INTRODUCTION TO THE THEORY OF GRADED BUNDLES I – BASICS

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Miraflores, June, 2016

What is a vector space?

- A (real) vector space is a set E with a distinguished element 0^E, equipped with two operations:
  1. an addition
  $$+ : E \times E \to E\,, \quad (u,v) \mapsto u+v\,,$$
  2. and a multiplication by scalars
  $$h : \mathbb{R} \times E \to E\,, \quad h(t,v) = h_t(v) = t \cdot v = tv\,,$$

  satisfying a list of axioms.
- For instance, $(E,+)$ is a commutative group with 0^E being the neutral element, the homotheties h_t satisfy
  $$h_t \circ h_s = h_{ts}\,,$$
  and $h_0(v) = 0^E$ for all $v \in E$.

One operation is enough

- To distinguish finite-dimensional real vector spaces among differentiable manifolds, a single operation of the above two is enough.
- If we know the addition, we get the multiplication by natural numbers in the obvious way:

$$nv = v + \cdots + v\,,$$

  and we easily extend it to integers by $(-n)v = n(-v)$. The multiplication by rational numbers, $(m/n)v$ we obtain as the solution of the equation $nx = mv$. Assuming differentiability (in fact, continuity) of h, we extend this multiplication to all reals uniquely.
- If we know the multiplication by reals h instead, we use a version of Euler's Homogeneous Function Theorem: any differentiable $f : \mathbb{R}^n \to \mathbb{R}$ is homogeneous of degree 1, i.e.

$$f(t \cdot x) = t \cdot f(x)\,,$$

  if and only if f is linear. Thus, from the multiplication by reals on E we get the dual space E^*, where the addition is well defined, and consequently the addition on $E = (E^*)^*$.

Graded spaces

Consider now a general (smooth) action $h : \mathbb{R} \times F \to F$ of the monoid $(\mathbb{R}, \cdot)$ on a manifold F (such an action we will call a homogeneity structure) and assume that $h_0(F) = 0^F$ for some element $0^F \in F$. Such a structure we will call a graded space by the following reasons.

Theorem (Grabowski-Rotkiewicz)

Any graded space (F, h) is diffeomorphically equivalent (isomorphic) to a certain $(\mathbb{R}^{\mathbf{d}}, h^{\mathbf{d}})$, where $\mathbf{d} = (d_1, \dots, d_k)$, with positive integers d_i, and $\mathbb{R}^{\mathbf{d}} = \mathbb{R}^{d_1}[1] \times \cdots \times \mathbb{R}^{d_k}[k]$ is equipped with the action $h^{\mathbf{d}}$ of multiplicative reals given by

$$h^{\mathbf{d}}_t(y_1, \dots, y_k) = (t \cdot y_1, \dots, t^k \cdot y_k), \quad y_i \in \mathbb{R}^{d_i}.$$

In other words, F can be equipped with a system of (global) coordinates (y^j_i), $i = 1 \dots, k$, $j = 1, \dots, d_i$, such that linear coordinates y^j_i in $\mathbb{R}^{d_i}[i]$ are homogeneous of degree i with respect to the homogeneity structure h, i.e.

$$y^j_i \circ h_t = t^i \cdot y^j_i.$$

Of course, in these coordinates $0^F = (0, \dots, 0)$.

How to recognize vector spaces?

- Note that the isomorphism in the above theorem is generally non-canonical. The number k, however, is uniquely determined and called the minimal degree of the graded space. By convention, the degree is any natural $k' \geq k$.
- How to recognize a vector space among graded spaces?
- Answer: Vector spaces are graded spaces of degree 1.
- Regularity condition: For any $y \in F$,

$$\frac{\mathrm{d}}{\mathrm{d}t_{\,|\,t=0}}(h_t(y)) = 0^F \;\Leftrightarrow y = 0^F\,.$$

Corollary

The homogeneity structure in a graded space comes from a vector space structure if and only if it is regular. In this case, the vector space structure is uniquely determined.

Weight vector field

- It is natural to call a morphism between graded spaces (F_a, h^a), $a = 1, 2$, a smooth map $\Phi : F_1 \to F_2$ which intertwines the homogeneity structures: $\Phi \circ h^1_t = h^2_t \circ \Phi$.
- The $(\mathbb{R}, \cdot)$ action restricted to positive reals gives a one-parameter group of diffeomorphism of F, thus is generated by a vector field ∇_F. It is called the weight vector field as it completely determines the weights (degrees) of coordinates. It reads

$$\nabla_F = \sum_w w\, y^j_w \partial_{y^j_w}\,.$$

- A function on F is homogeneous of degree w (has weight w) if and only if $\nabla_F(f) = w \cdot f$, and a smooth map $\Phi : F_1 \to F_2$ is a morphism of graded spaces iff it relates the corresponding weight vector fields.
- Note that morphisms need not to be linear, so the category of graded spaces is different from that of vector spaces. For instance, if $(y, z) \in \mathbb{R}^2$ are coordinates of degrees $1, 2$, respectively, then the map $(y, z) \mapsto (y, z + y^2)$ is an automorphism of the structure, but is nonlinear.

Vector bundles as graded bundles

- A vector bundle is a locally trivial fibration $\tau: E \to M$ which, locally over $U \subset M$, reads $\tau^{-1}(U) \simeq U \times \mathbb{R}^n$ and admits an atlas in which local trivializations transform linearly in fibers

  $$U \cap V \times \mathbb{R}^n \ni (x, y) \mapsto (x, A(x)y) \in U \cap V \times \mathbb{R}^n\,,$$

  $A(x) \in \mathrm{GL}(n, \mathbb{R})$.
- The latter property can also be expressed in the terms of the gradation in which base coordinates x have degrees 0, and 'linear coordinates' y have degree 1. Linearity in y's is now equivalent to the fact that changes of coordinates respect the degrees.
- Morphisms in the category of vector bundles are represented by commutative diagram of smooth maps

  $$\begin{array}{ccc} E_1 & \xrightarrow{\ \Phi\ } & E_2 \\ \downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \tau_2} \\ M_1 & \xrightarrow{\ \varphi\ } & M_2 \end{array}$$

  being linear (homogeneous) in fibres.

Graded bundles

- A straightforward generalization is the concept of a graded bundle $\tau : F \to M$ of rank $\mathbf{d}$, with a local trivialization by $U \times \mathbb{R}^{\mathbf{d}}$, and with the difference that the transition functions of local trivializations:
$$U \cap V \times \mathbb{R}^{\mathbf{d}} \ni (x, y) \mapsto (x, A(x, y)) \in U \cap V \times \mathbb{R}^{\mathbf{d}},$$
respect the weights of coordinates $(y^1, \ldots, y^{|\mathbf{d}|})$ in the fibres. In other words, a graded bundle of rank $\mathbf{d}$ is a locally trivial fibration with fibers modelled on the graded space $\mathbb{R}^{\mathbf{d}}$.

Theorem

$A(x, y)$ must be polynomial in homogeneous fiber coordinates y's, i.e. any graded bundle is a polynomial bundle.

- As these polynomials need not to be linear, graded bundles do not have, in general, vector space structure in fibers.
- If all $w_i \leq r$, we say that the graded bundle is of degree r.
- In the above terminology, vector bundles are just graded bundles of degree 1.

Graded bundles - examples

- Note that, according to our convention, any differential manifold M can be viewed as a graded bundle of degree 0.
- A trivial example is of course
$$F = M \times \mathbb{R}^{\mathbf{d}} = M \times \left(\mathbb{R}^{d_1}[1] \oplus \cdots \oplus \mathbb{R}^{d_k}[k]\right).$$
- Another trivial example, is a split graded bundle, i.e. a graded vector bundle
$$F = E^1[1] \oplus_M \cdots \oplus_M E^k[k],$$
where E^i are vector bundles over M.
- For vector bundles E^0, E^1 over M, we can consider the vector bundle $E = E^0[0] \oplus E^1[1]$ as a vector bundle over E^0.The wedge product $\wedge^2 E = \wedge^2 E^0 \oplus (E^0 \otimes E^1) \oplus \wedge^2 E^1$ can be then viewed as a graded vector bundle over $\wedge^2 E^0$ of degree 2, with $(E^0 \otimes E^1)$ being its part of degree 1 and $\wedge^2 E^1$ being of degree 2.
- Note that objects similar to graded bundles have been used in supergeometry by Ševera, Voronov, Roytenberg et al. under the name N-manifolds. However, during this course we will work exclusively with classical purely even manifolds.

Homogeneity structure of a graded bundle

- Note that the homogeneity structure in the typical fiber of a graded bundle F, i.e. the action $h:\mathbb{R}\times\mathbb{R}^{\mathbf{d}}\to\mathbb{R}^{\mathbf{d}}$, is preserved under the transition functions, that defines a globally defined homogeneity structure $h:\mathbb{R}\times F\to F$.
- In local homogeneous coordinates,

$$h_t(x^A,y^a_w)=(x^A,t^w y^a_w)\,.$$

- We call a function $f:F\to\mathbb{R}$ homogeneous of degree (weight) w if

$$f\circ h_t=t^w f\,.$$

- The whole information about the degree of homogeneity is contained in the weight vector field (called for vector bundles the Euler vector field)

$$\nabla_F=\sum_a w y^a_w\partial_{y^a_w}\,.$$

- A function $f:F\to\mathbb{R}$ is homogeneous of degree w if and only if

$$\nabla_F(f)=w\,f\,.$$

The category of graded bundles

Mimicking the definition of a vector bundle morphism, we get the following.

Definition

Morphisms in the category of graded bundles are represented by commutative diagram of smooth maps

$$\begin{array}{ccc} F^1 & \xrightarrow{\Phi} & F^2 \\ \downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \tau_2} \\ M_1 & \xrightarrow{\varphi} & M_2 \end{array}$$

which are morphisms of graded spaces in fibers, i.e. which locally preserve the weight of homogeneous coordinates.

One can equivalently say that the fiber bundle morphism Φ is a smooth map which relates the weight vector fields ∇_{F^1} and ∇_{F^2}.

Example. Morphisms $\Phi : F \to F$, for $F = \mathbb{R} \times \mathbb{R}^{(1,1)}$ with local coordinates (x, y, z) of degrees $(0, 1, 2)$, respectively, are of the form $\Phi(x, y, z) = (\phi(x), a(x)y, b(x)z + d(x)y^2)$.

Graded bundle = homogeneity structure

The fundamental fact (cf. [Grabowski-Rotkiewicz]) says that graded bundles and homogeneity structures are in fact equivalent concepts.

Theorem

Associating the homogeneity structure with a graded bundle is an isomorphism of categories.In particular, for any homogeneity structure h *on a manifold* F*, there is a smooth submanifold* $M = h_0(F) \subset F$ *and a non-negative integer* $k \in \mathbb{N}$ *such that* $h_0 : F \to M$ *is canonically a graded bundle of degree* k *whose homogeneity structure coincides with* h*. In other words, there is an atlas on* F *consisting of local homogeneous functions.*

Since morphisms of two homogeneity structures are defined as smooth maps $\Phi : F_1 \to F_2$ intertwining the $\mathbb{R}$-actions: $\Phi \circ h^1_t = h^2_t \circ \Phi$, this describes also morphism of graded bundles.

Consequently, a graded subbundle of a graded bundle F is a smooth submanifold S of F which is invariant with respect to homotheties, $h_t(S) \subset S$ for all $t \in \mathbb{R}$.

Consequences for vector bundles

Vector bundles can be recognized as graded bundles $\tau : F \to M$ of degree 1, i.e. satisfying the following regularity condition:

$$\frac{\mathrm{d}}{\mathrm{d}t_{\,|\,t=0}} h_t(p) = 0 \;\Leftrightarrow\; p \in M\,.$$

The principle multiplication by reals is enough has now the following consequences for vector bundles.

Corollary

A smooth map $\Phi : E_1 \to E_2$ between the total spaces of two vector bundles $\pi_i : E_i \to M_i$, $i = 1, 2$ is a morphism of vector bundles if and only if it intertwines the multiplications by reals:

$$\Phi(t \cdot v) = t \cdot \Phi(v)\,.$$

In this case the map $\phi = \Phi_{|M_1}$ is a smooth map between the base manifolds covered by Φ.

Graded bundles - further examples

- Example. Consider the second-order tangent bundle $\mathsf{T}^2 M$, i.e. the bundle of second jets of smooth maps $(\mathbb{R},0)\to M$. Writing paths in local coordinates (x^A) on M:
$$x^A(t)=x^A(0)+\dot{x}^A(0)t+\ddot{x}^A(0)\frac{t^2}{2}+o(t^2)\,,$$
we get local coordinates $(x^A,\dot{x}^B,\ddot{x}^C)$ on $\mathsf{T}^2 M$, which transform
$$\begin{aligned} x'^A &= x'^A(x)\,,\\ \dot{x}'^A &= \frac{\partial x'^A}{\partial x^B}(x)\,\dot{x}^B\,,\\ \ddot{x}'^A &= \frac{\partial x'^A}{\partial x^B}(x)\,\ddot{x}^B+\frac{\partial^2 x'^A}{\partial x^B\partial x^C}(x)\,\dot{x}^B\dot{x}^C\,. \end{aligned}$$
This shows that associating with $(x^A,\dot{x}^B,\ddot{x}^C)$ the weights $0,1,2$, respectively, will give us a graded bundle structure of degree 2 on $\mathsf{T}^2 M$. Due to the quadratic terms above, this is not a vector bundle!
- All this can be generalised to higher tangent bundles $\mathsf{T}^k M$. Note that any smooth map $\phi: M_1\to M_2$ induces a canonical morphism of graded bundles $\mathsf{T}^k\phi:\mathsf{T}^k M_1\to\mathsf{T}^k M_2$.

Graded bundles - further examples

- Another canonical example. If $\tau : E \to M$ is a vector bundle, then $\wedge^2 \mathsf{T}E$ is canonically a graded bundle of degree 2 with respect to the projection

$$\wedge^2\mathsf{T}\tau : \wedge^2\mathsf{T}E \to \wedge^2\mathsf{T}M\,.$$

- The adapted coordinates $(x^\rho, y^a, \dot{x}^{\mu\nu}, y^{\sigma b}, z^{cd})$ on $\wedge^2 E$, with $\dot{x}^{\mu\nu} = -\dot{x}^{\nu\mu}$, $z^{cd} = -z^{dc}$, coming from the decomposition of a bivector

$$\wedge^2\mathsf{T}E \ni u = \frac{1}{2}\dot{x}^{\mu\nu}\frac{\partial}{\partial x^\mu} \wedge \frac{\partial}{\partial x^\nu} + y^{\sigma b}\frac{\partial}{\partial x^\sigma} \wedge \frac{\partial}{\partial y^b} + \frac{1}{2}z^{cd}\frac{\partial}{\partial y^c} \wedge \frac{\partial}{y^d}\,,$$

  are of degrees $0, 1, 0, 1, 2$, respectively.

- All this can be generalized to a graded bundle structure of degree r on $\wedge^r\mathsf{T}E$:

$$\wedge^r\mathsf{T}\tau : \wedge^r\mathsf{T}E \to \wedge^r\mathsf{T}M\,.$$

Transition functions for graded bundles

- One can pick an atlas of F consisting of charts for which we have homogeneous local coordinates (x^A, y^a_w) with weight $\deg$, where $\deg(x^A) = 0$ and $\deg(y^a_w) = w$ with $1 \leq w \leq k$, where k is the degree of the graded bundle.
- The local changes of coordinates are of the form

$$\begin{aligned} x'^A &= x'^A(x), \\ y'^a_w &= y^b_w\, T_b{}^a(x) + \sum_{\substack{1<n \\ w_1+\cdots+w_n=w}} \frac{1}{n!} y'^{b_1}_{w_1} \cdots y'^{b_n}_{w_n}\, T_{b_n\cdots b_1}{}^a(x), \end{aligned} \tag{1}$$

  where $T_b{}^a$ are invertible and $T_{b_n\cdots b_1}{}^a$ are symmetric in indices b.
- In particular, the transition functions of coordinates of degree r involve only coordinates of degree $\leq r$, defining a reduced graded bundle F_r of degree r (we simply 'forget' coordinates of degrees $> r$).

Graded bundles - the tower of affine fibrations

- Transformations for the canonical projection $F_r \to F_{r-1}$ are linear modulo a shift by a polynomial in variables of degrees $< r$,

$$y_r'^a = y_r^b T_b{}^a(x) + \sum_{\substack{1<n \\ w_1+\cdots+w_n=r}} \frac{1}{n!} y_{w_1}'^{b_1} \cdots y_{w_n}'^{b_n} T_{b_n\cdots b_1}^{\;\;a}(x)\,,$$

  so the fibrations $F_r \to F_{r-1}$ are affine. The linear part of F_r corresponds to a vector subbundle $\bar{F}_r$ over M (we put y_w^a, with $0 < w < r$, equal to 0).
- In this way we get for any graded bundle F of degree k, like for jet bundles, a tower of affine fibrations

$$F = F_k \xrightarrow{\tau^k} F_{k-1} \xrightarrow{\tau^{k-1}} \cdots \xrightarrow{\tau^3} F_2 \xrightarrow{\tau^2} F_1 \xrightarrow{\tau^1} F_0 = M\,.$$

- Example. In the case of the canonical graded bundle $F = \mathrm{T}^k M$, we get exactly the tower of projections of jet bundles

$$\mathrm{T}^k M \xrightarrow{\tau^k} T^{k-1} M \xrightarrow{\tau^{k-1}} \cdots \xrightarrow{\tau^3} \mathrm{T}^2 M \xrightarrow{\tau^2} \mathrm{T}M \xrightarrow{\tau^1} F_0 = M\,.$$

Further constructions

- The reduced manifold F_r will also be denoted $F[\nabla \leq r]$ if we want to stress which weight vector field ∇ we have in mind (sometimes we will work with many).
- There is also a "dual" sequence of submanifolds and their inclusions

$$M := F_0 = F^{[k]} \hookrightarrow F^{[k-1]} \hookrightarrow \cdots \hookrightarrow F^{[0]} = F_k, \tag{2}$$

where we define, locally but correctly,

$$F^{[i]} := \{ p \in F_k \,|\, y^a_w = 0 \ \text{ if } \ w \leq i \}.$$

- In words, "you project higher to lower, but set to 0 lower to higher".
- Note that the $C^\infty(M)$-module $\mathcal{A}^r(F)$ of homogeneous functions of degree r on F is finitely generated and projective, so it corresponds to sections of a vector bundle $A^r(F)$ over M. The graded algebra

$$\mathcal{A}(F) = \bigoplus_{i=0}^{\infty} \mathcal{A}^i(F)$$

generated by homogeneous functions is called the polynomial algebra of F.

Splitting of graded bundles

- Homogeneous local coordinates (y_i^a) of degree $i > 0$ represent locally a basis of the quotient vector bundle

$$\mathcal{A}^i(F)/\bar{\mathcal{I}}^i(F) \simeq \bar{F}_i^*\,,$$

where $\bar{\mathcal{I}}^i(F) = (\mathcal{I}(F)\cdot\mathcal{I}(F))^i$ is the degree i part of the algebraic square $\mathcal{I}(F)\cdot\mathcal{I}(F)$ of the ideal $\mathcal{I}(F)$ in the polynomial algebra $\mathcal{A}(F)$ generated by homogeneous functions of degrees > 0,

$$\mathcal{I}(F) = \oplus_{i>0}\mathcal{A}^i(F)\,.$$

- Choosing vector subbundles $\bar{\mathcal{A}}^i(F) \subset \mathcal{A}^i(F)$ complementary to $\bar{\mathcal{I}}^i(F)$, we can pick up local coordinates of degree i from $\bar{\mathcal{A}}^i(F) \subset \mathcal{A}^i(F)$, killing therefore the higher order polynomial parts in transition rules. This gives the following.

Theorem

Any graded bundle F of degree k is isomorphic with the split graded bundle $\bar{F} = \bar{F}^1 \oplus \cdots \oplus \bar{F}^k$.

Splitting of graded bundles – comments

- The point is that this isomorphism is not canonical. Also the morphism of graded vector bundles in the category of graded bundles differ from graded vector bundle morphism which makes these categories different.
- The situation is similar to the celebrated Batchelor Theorem in supergeometry stating that any supermanifold is (non-canonically) diffeomorphic with the 'superization' ΠE of a vector bundle E. Of course, morphisms of such supermanifolds are different from that of vector bundles, so these categories are completely different.
- The Betchelor Theorem was actually first proved by Polish physicist Gawędzki, that provides therefore another example of the Arnold's law saying that *"Discoveries are rarely attributed to the correct person"*.
- Of course Arnold's law is self-referential, as e.g. Whitehead claimed earlier that *"Everything of importance has been said before by someone who did not discover it"*.

Tangent lifts of graded structures

- Consider an arbitrary graded bundle F_k of degree k over M with homogeneous coordinates (x^A, y^a_w), $1 \leq w \leq k$. The corresponding homogeneity structure is then

$$h_t(x^A, y^a_w) = (x^A,\, t^w y^a_w)$$

and the weight vector field: $\nabla_F := \sum_w w y^a_w \frac{\partial}{\partial y^a_w}$.

- Applying the tangent functor to all h_t, we get a homogeneity structure $(\mathrm{d}_{\mathsf{T}} h)_t = \mathsf{T} h_t$ on $\mathsf{T}F$:

$$\mathrm{d}_{\mathsf{T}} h_t(x^A, y^a_w, \dot{x}^B, \dot{y}^b_w) = (x^A\,, t^w y^a_w\,, \dot{x}^B, t^w \dot{y}^b_w)\,.$$

- The corresponding weight vector field is the tangent lift of ∇_F:

$$\nabla_{\mathsf{T}F} = \mathrm{d}_{\mathsf{T}} \nabla_F = \sum_w w y^a_w \frac{\partial}{\partial y^a_w} + \sum_w w \dot{y}^a_w \frac{\partial}{\partial \dot{y}^a_w}\,.$$

Phase lifts of graded structures

- Similarly we can try to lift h_t to the cotangent bundle T^*F with the adapted coordinates (x^A, y^a_w, p_B, p^w_b); for $t \neq 0$:

$$\mathsf{T}^*h_t(x^A, y^a_w, p_B, p^w_b) = (x^A, t^{-w}y^a_w, p_B, t^w p^w_b)\,.$$

- As this cannot be directly extended to an action of $\mathbb{R}$, we define the k-phase lift as $(\mathrm{d}^k_{\mathsf{T}^*}h)_t = t^k\mathsf{T}^*h_{t^{-1}}$:

$$(\mathrm{d}^k_{\mathsf{T}^*}h)_t(x^A, y^a_w, p_B, p^w_b) = (x^A, t^w y^a_w, t^k p_B, t^{k-w}p^w_b)\,.$$

- The associated weight vector field reads

$$\nabla_{\mathsf{T}^*F} = \mathrm{d}^k_{\mathsf{T}^*}\nabla_F = \sum_w w y^a_w\frac{\partial}{\partial y^a_w} + k p_B\frac{\partial}{\partial p_B} + \sum_w (k-w)p^w_a\frac{\partial}{\partial p^w_a}\,.$$

- In this way, the tangent and cotangent bundles are canonically graded bundles of degree k over F and $\bar{F}^*_k$, respectively.

Higher lifts and canonical isomorphisms

- Using higher tangent functors T^k, we can lift homogeneity structures on F to homogeneity structures on $\mathsf{T}^k F$ simply putting

$$(\mathrm{d}_{\mathsf{T}^k} h)_t = \mathsf{T}^k(h_t) : \mathsf{T}F \to \mathsf{T}^k F\,.$$

- We have fundamental isomorphisms between iterated tangent and cotangent functors.

Theorem (Cantrijn-Crampin-Sarlet-Saunders-Tulczyjew)

For any manifold M and any $k \in \mathbb{N}$, there is a canonical isomorphism

$$\mathsf{T}^*\mathsf{T}^k M \simeq \mathsf{T}^k\mathsf{T}^* M\,.$$

- The corresponding graded bundle structure $\mathsf{T}^k\mathsf{T}^* M \to \mathsf{T}^* M$ and the vector bundle structure $\mathsf{T}^*\mathsf{T}^k M \to \mathsf{T}^k M$ are compatible in a natural sense, so that $\mathsf{T}^*\mathsf{T}^k M \simeq \mathsf{T}^k\mathsf{T}^* M$ is a canonical example of a double graded bundle, which will be discussed in the next talk.

References

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Homework

- Problem 1. Prove that any real vector space structure on $\mathbb{R}^n$, with zero at $0 \in \mathbb{R}^n$, coincides with the standard one.
- Problem 2. Prove directly that any smooth function $f : \mathbb{R}^n \to \mathbb{R}$, which satisfies $f(t \cdot x) = t^k \cdot f(x)$ for some $k \in \mathbb{N}$ and all $t \in \mathbb{R}$, is a polynomial.
- Problem 3. Show that a submanifold E_0 of a vector bundle E over M is a vector subbundle (possibly covering a submanifold $M_0 \subset M$) if and only if it E_0 is invariant with respect to all homotheties, i.e. $h_t(E_0) \subset E_0$ for all $t \in \mathbb{R}$.
- Problem 4. Find a split graded bundle isomorphic to the graded bundle $\mathrm{T}^2 M$.
- Problem 5. Let $\tau : E \to M$ be a vector bundle. What is the base of the vector bundle structure on $\mathrm{T}^* E$ being the 1-phase lift of the vector bundle (graded bundle of degree 1) structure on E?