicit dynamics, but the geometric model is the same.

Algebroid setting



$H : E^* \longrightarrow \mathbb{R}$ $\qquad$ $\mathcal{D} = \mathcal{T}L(E)$ $\qquad$ $L : E \longrightarrow \mathbb{R}$

$\mathcal{D}_H \subset \mathsf{T}^*E^*$ $\qquad$ $\mathcal{D} = \Pi^\#(\mathrm{d}H(E^*))$ $\qquad$ $\mathcal{D}_L \subset \mathsf{T}^*E$

The Euler-Lagrange equations read $\mathcal{T}L \circ \gamma = \mathrm{t}(\lambda_L \circ \gamma)$.

E-L equations for algebroids

If (q^a) are local coordinates in M,
(y^i) i (ξ_i) are linear coordinates in fibers of, respectively, E and E^*, and

$$P = c^k_{ij}(q)\xi_k\partial_{\xi_i}\otimes\partial_{\xi_j} + \rho^b_i(q)\partial_{\xi_i}\otimes\partial_{q^b} - \sigma^a_j(q)\partial_{q^a}\otimes\partial_{\xi_j}\,,$$

then the Euler-Lagrange equations read

$$\begin{aligned} (1) \qquad \frac{\mathrm{d}q^a}{\mathrm{d}t} &= \rho^a_k(q)y^k\,, \\ (2) \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial y^j}\right)(q,y) &= c^k_{ij}(q)y^i\frac{\partial L}{\partial y^k}(q,y) + \sigma^a_j(q)\frac{\partial L}{\partial q^a}(q,y)\,. \end{aligned}$$

They are first-order differential equations (!) but for admissible curves in E, i.e. for curves satisfying (1). For $E = \mathsf{T}M$, they are exactly the tangent prolongations of curves in M.

E-L equations for algebroids (continued)

A particular example of the equation (2) is not only the classical Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^a}(q,\dot{q}) = \frac{\partial L}{\partial q^a}(q,\dot{q})\,.$$

but also the Lagrange-Poincare equation for G-invariant Lagrangians on principal G-bundle

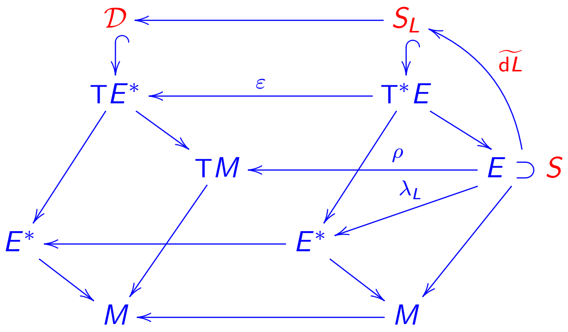
$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^a} - \frac{\partial L}{\partial q^a}\right)(q,\dot{q},v) - \left(B^k_{ba}(q)\dot{q}^b + D^k_{ia}(q)v^i\right)\frac{\partial L}{\partial v^k}(q,\dot{q},v) = 0\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial v^j}(q,\dot{q},v) - \left(D^k_{aj}(q)\dot{q}^a + C^k_{ij}v^i\right)\frac{\partial L}{\partial v^k}(q,\dot{q},v) = 0\,,$$

and the Euler-Poincare equations, for instance the rigid body equations,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial v^j}(v) - C^k_{ij}v^i\frac{\partial L}{\partial v^k}(v) = 0\,.$$

Algebroid setting with vakonomic constraints



where S_L is the lagrangian submanifold in T^*E induced by the Lagrangian on the constraint S, and $\widetilde{\mathrm{d}L} : S \to \mathsf{T}^*E$ is the corresponding relation,

$$S_L = \{\alpha_e \in \mathsf{T}^*_e E : e \in S \text{ and } \langle \alpha_e, v_e \rangle = \mathrm{d}L(v_e) \text{ for every } v_e \in \mathsf{T}_e S\}\,.$$

The vakonomically constrained phase dynamics is just $\mathcal{D} = \varepsilon(S_L) \subset \mathsf{T}E^*$.

Vakonomic equations in coordinates

Suppose that the vakonomic constraint S is defined as the zero-set of functions Φ^k.
Then, for a Lagrangian $L(x,y)$ on E, we have

$$S_L = \left\{ \left(x, y, \frac{\partial L}{\partial x}(x,y), \frac{\partial L}{\partial y}(x,y) - \mu_k(x,y)\frac{\partial \Phi^k}{\partial y}(x,y) \right) \mid \Phi^k(x,y) = 0 \right\} .$$

where $\mu_k \in C^\infty(S)$ are 'Lagrange multipliers'.
Looking for curves in S_L which are mapped by $\varepsilon : \mathsf{T}^*E \to \mathsf{T}E^*$,

$$\varepsilon(x^a, y^i, p_b, \xi_j) = (x^a, \xi_i, \rho^b_k(x)y^k, c^k_{ij}(x)y^i\xi_k + \sigma^a_j(x)p_a) ,$$

into admissible curves, we get the vakonomic E-L equations

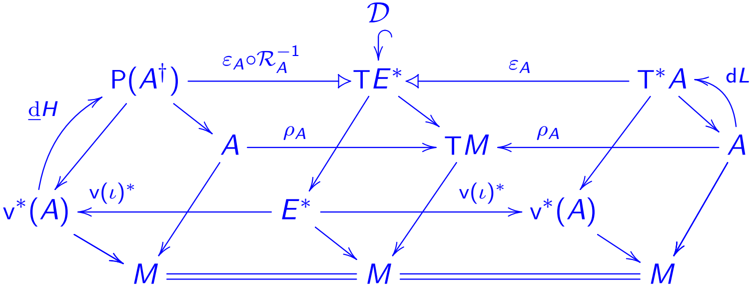
$$\Phi^k(x,y) = 0 , \quad \frac{\mathrm{d}x^a}{\mathrm{d}t} = \rho^a_k(x)y^k ,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial y^j}(x,y,t) - c^l_{ij}(x)y^i\frac{\partial L}{\partial y^l}(x,y,t) - \sigma^a_j(x)\frac{\partial L}{\partial x^a}(x,y,t) =$$

$$\dot{\mu}_k(t)\frac{\partial \Phi^k}{\partial y^j}(x,y) + \mu_k(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \Phi^k}{\partial y^j}(x,y) - c^l_{ij}(x)y^i\frac{\partial \Phi^k}{\partial y^l}(x,y) - \sigma^a_j(x)\frac{\partial \Phi^k}{\partial x^a}(x,y)\right)$$

Affine vakonomic constraints

In the case when $S = A$ is an affine subbundle of an algebroid E (assume for simplicity that A is supported on the whole M), we get the *reduced Tulczyjew triple* for an affine vakonomic constraint:



Here, $A^\dagger$ is the affine dual bundle, i.e. the bundle of affine functions on fibers of A, and Hamiltonians are sections of the affine phase bundle $\mathsf{P}(A^\dagger)$ over $\mathsf{v}^*(A)$ – the dual of the linear model $\mathsf{v}(A)$ of A.