Nonlinear Stability of Riemann Ellipsoids with Symmetric Configurations

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The Hydrodynamic Equation and Dirichlet's Problem

Motion of a self-gravitating ideal fluid (homogeneous and incompressible):

$$\rho \frac{du_i}{dt} = -\frac{\partial p}{\partial x_i} + \rho \frac{\partial \mathcal{B}}{\partial x_i},$$

ρ : density, $p(\vec{x})$: pressure, $\vec{u}(\vec{x})$: velocity field and $\mathcal{B}(\vec{x})$ gravitational potential due to instantaneous configuration $V \subset \mathbb{R}^3$,

$$\mathcal{B}(\vec{x}) = G \int_V \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3\vec{x}'.$$

Dirichlet's Problem: *When a homogeneous self-gravitating incompressible fluid mass can maintain at all time an ellipsoidal figure (which may be variable) in which the motion is, in an inertial frame, a linear function of the coordinates?.*

$\rightarrow$ ... explanation of the figure of the Earth and planets. (Newton, Lagrange, Lyapunov, Poincaré,...).

Riemann-Lebovitz Solution to Dirichlet's Problem

- Instantaneous configuration at time t:

$$V_t = Q(t)\left(B(\vec{0},1)\right),$$

where $Q(t) \in \mathrm{SL}(3)$, and $B(\vec{0},1) \subset \mathbb{R}^3$ unit ball.

- Singular Value Decomposition:

$$Q = L^T A S,$$

with $L, S \in \mathrm{SO}(3)$ and $A = \mathrm{diag}(a_1, a_2, a_3)$.

- Angular velocity and vorticity:

$$\Omega = \dot{L}L^T,\ \ \Lambda = \dot{S}S^T, \quad (\Omega,\, \Lambda \in \mathfrak{so}(3)).$$

$$Q \Leftrightarrow (A, L, S) \Leftrightarrow (A, \Omega, \Lambda)$$

Riemann-Lebovitz:

V_t is a solution for Dirichlet's problem iff $Q(t)$ satisfies Riemann's equations:

$$\begin{cases} (1) & \ddot{A} + \Omega(\Omega A - \dot{A} - A\Lambda) + (-\Omega A + \dot{A} + A\Lambda)\Lambda \\ & + \frac{d}{dt}(A\Lambda - \Omega A) - L\left(\frac{\partial V}{\partial Q}\right)_{Q=L^T AS} S^T \\ & = \lambda\, L\left(\frac{\partial(\det Q)}{\partial Q}\right)_{Q=L^T AS} S^T \\ (2) & \det A = 1 \end{cases}$$

where $V : \mathrm{GL}^+(3) \to \mathbb{R}^3$, is the self-gravitating potential:

$$V(Q) = -\beta \int_0^\infty \frac{ds}{\Delta}, \quad \beta = \frac{8}{15}\pi^2 G\rho^2,$$

and

$$\begin{aligned} \Delta &= \left[s^3 + I_1(Q)s^2 + I_2(Q)s + 1\right]^{\frac{1}{2}} \\ I_1(Q) &= \mathrm{tr}\,(QQ^T) \\ I_2(Q) &= \frac{1}{2}\left(\mathrm{tr}^2(QQ^T) - \mathrm{tr}(QQ^T)\right). \end{aligned}$$

Symmetries of Riemann's equations:

If $L_1, L_2 \in \mathrm{SO}(3)$, a solution $Q(t)$ is invariant under :

- $(A, \Omega, \Lambda) \longrightarrow (L_1 A L_2{}^T, L_1 \Omega L_1^T, L_2 \Lambda L_2{}^T)$,
- $(A, \Omega, \Lambda) \longrightarrow (A, \Lambda, \Omega)$ (Dedekind's Theorem).

Therefore, Riemann's equations (as well as V, I_1, I_2) are invariant under the action of the semidirect product group

$$\mathbb{Z}_2 \ltimes (\mathrm{SO}(3) \times \mathrm{SO}(3)) \, .$$

Riemann ellipsoids:

Solutions of Dirichlet's problem with: shape, angular velocity and vorticity constants.Then

$Q(t)$ Riemann ellipsoid $\Leftrightarrow$ constants (A, Ω, Λ) solution of Riemann's equations.

Examples:

- Spherical equilibrium: $(A, \Omega, \Lambda) = (\mathrm{Id}, 0, 0)$. This equilibrium is Lyapunov stable, as can be seen using the potential energy V as a Lyapunov function.
- Jacobi ellipsoid:
  $A = \mathrm{diag}(a_1, a_2, a_3)$, $\Omega = \omega\, \widehat{\mathbf{e_3}}$, $\Lambda = 0$.
- Dedekind ellipsoid:
  $A = \mathrm{diag}(a_1, a_2, a_3)$, $\Omega = 0$, $\Lambda = \omega\, \widehat{\mathbf{e_3}}$.

  Note that the Jacobi and Dedekind ellipsoids are interchanged by the $\mathbb{Z}_2$-symmetry.
- MacLaurin Spheroids:

$$(A, \Omega, \Lambda) = (\mathrm{diag}(a, a, c), \frac{|\omega|}{2}\widehat{\mathbf{e_3}}, \frac{|\omega|}{2}\widehat{\mathbf{e_3}}).$$

MacLaurin's condition (1742):

This family of solutions exist if $a > c$ and

$$\omega^2 = \frac{2\pi G\rho}{e^3}\left((1-e^2)^{\frac{1}{2}}(3-2e^2)\arcsin e - 3e(1-e^2)\right),$$

where $e = \left(1 - \left(\frac{c}{a}\right)^2\right)^{\frac{1}{2}}$ is the eccentricity.

Chandrasekhar:

Linearization of Riemann's equations shows that this solution is spectrally stable in the range $0 < e < 0.953887$.

Objective:

Study the geometry of Dirichlet's problem and use it to improve these results.

Symmetric Natural Systems with Holonomic Constraints

- $(M, \ll \cdot, \cdot \gg, V, G)$ symmetric natural system on M.

$$H_M(p_x) = \frac{1}{2}\|p_x\|^2 + V(x) \quad \in C^G(T^*M).$$

  Important: Assume G acts properly on M (for instance if G is compact).

- $f : M \to \mathbb{R}$ G-invariant constraint.
- $(N, \ll \cdot, \cdot \gg_N,\ V_N, G)$ induced system on $N = f^{-1}(1)$.
  Induced Hamiltonian

$$H_N(p_x) = \frac{1}{2}\|p_x\|_N^2 + V_N(x) \quad \in C^G(T^*N).$$

Dirichlet's Problem as a Constrained System

- For $A_1, A_2 \in T_F\mathrm{GL}^+(3) \simeq \mathrm{L}(3)$

$$\ll A_1, A_2 \gg := \alpha \, \mathrm{tr}\,(A_1^T A_2), \quad \alpha = \frac{4\pi}{15}\rho.$$

- $V(Q) = -\beta \int_0^\infty \frac{ds}{\Delta}$ (self-gravitating potential).

- Holonomic constraint: $f(Q) = \det(Q)$.

- $N = f^{-1}(1) = \mathrm{SL}(3) \subset \mathrm{GL}^+(3)$.

- Action of $G = \mathbb{Z}_2 \ltimes (\mathrm{SO}(3) \times \mathrm{SO}(3))$:

$$\begin{aligned} (1; (L_1, L_2)) \cdot Q &= L_1 Q L_2^T, \\ (\tau; (L_1, L_2)) \cdot Q &= L_2 Q^T L_1^T. \end{aligned}$$

- $\ll \cdot, \cdot \gg$, V and f are G-invariant.

$\Rightarrow$

Induced symmetric natural system in $\mathrm{SL}(3)$ which is equivalent to the constrained system:

$$\begin{aligned} \ddot{Q} - \mathrm{grad}\, V(Q) &= \lambda\, \mathrm{grad}\, \det Q && (1) \\ \det Q &= 1 && (2) \end{aligned}$$

Using $Q = L^T A S$, $\Omega = \dot{L} L^T$ and $\Lambda = \dot{S} S^T$, this set of equations is exactly the same as Riemann's equations.

$$\begin{cases} (1) \quad \ddot{A} + \Omega(\Omega A - \dot{A} - A\Lambda) + (-\Omega A + \dot{A} + A\Lambda)\Lambda \\ \qquad + \frac{d}{dt}(A\Lambda - \Omega A) - L \left(\frac{\partial V}{\partial Q}\right)_{Q = L^T A S} S^T \\ \qquad = \lambda\, L \left(\frac{\partial (\det Q)}{\partial Q}\right)_{Q = L^T A S} S^T \\ \\ (2) \quad \det A = 1 \end{cases}$$

Relative Equilibria

A relative equilibrium of a G-symmetric dynamical system on N is a point x with orbit

$$\gamma(t) = e^{t\xi} \cdot x, \ \gamma(0) = x,$$

where $\xi \in \mathfrak{g}$ is the velocity.
For a constrained symmetric natural system $(M, \ll \cdot, \cdot \gg, V, G, f)$, a point $x \in N = f^{-1}(1)$ is a R.E. with velocity $\xi \in \mathfrak{g}$ iff

$$\begin{aligned} \mathbf{d} V_{\lambda,\xi}(x) &= 0 \quad \text{and} \\ f(x) &= 1, \quad \text{where} \end{aligned}$$

- $V_{\lambda,\xi}(x) = V(x) - \lambda \operatorname{grad} f - \frac{1}{2}\mathbb{I}(x)(\xi, \xi)$
  augmented potential.
- $\mathbb{I}(x)(\xi, \eta) = \ll \xi_M(x), \eta_M(x) \gg \qquad \xi, \eta \in \mathfrak{g}$.
  locked inertia tensor
- Its momentum value is $\mu = \mathbb{I}(x)(\xi) \in \mathfrak{g}^*$.

Stability of Relative Equilibria.

Rodríguez-Olmos, M. "Stability of Relative Equilibria with Singular Momentum Values in Simple Mechanical Systems". Nonlinearity **19** *(2006) 853–877.*

Let $x \in N$ be a R.E. with velocity $\xi \in \mathfrak{g}$, momentum $\mu \in \mathfrak{g}^*$.

- Define $G_x = \{g \in G \,:\, g \cdot x = x\}$,

$$\begin{aligned} G_\mu &= \{g \in G \,:\, \mathrm{Ad}^*_g \mu = \mu\}, \text{(assume compact)} \\ G_p &= G_x \cap G_\mu. \end{aligned}$$

- Choose a G_p-invariant splitting

$$\mathfrak{g} = \mathfrak{g}_x \oplus \mathfrak{p} \oplus \mathfrak{t}$$

  satisfying $\mathfrak{g}_\mu = \mathfrak{g}_p \oplus \mathfrak{p}$ and $\mathbb{I}(x)(\mathfrak{p}, \mathfrak{t}) = 0$.
- Let $\xi^\perp$ be the projection of ξ onto $\mathfrak{p}$.
- $\hat{\mathbb{I}}_0 = \mathbb{I}(x)|_{\mathfrak{p}\oplus\mathfrak{t}}$ is non-degenerate and for $v_1, v_2 \in T_x M$.

$$\mathrm{corr}_{\,\xi}(v_1, v_2) := \frac{1}{2}\langle \mathbb{P}_{\mathfrak{p}^*\oplus\mathfrak{t}^*}\left[(\mathbf{D}\mathbb{I}\cdot v_1)(\xi)\right], \hat{\mathbb{I}}_0^{-1}\left(\mathbb{P}_{\mathfrak{p}^*\oplus\mathfrak{t}^*}\left[(\mathbf{D}\mathbb{I}\cdot v_2)(\xi)\right]\right)\rangle.$$

Define $\mathbf{S}, \mathfrak{q}^\mu$ and Σ_{int} by

- $\mathbf{S} = (\mathfrak{g} \cdot x)^{\perp_N}, \quad T_x N = \mathfrak{g} \cdot x \oplus \mathbf{S}$,
- $\mathfrak{q}^\mu = \left\{ \lambda \in \mathfrak{t} \,:\, \mathbb{P}_{\mathfrak{g}_x^*} \left[\mathrm{ad}^*_\lambda \mu \right] = 0 \right\}$
- $\Sigma_{\text{int}} = \left\{ \lambda_N(x) + a \,:\, \lambda \in \mathfrak{q}^\mu, a \in \mathbf{S}, \left(\mathbf{D}\mathbb{I} \cdot (\lambda_N(x) + a) \right) (\xi^\perp) \in \mathfrak{p}^* \right\}.$
- Define the Arnold form, $\mathrm{Ar} : \mathfrak{q}^\mu \times \mathfrak{q}^\mu \to \mathbb{R}$ as

$$\mathrm{Ar}(\lambda_1, \lambda_2) = \langle \mathrm{ad}^*_{\lambda_1} \mu, \hat{\mathbb{I}}_0^{-1} (\mathrm{ad}_{\lambda_2} \mu) + \mathbb{P}_{\mathfrak{p}^* \oplus \mathfrak{t}^*} \left[\mathrm{ad}_\lambda \left(\hat{\mathbb{I}}_0^{-1} \mu \right) \right] \rangle$$

Theorem. If

$$\mathrm{Ar} > 0 \quad \text{and} \quad \left(\mathbf{d}_x^2 V_{\lambda, \xi^\perp} + \mathrm{corr}_{\,\xi^\perp}(x) \right) \Big|_{\Sigma_{\text{int}}} > 0$$

then x is nonlinearly stable.

Nonlinear stability $\Rightarrow$ ($\not\Leftarrow$) Spectral stability.

Riemann Ellipsoids as Relative Equilibria

Let $A(t) = \mathrm{diag}(a_1, a_2, a_3)$, $\Omega, \Lambda \in \mathfrak{so}(3)$.

Theorem: *A is a relative equilibrium with velocity $(\Omega, \Lambda) \in \mathfrak{g}$ for the geometric formulation of Dirichlet's problem iff*

$$Q(t) = \left(e^{\Omega t}\right)^T A e^{\Lambda t}$$

is a Riemann ellipsoid. (i.e. A, Ω, Λ are constants and solutions of Riemann's equations).

If $A = \mathrm{diag}(a, a, c)$, $a > c$, $\Omega = \Lambda = \frac{\omega}{2}\widehat{e_3}$ the relative equilibrium conditions

$$\mathbf{d}_A V_{\lambda,(\Omega,\Lambda)} = 0, \ \det A = 1$$

are equivalent to MacLaurin's formula

$$\omega^2 = \frac{2\pi G\rho}{e^3}\left((1-e^2)^{\frac{1}{2}}(3-2e^2)\arcsin e - 3e(1-e^2)\right),$$

Nonlinear Stability of the MacLaurin Spheroid

- $A = \mathrm{diag}(a, a, c)$, $\Omega = \Lambda = \frac{\omega}{2}\mathbf{e_3}$.
- $\mathbb{I}(A) = \alpha \begin{pmatrix} \mathrm{tr}\,(A^2)I - A^2 & -2\mathrm{det}\,(A)A^{-1} \\ -2\mathrm{det}\,(A)A^{-1} & \mathrm{tr}\,(A^2)I - A^2 \end{pmatrix}$.
- $\mu = \mathbb{I}(A)(\frac{\omega}{2}(\mathbf{e_3}, \mathbf{e_3})) = \frac{8\pi\rho\omega}{15(1-e^2)^{\frac{1}{6}}}(\mathbf{e_3}, -\mathbf{e_3})$.
- $G_x = \mathbb{Z}_2 \times \mathrm{SO}(2)^D_{\mathbf{e_3}}$,
- $G_\mu = \mathrm{SO}(2)_{\mathbf{e_3}} \times \mathrm{SO}(2)_{\mathbf{e_3}}$,
- Then $G_p = G_x \cap G_\mu = \mathrm{SO}(2)^D_{\mathbf{e_3}}$, $\quad \mathfrak{g}_p \simeq \mathbb{R}\langle \mathbf{e_3} \rangle$.
- Splitting of $\mathfrak{g} \simeq \mathbb{R}^3 \times \mathbb{R}^3$:

$$\begin{aligned} \mathfrak{g}_x &= \{\frac{1}{\sqrt{2}}(\mathbf{e_3}, \mathbf{e_3})\}, \\ \mathfrak{p} &= \{\frac{1}{\sqrt{2}}(\mathbf{e_3}, -\mathbf{e_3})\}, \\ \mathfrak{t} &= \{(\mathbf{e_1}, 0), (0, \mathbf{e_1}), (\mathbf{e_2}, 0), (0, \mathbf{e_2})\}. \end{aligned}$$

- $$\hat{\mathbb{I}}_0 = \begin{pmatrix} i_0 & 0 & 0 & 0 & 0 \\ 0 & i_1 & i_2 & 0 & 0 \\ 0 & i_2 & i_1 & 0 & 0 \\ 0 & 0 & 0 & i_1 & i_2 \\ 0 & 0 & 0 & i_2 & i_1 \end{pmatrix},$$

  $$i_0 = \frac{4\alpha}{(1-e^2)^{\frac{1}{3}}},\; i_1 = \frac{(2-e^2)\alpha}{(1-e^2)^{\frac{1}{3}}},\; i_2 = -2(1-e^2)^{\frac{1}{6}}\alpha.$$

- $$\mathrm{Ar} = \begin{pmatrix} A_1 & -A_2 & 0 & 0 \\ -A_2 & A_1 & 0 & 0 \\ 0 & 0 & A_1 & -A_2 \\ 0 & 0 & -A_2 & A_1 \end{pmatrix},$$

  $$A_1 = \frac{4(8-4e^2+e^4)\alpha\omega^2}{e^4(1-e^2)^{\frac{1}{3}}}\;,\; A_2 = \frac{32(1-e^2)^{\frac{1}{6}}\alpha\omega^2}{e^4}.$$

Ar is positive-definite.

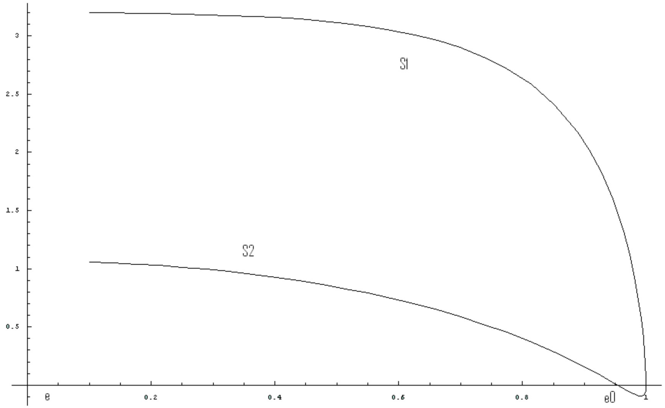
- Test space:

$$\Sigma_{\text{int}} = \left\{ \begin{pmatrix} s_1 + s_2 & s_3 & 0 \\ s_3 & s_1 - s_3 & 0 \\ 0 & 0 & -2\sqrt{1-e^2}s_1 \end{pmatrix} : s_1, s_2, s_3 \in \mathbb{R} \right\}.$$

Stability test:

$$\left(\mathbf{d}_x^2 V_{\lambda,(\Omega,\Lambda)^\perp} + \mathrm{corr}_{(\Omega,\Lambda)^\perp}(x)\right)\Big|_{\Sigma_{\text{int}}} = \mathrm{diag}\,(S_1, S_2, S_2),$$

$$\begin{aligned} S_1 &= \frac{16\pi^2 G\rho^2(9e(3-5e^2+2e^4)-\sqrt{1-e^2}(27-36e^2+8e^4)\arcsin e)}{15e^5} \\ S_2 &= \frac{8\pi^2 G\rho^2((1-e^2)e(3+4e^2)-\sqrt{1-e^2}(3+2e^2-4e^4)\arcsin e)}{15e^5}. \end{aligned}$$



with $e_0 = 0.953887$.

It follows that in the range $e \in (0, 0.953887)$, MacLaurin spheroids are nonlinearly stable.