Symmetric Hamiltonian Systems. Stability Methods and Applications. Part II.

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Outline

1. The different notions of stability
2. Hamiltonian stability and intrinsic difficulties
3. Perturbation theory and energy based stability methods
4. Symmetric Hamiltonian systems: momentum maps and reduction theory
5. Relative dynamical elements and their stability: the energy-momentum method
6. Bifurcation and persistence of relative dynamical elements
7. Stability in stochastic Hamiltonian dynamical systems

Relative equilibria

M is a G-manifold and $X \in \mathfrak{X}(M)^G$ a G-equivariant vector field with G-equivariant flow F_t.

- m is a **relative equilibrium (RE)** when there exists a **velocity** $\xi \in \mathfrak{g}$ such that :

$$X(m) = \xi_M(m) \quad \text{or equivalently} \quad F_t(m) = \exp t\xi \cdot m.$$

- m is a **relative periodic orbit (RPO)** when there exists an element $g \in G$ (**phase shift**) and a positive constant $\tau > 0$ (**relative period**) such that

$$F_{t+\tau}(m) = g \cdot F_t(m).$$

If the action is free and proper and X can be projected to a vector field $X^G \in \mathfrak{X}(M/G)$. REs and RPOs of X amount to equilibria and periodic orbits of X^G, respectively.

Drifts and neutral directions

- REs come in orbits.
- If m is a RE with velocity $\xi \in \mathfrak{g}$ then so is $g \cdot m$ with velocity $\mathrm{Ad}_g \xi$:

$$F_t(g \cdot m) = g \cdot F_t(m) = g \exp t\xi \cdot m = g \exp t\xi g^{-1} g \cdot m = \exp t \left(\mathrm{Ad}_g \xi\right) \cdot m$$

The Hamiltonian case

(M, ω) a symplectic manifold and G a Lie group acting properly on M in a globally Hamiltonian fashion with associated equivariant momentum map $\mathbf{J} : M \to \mathfrak{g}^*$.

- $\mathbf{J}(m) = \mu \in \mathfrak{g}^*$.
- G_μ the isotropy of μ under the coadjoint action of G.
- $H := G_m$.

Let $h \in C^\infty(M)^G$ be a G–invariant Hamiltonian. A point $m \in M$ is a **relative equilibrium** (respectively **relative periodic orbit (RPO)**) of h with respect to the G–symmetry of M, if the point $[m]_\mu^{(H)} = \pi_\mu^{(H)}(m)$ is an equilibrium (respectively periodic point) of the Hamiltonian dynamical system $(M_\mu^{(H)}, \omega_\mu^{(H)}, h_\mu^{(H)})$.

The following statements are equivalent:

(i) $m \in \mathbf{J}^{-1}(\mu) \cap M_{(H)}^{G_\mu}$ with $H := G_m$ is a relative equilibrium.

(ii) There is a unique $\lambda \in \mathrm{Lie}(N_{G_\mu}(H)/H) \subset \mathfrak{l}$ such that

$$F_t(m) = \exp_L t\lambda \cdot m \qquad \text{for all} \qquad t \in \mathbb{R},$$

with $\exp_L : \mathfrak{l} \to L$ the exponential map associated to $L := N(H)/H$. $\lambda \in \mathfrak{l}$ is called the **canonical velocity** of m.

(iii) There is a $\xi \in \mathrm{Lie}(N_{G_\mu}(H))$ such that

$$F_t(m) = \exp t\xi \cdot m \qquad \text{for all} \qquad t \in \mathbb{R}.$$

$\xi \in \mathrm{Lie}(N_{G_\mu}(H))$ is called a **velocity** of m. The set of all possible velocities coincides with the set of representatives of the canonical velocity in the Lie algebra of $N_{G_\mu}(H)$, that is, $\xi \in \mathrm{Lie}(N_{G_\mu}(H))$ is a velocity if and only if $[\xi] = \lambda$.

(iv) There is a $\xi \in \mathrm{Lie}(N_{G_\mu}(H))$ such that $X_h(m) = \xi_M(m)$.

(v) There is a $\xi \in \mathrm{Lie}(N_{G_\mu}(H))$ such that the **augmented Hamiltonian** $L^\xi := h - \mathbf{J}^\xi$ satisfies

$$\mathbf{d}\, L^\xi(m) = 0.$$

The RPO case

The conservation of isotropy and momentum implies that the phase shift of a Hamiltonian RPO satisfies:

$$g \in N_{G_\mu}(H) := N(H) \cap G_\mu.$$

Indeed:

- $F_{t+\tau}(m) = g \cdot F_t(m)$ implies that $\mathbf{J}(F_{t+\tau}(m)) = \mathbf{J}(g \cdot F_t(m)) = g \cdot \mathbf{J}(F_t(m))$ or equivalently $\mu = g \cdot \mu$.
- By equivariance $H := G_m = G_{F_t(m)}$ and hence

$$H := G_m = G_{F_\tau(m)} = G_{g \cdot m} = g G_m g^{-1} = g H g^{-1}.$$

Stability modulo a subgroup

Definition

Let $X \in \mathfrak{X}(M)$ be a G–equivariant vector field on the G–manifold M and let G' be a subgroup of G.

- A relative equilibrium $m \in M$ of X, is called **G'–stable**, or **stable modulo** G', if for any G'–invariant open neighborhood V of the orbit $G' \cdot m$, there is an open neighborhood $U \subset V$ of m, such that if F_t is the flow of the vector field X and $u \in U$, then $F_t(u) \in V$ for all $t \geq 0$.
- The RPO m is **G'-stable**, or **stable modulo** G', if for any G'–invariant open neighborhood V of the set $G' \cdot \{F_t(m)\}_{t>0}$, there is an open neighborhood $U \subseteq V$ of m such that $F_t(U) \subset V$, for any $t > 0$.

Orthogonal velocities

$m \in M$ relative equilibrium and unique $\lambda \in \mathrm{Lie}(N_{G_\mu}(H)/H)$ such that

$$F_t(m) = \exp_L t\lambda \cdot m$$

where the dot denotes the free action of $N_{G_\mu}(H)/H$ on M_H. The properness of the G–action allows us to choose an Ad_H–invariant inner product in $\mathfrak{n}_\mu := \mathrm{Lie}(N_{G_\mu}(H))$ and we have an orthogonal direct sum decomposition

$$\mathfrak{n}_\mu = \mathfrak{h} \oplus \mathfrak{p}_\mu.$$

From here it follows that

$$\mathrm{Lie}(N_{G_\mu}(H)/H) \simeq \mathfrak{n}_\mu/\mathfrak{h} \simeq \mathfrak{p}_\mu.$$

Let $\xi \in \mathfrak{p}_\mu \subset \mathfrak{n}_\mu$ be the unique image of $\lambda \in \mathrm{Lie}(N_{G_\mu}(H))$ under this isomorphism. We have

$$F_t(m) = \exp_L t\lambda \cdot m = \exp t\xi \cdot m.$$

Definition

The unique element $\xi \in \mathfrak{p}_\mu$ just defined is called the **orthogonal velocity** of the relative equilibrium $m \in M$, relative to the splitting $\mathfrak{n}_\mu = \mathfrak{h} \oplus \mathfrak{p}_\mu$.

Important: the orthogonal velocity depends on the splitting and is unique only if this splitting is specified. In applications, probing the stability of the system with all its possible orthogonal velocities, that is, considering all possible splittings, is the way to obtain optimal stability conditions.

The energy-momentum method

Theorem

Let $(M, \{\cdot,\cdot\}, h)$ *be a Poisson system. Lie group* G *acting properly on* M *with equivariant momentum map* $\mathbf{J} : M \to \mathfrak{g}^*$*. The Hamiltonian* $h \in C^\infty(M)$ *is* G*–invariant.* $m \in M$ *relative equilibrium such that* $\mathbf{J}(m) = \mu \in \mathfrak{g}^*$*,* $\mathfrak{g}^*$ *admits an* $\mathrm{Ad}^*_{G_\mu}$*–invariant inner product,* $H := G_m$*, and* $\xi \in \mathrm{Lie}(N_{G_\mu}(H))$ *is its orthogonal velocity, relative to a given* Ad_H*–invariant splitting. If the quadratic form*

$$\mathbf{d}^2(h - \mathbf{J}^\xi)(m)|_{W \times W}$$

is definite for some (and hence for any) subspace W *such that*

$$\ker T_m\mathbf{J} = W \oplus T_m(G_\mu \cdot m),$$

then m *is a* G_μ*–stable relative equilibrium.* $\mathbf{d}^2(h - \mathbf{J}^\xi)(m)|_{W \times W}$ *will be called the* **stability form** *of the relative equilibrium* m*.*

Proof of the energy-momentum method

Let S be a slice at the point m associated to the Hamiltonian action of G_μ on M. Let now $T := G_\mu \cdot S$, be a tube around the orbit $G_\mu \cdot m$. By definition,

$$T_m M = T_m S \oplus T_m(G_\mu \cdot m). \tag{1}$$

Let now be

$$Z := T_m S \cap \ker T_m \mathbf{J}. \tag{2}$$

Since $T_m(G_\mu \cdot m) \subset \ker T_m \mathbf{J}$, we have that

$$\ker T_m \mathbf{J} = Z \oplus T_m(G_\mu \cdot m);$$

and hence Z satisfies the requirements of W in the statement of the theorem. Importance of the orthogonal velocity:

Lemma

Fix a splitting and let $\xi \in \mathfrak{p}_\mu$ be the corresponding orthogonal velocity of the relative equilibrium $m \in M$ whose symmetry group is $H := G_m$. Then $\mathrm{Ad}_h \xi = \xi$ for any $h \in H$.

We now introduce a singular **Patrick velocity map**. Let r be the G_μ–equivariant retraction associated to the slice S

$$\begin{array}{rccc} r: & G_\mu \cdot S & \longrightarrow & G_\mu \cdot m \\ & g \cdot z & \longmapsto & g \cdot m. \end{array}$$

We define

$$\begin{array}{rccc} \tilde{\Psi}: & G_\mu \cdot m & \longrightarrow & G_\mu \cdot \xi \\ & g \cdot m & \longmapsto & \mathrm{Ad}_g \xi \end{array}$$

with ξ the orthogonal velocity of the relative equilibrium. The previous lemma guarantees that $\tilde{\Psi}$ is well-defined: if $g \cdot m = g' \cdot m$ then $g^{-1} g' \in H$ and therefore $g^{-1} g' \cdot \xi = \xi$ and so $g' \cdot \xi = g \cdot \xi$.

Patrick velocity map: $\Psi := \tilde{\Psi} \circ r : g \cdot z \in G_\mu \cdot S \mapsto \mathrm{Ad}_g \xi \in G_\mu \cdot \xi$. Note that $\Psi(m) = \tilde{\Psi}(m) = \xi$ and that for any $g \in G_\mu$ and any $z = g' \cdot z' \in G_\mu \cdot S$,

$$\Psi(g \cdot z) = \Psi(g g' \cdot z') = \mathrm{Ad}_{gg'} \xi = \mathrm{Ad}_g(\mathrm{Ad}_{g'} \xi) = \mathrm{Ad}_g \Psi(g' \cdot z') = \mathrm{Ad}_g \Psi(z).$$

Also, $\mathrm{Im}\, \Psi = G_\mu \cdot \xi$ and $\langle \mu, \Psi(z) \rangle = \langle \mu, \xi \rangle$, for any $z \in G_\mu \cdot S$.

Let f_1 and f_2 be the functions defined by

$$f_1 = (h - h(m)) + (\langle \mathbf{J}, \Psi \rangle - \langle \mu, \xi \rangle),$$
$$f_2 = \|\mathbf{J} - \mu\|^2,$$

where in f_2, the modulus is taken using the norm associated to some $\mathrm{Ad}^*_{G_\mu}$–invariant inner product in $\mathfrak{g}^*$ (always available by hypothesis).

- f_2 is a G_μ–invariant conserved quantity.
- f_1 is G_μ–invariant but in general not conserved.
- $h - \mathbf{J}^\xi$ and $f_1|_S$ differ on S by a constant, which implies that $\mathbf{d}(f_1|_S)(m) = 0$ and $\mathbf{d}^2(f_1|_S)(m)$ is well–defined. Moreover,

$$\mathbf{d}^2(f_1|_S)(m)|_{Z \times Z} = \mathbf{d}^2(h - \mathbf{J}^\xi)(m)|_{Z \times Z}.$$

- Since Z satisfies the requirements of W, $\mathbf{d}^2(f_1|_S)(m)|_{Z \times Z}$ is definite.
- Z is the kernel of $\mathbf{d}^2(f_2|_S)(m)$.
- Patrick's lemma guarantees the existence of a positive constant $a > 0$ for which $f := af_1 + f_2$ and such that $\mathbf{d}^2(f|_S)(m)$ is positive definite and $f \geq 0$ in a given neighborhood of the point m.

Note that f is G_μ–invariant but, in general, it is not a constant of the motion since $\langle \mathbf{J}, \Psi \rangle$ is not conserved. It can be shown:

$$\frac{1}{a}(f(F_t(z)) - f(z)) = \langle \mathbf{J}(z) - \mu, \Psi(F_t(z)) - \xi \rangle ,$$

where we used Noether's Theorem, $\Psi(z) = \xi$ because $z \in S$, and $\langle \mu, \Psi(z) \rangle = \langle \mu, \xi \rangle$, for any $z \in G_\mu \cdot S$. Hence, for any $z \in S$ such that $F_t(z) \in G_\mu \cdot S$,

$$\begin{aligned} 0 \leq f(F_t(z)) &\leq f(z) + a \left| \langle \mathbf{J}(z) - \mu, \Psi(F_t(z)) - \xi \rangle \right| \\ &\leq f(z) + a \| \mathbf{J}(z) - \mu \| (\| \Psi(F_t(z)) \| + \| \xi \|) \\ &= f(z) + 2a \| \xi \| \, \| \mathbf{J}(z) - \mu \|, \end{aligned} \tag{3}$$

where we used that $\mathrm{Im}\Psi = G_\mu \cdot \xi$, and the G_μ–invariance of the norm $\| \cdot \|$. These tools suffice to prove the G_μ–stability of m by thinking of f as a distance function to the relative equilibrium that we are studying.

Remarks and improvements

- The hypothesis on the existence of a $\mathrm{Ad}^*_{G_\mu}$–invariant inner product on $\mathfrak{g}^*$ cannot be dropped. See $SL(2,\mathbb{R})$ example in [OR99] paper.
- Montaldi and Rodríguez-Olmos [2011] drop orthogonal velocities when G_μ is compact. It is relevant in the absence of non-trivial orthogonal velocities.
- Poster by Miquel Teixidó-Román.

T_2-energy-momentum method

- G proper and free action on (M, ω). Momentum map $\mathbf{J} : M \to \mathfrak{g}^*$ is a submersion. $h \in C^\infty(M)^G$.
- m relative equilibrium such that $\mathbf{J}(m) = \mu$ and S slice at m.
- If there exists an open neighborhood U_S of m in S and an open neighborhood U_μ of μ in $\mathfrak{g}^*$ such that

$$h^{-1}(h(m)) \cap \mathbf{J}^{-1}(T_2^{U_\mu}(\mu)) \cap U_S = \{m\},$$

  then m is G-stable.
- Sketch of the proof: due to the freeness of the action $M \simeq G \times S$ locally and hence $M/G \simeq S$. Hence the G-stability of m amounts to the stability of m as an equilibrium of $(S, \{\cdot, \cdot\}_S, h|_S)$. The statement follows from the T_2-energy-Casimir theorem by noticing that J induces a homeomorphism between the leaf space of U_S an that of U_μ. In particular:

$$T_2^{U_S}(m) = \mathbf{J}^{-1}\left(T_2^{U_\mu}(\mu)\right) \cap U_S.$$

- In this case the symplectic leaf space of M/G at $G \cdot m$ is Hausdorff iff $\mathfrak{g}^*/G$ is Hausdorff at $G \cdot \mu$. If this holds then the stability condition reduces to
$$h^{-1}(h(m)) \cap \mathbf{J}^{-1}(\mu) \cap U_S = \{m\},$$
which in general only ensures leafwise stability.
- In the case of the existence of G_μ-invariant inner products in $\mathfrak{g}^*$ the G-invariance can be improved to G_μ-invariance.

The heavy top

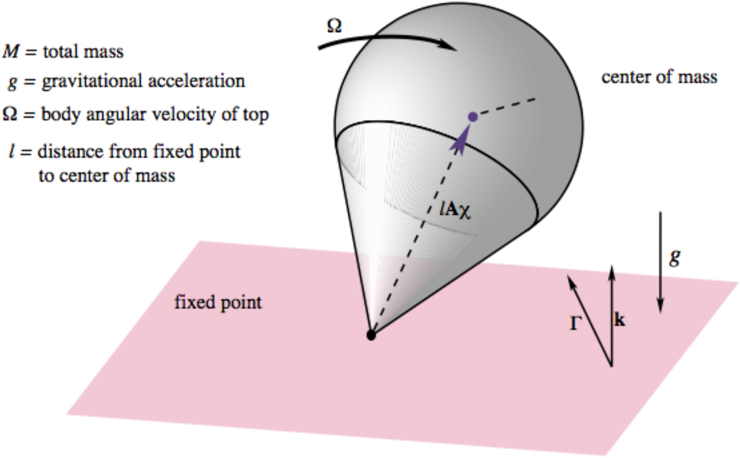


Figure : Taken from *Introduction to Mechanics and Symmetry*

Sleeping and hanging tops

- Elements of $TSO(3)$ in spatial representation (that is, right trivialization) will be expressed as $(\Lambda, \widehat{\boldsymbol{\delta\theta}}\Lambda)$ where $\Lambda \in SO(3)$ and $\widehat{\boldsymbol{\delta\theta}}$ is the skew–symmetric matrix associated to $\boldsymbol{\delta\theta} \in \mathbb{R}^3$ via the relation $\widehat{\boldsymbol{\delta\theta}}\,\mathbf{x} = \boldsymbol{\delta\theta} \times \mathbf{x}$.
- Analogously, the elements of $T^*SO(3)$ have the form $(\Lambda, \widehat{\boldsymbol{\pi}}\Lambda)$ with $\boldsymbol{\pi} \in \mathbb{R}^3$.
- $\mathbf{g} = g\mathbf{e}_3$ denotes the gravity vector, where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a spatial orthonormal basis of $\mathbb{R}^3$.
- The mass vector by $\mathbf{M} := \int_{\mathcal{B}} \rho_{ref}(X) X d^3X$, where $\mathcal{B}$ is a reference configuration.
- If m is the total mass of the body and l is the distance from the fixed point to the center of mass, then $|\ \mathbf{M}\ | = ml$.
- The *reference inertia tensor* $\mathbb{I}_{ref}$ is defined as

$$\mathbb{I}_{ref} := \int_{\mathcal{B}} \rho_{ref}(X)(|\ X\ |^2\ \mathbb{I}_3 - X \otimes X) d^3X$$

- The *current spatial inertia tensor* is given by

$$\mathbb{I}_\Lambda := \Lambda \mathbb{I}_{ref} \Lambda^T.$$

- If $\mathbf{m} := \Lambda \mathbf{M}$ is the spatial representation of the mass vector, in these variables, the Hamiltonian of the heavy top is given by

$$h(\Lambda, \boldsymbol{\pi}) := \mathbf{m} \cdot \mathbf{g} + \frac{1}{2} \boldsymbol{\pi} \cdot \mathbb{I}_\Lambda^{-1} \boldsymbol{\pi}.$$

- Choose, without loss of generality, $\mathbb{I}_{ref} = \operatorname{diag}[I_1, I_1, I_3]$ for some constants I_1 and I_3 and $\mathbf{M} = ml\mathbf{e}_3$.
- Symmetries: $G = S^1 \times S^1$. Using spatial variables, the G–action on the phase space

$$\begin{array}{ccc} G \times T^*SO(3) & \longrightarrow & T^*SO(3) \\ ((\theta_1, \theta_2), (\Lambda, \boldsymbol{\pi})) & \longmapsto & \left(\exp(\theta_1 \widehat{\mathbf{e}_3}) \Lambda \exp(-\theta_2 \widehat{\mathbf{e}_3}),\ \exp(-\theta_1 \widehat{\mathbf{e}_3}) \boldsymbol{\pi}\right). \end{array}$$

- Infinitesimal generators:

$$(\xi, \omega)_Q(\Lambda) = (\Lambda, \xi \mathbf{e}_3 - \omega \Lambda \mathbf{e}_3).$$

- Momentum map: $\mathbf{J} : T^*SO(3) \to \mathfrak{g}^* = \mathbb{R}^2$:

$$\langle \mathbf{J}(\Lambda, \boldsymbol{\pi}), (\xi, \omega) \rangle = \langle (\Lambda, \boldsymbol{\pi}), (\Lambda, \xi \mathbf{e}_3 - \omega \Lambda \mathbf{e}_3) \rangle = \xi \boldsymbol{\pi} \cdot \mathbf{e}_3 - \omega \boldsymbol{\pi} \cdot \Lambda \mathbf{e}_3,$$

hence

$$\mathbf{J}(\Lambda, \boldsymbol{\pi}) = (\boldsymbol{\pi} \cdot \mathbf{e}_3, \; -\boldsymbol{\pi} \cdot \Lambda \mathbf{e}_3).$$

- We show how any sleeping top is a relative equilibrium, in other words, for every point in $T^*SO(3)$ of the form $z = (I, \lambda I_3 \mathbf{e}_3)$ there is an element $(\alpha_1, \alpha_2) \in \mathfrak{g} = \mathbb{R}^2$ for which

$$\mathbf{d}(h - \mathbf{J}^{(\alpha_1, \alpha_2)})(z) = 0.$$

- The derivative of the augmented Hamiltonian equals to

$$\mathbf{d}(h - \mathbf{J}^{(\alpha_1, \alpha_2)})(z)(\delta \Lambda, \boldsymbol{\delta\pi}) = ((\xi - \omega) - \lambda) \boldsymbol{\delta\pi} \cdot \mathbf{e}_3.$$

where $\delta\Lambda := \widehat{\boldsymbol{\delta\theta}}\Lambda$. Therefore, in order to prove that z is a relative equilibrium we just need to take $\lambda = (\alpha_1 - \alpha_2)$.

- z has non–trivial symmetry. Indeed if $(\theta_1, \theta_2) \in G$ is such that $(\theta_1, \theta_2) \cdot z = z$, that is, $(\exp((\theta_1, \theta_2)\widehat{\mathbf{e}_3}), (\xi - \omega)I_3\mathbf{e}_3) = (I, (\xi - \omega)I_3\mathbf{e}_3)$, then $\theta_1 = \theta_2$ and thus

$$H = \{(\theta_1, \theta_2) \in G \mid \theta_1 = \theta_2\}.$$

- It is easy to check that

$$(T^*SO(3))_H = \{(\exp(\psi\widehat{\mathbf{e}_3}), \pi\mathbf{e}_3) \mid \psi \in Lie(S^1) = \mathbb{R}, \pi \in \mathbb{R}\}.$$

- Additionally,

$$\ker T_z\mathbf{J} = \{(\delta\Lambda, \boldsymbol{\delta\pi}) \in T_z(T^*SO(3)) \mid \boldsymbol{\delta\pi} \cdot \mathbf{e}_3 = 0\}.$$

One computes similarly

$$T_z(G_\mu \cdot z) = T_z(G \cdot z) = \mathrm{span}\left\{(\widehat{\mathbf{e}_3}, 0)\right\}.$$

$$\begin{aligned} W &= \ker T_z\mathbf{J} \cap T_z(G \cdot z)^\perp \\ &= \{(\boldsymbol{\delta\Lambda}, \boldsymbol{\delta\pi}) \in T_z(T^*SO(3)) \mid \boldsymbol{\delta\Lambda} \cdot \mathbf{e}_3 = \boldsymbol{\delta\pi} \cdot \mathbf{e}_3 = 0\}. \end{aligned}$$

- We write the second summand of the augmented Hamiltonian using an orthogonal velocity, that is, the projection of (ξ, ω) on the orthogonal complement of $\mathfrak{h} = \mathrm{Lie}(\mathrm{H}) = \mathrm{span}\,\{(1,1)\}$ with respect to an Ad_G–invariant metric on $\mathfrak{g}$. Since G is Abelian, any metric is Ad_G–invariant, hence the most general situation consists of taking the inner product in $\mathfrak{g}$ given by the quadratic form

$$g = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

subject to the condition $\det g = ac - b^2 > 0$, which ensures the positive definiteness of g.

- The orthogonal complement $\mathfrak{p}_\mu$ of $\mathfrak{h}$ with respect to g is

$$\mathfrak{p}_\mu = \mathrm{span}\,\{(1, -k)\}$$

where $k = (a+b)/(b+c)$. This implies that the orthogonal velocity v_c of z with respect to the splitting determined by g is

$$v_c(k) = \lambda \left(\frac{1}{1+k}, \frac{-k}{1+k} \right).$$

- The matrix of the Hessian $\mathbf{d}^2(h - \mathbf{J}^{v_c(k)})(z)$ restricted to W is

$$\begin{pmatrix} -mgl - \lambda^2 I_3\left(\frac{1}{1+k} - \frac{I_3}{I_1}\right) & 0 & 0 & \lambda\left(\frac{I_3-I_1}{I_1} + \frac{k}{1+k}\right) \\ 0 & -mgl - \lambda^2 I_3\left(\frac{1}{1+k} - \frac{I_3}{I_1}\right) & -\lambda\left(\frac{I_3-I_1}{I_1} + \frac{k}{1+k}\right) & 0 \\ 0 & -\lambda\left(\frac{I_3-I_1}{I_1} + \frac{k}{1+k}\right) & \frac{1}{I_1} & 0 \\ \lambda\left(\frac{I_3-I_1}{I_1} + \frac{k}{1+k}\right) & 0 & 0 & \frac{1}{I_1} \end{pmatrix},$$

whose eigenvalues are

$$\sigma_\pm = A \pm \sqrt{-4I_1(1+k)^2 B + A^2},$$

with

$$A = (1+k)^2 - mglI_1(1+k)^2 + I_3\lambda^2(I_3(1+2k) - I_1(1+k))$$
$$B = \lambda^2(I_3 k + I_3 - I_1) - mgl(1+k)^2.$$

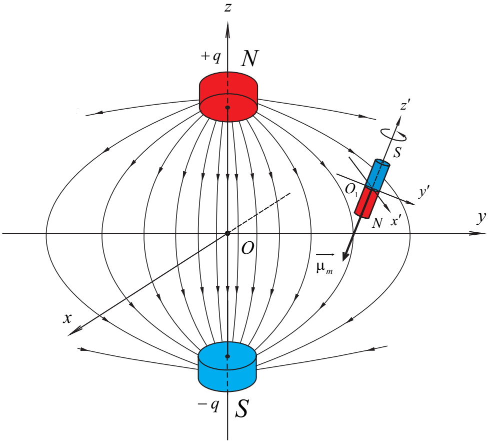
It is clear that $\mathbf{d}^2(h - \mathbf{J}^{v_c(k)})(z)$ is positive definite iff $B > 0$, that is

$$\lambda^2 > mgl\frac{(1+k)^2}{I_3 k + I_3 - I_1}.$$

- For each k (for each orthogonal velocity) we have a lower bound for the values of λ for which the sleeping top is stable. The optimal stability condition will be achieved when
$$\frac{(1+k)^2}{I_3 k + I_3 - I_1}$$
reaches a minimum.
- Taking the first and second derivatives of this function, one checks that this happens when
$$k = \frac{2I_1 - I_3}{I_3}$$
and therefore, the optimal stability condition is
$$\lambda^2 > \frac{4mglI_1^2}{I_3}.$$

The orbitron (Grigoryeva, JPO, Zub (2014))

Consider a small axisymmetric magnetized rigid body (permanent magnet or a current-carrying loop) with magnetic moment μ, in the permanent magnetic field created by two fixed magnetic poles/ "charges" placed at distance h in the absence of gravity.



Phase space

- The configuration space of the orbitron is $SE(3) = SO(3) \times \mathbb{R}^3$
- The orbitron is a simple mechanical system
- Phase space is the cotangent bundle $T^*SE(3)$ of its configuration space $SE(3)$ endowed with the canonical symplectic structure ω obtained as minus the differential of the corresponding Liouville one form
- Left/right trivializations provide an identification of the bundle $T^*SE(3)$ with the product $SE(3) \times \mathfrak{se}(3)^*$. We work in body coordinates and denote by $(A, \mathbf{x})$ the elements of $SE(3) = SO(3) \times \mathbb{R}^3$ and by $((A, \mathbf{x}), (\mathbf{\Pi}, \mathbf{p}))$ those of $T^*SE(3) \simeq SE(3) \times \mathfrak{se}(3)^*$.

The Hamiltonian of the orbitron is

$$h((A,\mathbf{x}),(\mathbf{\Pi},\mathbf{p})) = T(\mathbf{\Pi},\mathbf{p}) + V(A,\mathbf{x}) \tag{4}$$

with

$$T(\mathbf{\Pi},\mathbf{p}) := \frac{1}{2}\mathbf{\Pi}^T \mathbb{I}_{ref}^{-1}\mathbf{\Pi} + \frac{1}{2M}\|\mathbf{p}\|^2, \tag{5}$$

$$V(A,\mathbf{x}) := -\mu\langle \mathbf{B}(\mathbf{x}), A\mathbf{e}_3\rangle, \tag{6}$$

where M is the mass of the axisymmetric magnetic body, the reference inertia tensor $\mathbb{I}_{ref} = \mathrm{diag}(I_1, I_1, I_3)$, $\mathbf{x} = (x,y,z) \in \mathbb{R}^3$, μ is the magnetic moment of the axisymmetric rigid body/dipole, and $\mathbf{B}(\mathbf{x})$ is the strength of the magnetic field created by two magnetic poles/"charges" $\pm q$ placed at the points $(0,0,h)$ and $(0,0,-h)$, $h>0$, that is,

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0 q}{4\pi}\left(\frac{x}{D(\mathbf{x})_+^{3/2}} - \frac{x}{D(\mathbf{x})_-^{3/2}}, \frac{y}{D(\mathbf{x})_+^{3/2}} - \frac{y}{D(\mathbf{x})_-^{3/2}}, \frac{z-h}{D(\mathbf{x})_+^{3/2}} - \frac{z+h}{D(\mathbf{x})_-^{3/2}}\right), \tag{7}$$

with $D(\mathbf{x})_+ = x^2 + y^2 + (z-h)^2$, $D(\mathbf{x})_- = x^2 + y^2 + (z+h)^2$, and μ_0 the magnetic permeability of vacuum.

The standard and generalized orbitron

Definition

A small axisymmetric magnetized rigid body subjected to a external magnetic field of the form specified in (7) is called a **standard orbitron**.

The external magnetic field **B** in (7) has the following symmetry properties, namely:

(i) Equivariance with respect to rotations $R^Z_{\theta_S}$ around the OZ axis:

$$\mathbf{B}(R^Z_{\theta_S}\mathbf{x}) = R^Z_{\theta_S}\mathbf{B}(\mathbf{x}) \text{ for } \theta_S \in \mathbb{R}.$$

(ii) Behavior with respect to the mirror transformation $(x, y, z) \longmapsto (x, y, -z)$ according to the prescription

$$B_x(x, y, z) = -B_x(x, y, -z), B_y(x, y, z) = -B_y(x, y, -z), B_z(x, y, z) = B_z(x, y, -z).$$

Definition

A small axisymmetric magnetized rigid body subjected to the influence of an arbitrary magnetic field in the magnetostatic approximation in a domain free of other magnetic sources that satisfies these symmetry properties is called a **generalized orbitron**.

Equations of motion

The equations of motion of the orbitron are determined by Hamilton's equations:

$$\dot{A} = A\widehat{\mathbb{I}_{ref}^{-1}\mathbf{\Pi}}, \tag{8}$$

$$\dot{\mathbf{x}} = \frac{1}{M}A\mathbf{p}, \tag{9}$$

$$\dot{\mathbf{\Pi}} = \mathbf{\Pi} \times \mathbb{I}_{ref}^{-1}\mathbf{\Pi} + A^{-1}\mathbf{B}(\mathbf{x}) \times \mathbf{e}_3, \tag{10}$$

$$\dot{\mathbf{p}} = \mathbf{p} \times \mathbb{I}_{ref}^{-1}\mathbf{\Pi} + \mu A^{-1}D\mathbf{B}(\mathbf{x})^T A\mathbf{e}_3. \tag{11}$$

The symbol $\widehat{\mathbb{I}_{ref}^{-1}\mathbf{\Pi}}$ stands for the antisymmetric matrix associated to the vector $\mathbb{I}_{ref}^{-1}\mathbf{\Pi} \in \mathbb{R}^3$ via the Lie algebra isomorphism $\widehat{\ }: (\mathbb{R}^3, \times) \longrightarrow (\mathfrak{so}(3), [\cdot, \cdot])$ and D for the differential.

Toral symmetry and associated momentum map

The axial symmetry of the magnetic rigid body + the rotational spatial symmetry of the external magnetic field w.r.t. rotations around OZ = toral symmetry. The action on $SE(3)$:

$$\begin{array}{rccc} \Phi : & (\mathbb{T}^2 = S^1 \times S^1) \times SE(3) & \longrightarrow & SE(3) \\ & \left(\left(e^{i\theta_S}, e^{i\theta_B} \right), (A, \mathbf{x}) \right) & \longmapsto & (R^Z_{\theta_S} A R^Z_{-\theta_B}, R^Z_{\theta_S}\mathbf{x}). \end{array} \tag{12}$$

The cotangent lift Φ is a canonical symmetry given by

$$\begin{array}{rccc} \Phi : & (\mathbb{T}^2 = S^1 \times S^1) \times T^*SE(3) & \longrightarrow & T^*SE(3) \\ & \left(\left(e^{i\theta_S}, e^{i\theta_B} \right), ((A, \mathbf{x}), (\mathbf{\Pi}, \mathbf{p})) \right) & \longmapsto & ((R_{\theta_S} A R_{-\theta_B}, R_{\theta_S}\mathbf{x}), (R_{\theta_B}\mathbf{\Pi}, R_{\theta_B}\mathbf{p})) \end{array}$$

that has an invariant momentum map associated $\mathbf{J} : T^*SE(3) \longrightarrow \mathfrak{t}^*$:

$$\mathbf{J}\left((A, \mathbf{x}), (\mathbf{\Pi}, \mathbf{p}) \right) = \left(\langle A\mathbf{\Pi} + \mathbf{x} \times A\mathbf{p}, \mathbf{e}_3 \rangle, -\langle \mathbf{\Pi}, \mathbf{e}_3 \rangle \right). \tag{13}$$

Relative equilibria equations of the orbitron

Proposition

Consider the orbitron system whose Hamiltonian function is given by (4) and let $\mathbf{z} = ((A, \mathbf{x}), (\mathbf{\Pi}, \mathbf{p})) \in T^*SE(3)$. *Then:*

(i) *The point* $\mathbf{z}$ *is a relative equilibrium of the orbitron with velocity* $(\xi_1, \xi_2) \in \mathbb{R}^2$ *with respect to the introduced toral symmetry if and only if the following identities are satisfied:*

$$\mu \left[\mathbf{B}(\mathbf{x}) \times A\mathbf{e}_3\right] + \xi_1 \left[A\mathbf{p} \times (\mathbf{x} \times \mathbf{e}_3) - A\mathbf{\Pi} \times \mathbf{e}_3\right] = 0, \tag{14}$$

$$-\mu D\mathbf{B}(\mathbf{x})^T (A\mathbf{e}_3) - \xi_1 (A\mathbf{p} \times \mathbf{e}_3) = 0, \tag{15}$$

$$\mathbb{I}_{ref}^{-1}\mathbf{\Pi} + \xi_2\mathbf{e}_3 - \xi_1 A^{-1}\mathbf{e}_3 = 0, \tag{16}$$

$$\frac{1}{M}\mathbf{p} - \xi_1 A^{-1} (\mathbf{e}_3 \times \mathbf{x}) = 0. \tag{17}$$

Proposition (Continued)

(ii) *Consider now* $A_0 = R^Z_{\theta_0}$, $\mathbf{x}_0 = (x, y, 0)$, $\mathbf{\Pi}_0 = I_3\,(\xi_1 - \xi_2)\,\mathbf{e}_3$ *and* $\mathbf{p}_0 = M\xi_1 A_0^{-1}\,(-y, x, 0)$. *The point* $\mathbf{z}_0 = \left((A_0, \mathbf{x}_0), (\mathbf{\Pi}_0, \mathbf{p}_0)\right)$ *is a relative equilibrium of the standard orbitron with velocity* (ξ_1, ξ_2)*, where* ξ_2 *is an arbitrary real number and* ξ_1 *is either arbitrary when* $\mathbf{x}_0 = \mathbf{0}$ *or*

$$\xi_1 = \pm\left(-\frac{3h\mu q\mu_0}{2\pi MD(\mathbf{x}_0)^{5/2}}\right)^{1/2}, \tag{18}$$

when $\mathbf{x}_0 \neq \mathbf{0}$ *(the existence is only guaranteed when* $\mu q < 0$*).*

(iii) *In the case of the generalized orbitron:* $B_z(x, y, z) = f(x^2 + y^2, z)$ *for some* $f \in C^\infty(\mathbb{R}^2)$*, and the spatial velocity* ξ_1 *of the relative equilibria with* $\mathbf{x}_0 \neq \mathbf{0}$ *is*

$$\xi_1 = \pm\left(-\frac{2}{M}\mu f_1'\right)^{1/2}, \tag{19}$$

where $f_1' = \left.\dfrac{\partial f(v, z)}{\partial v}\right|_{v=x^2+y^2, z=0}$ *(exists only when* $\mu f_1' < 0$*).*

The relative equilibria for which $\mathbf{x}_0 \neq \mathbf{0}$ *(resp.* $\mathbf{x}_0 = \mathbf{0}$*) have trivial (resp. nontrivial* H*) isotropy and hence belong to the orbit type* $(T^*SE(3))_{\{e\}}$ *(resp.* $(T^*SE(3))_H$*); we refer to them as* **regular relative equilibria** *(resp.* **singular relative equilibria***).*

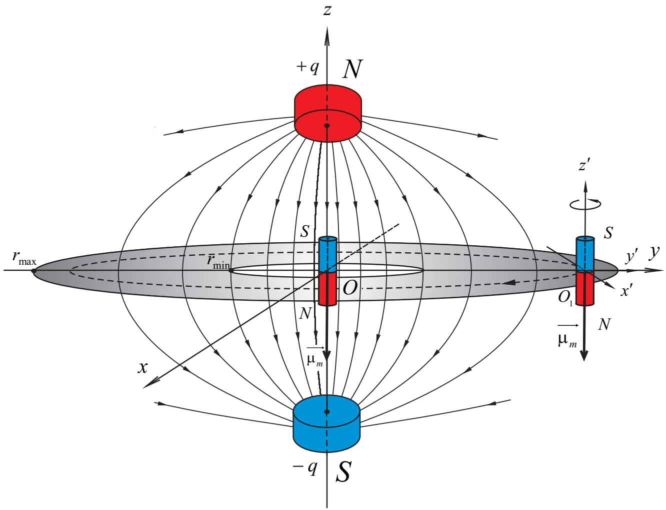
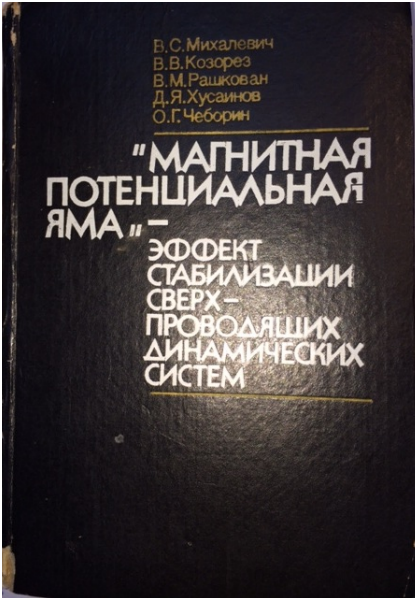


Figure : Regular and singular relative equilibria of the standard orbitron. r_{min} and r_{max} represent the stability region in config. space determined by the conditions below.


В.С.Михалевич
В.В.Козорез
В.М.Рашкован
Д.Я.Хусаинов
О.Г.Чеборин
"МАГНИТНАЯ ПОТЕНЦИАЛЬНАЯ ЯМА" – ЭФФЕКТ СТАБИЛИЗАЦИИ СВЕРХ-ПРОВОДЯЩИХ ДИНАМИЧЕСКИХ СИСТЕМ

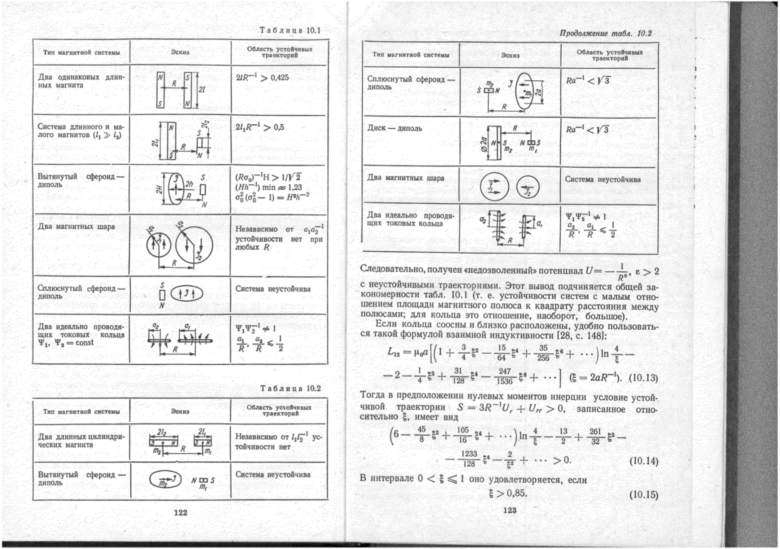


Таблица 10.1

Тип магнитной системы	Эскиз	Область устойчивых траекторий
Два одинаковых длинных магнита		$2lR^{-1} > 0{,}425$
Система длинного и малого магнитов ($l_1 \gg l_2$)		$2l_1R^{-1} > 0{,}5$
Вытянутый сфероид — диполь		$(R\sigma_0)^{-1}H > 1/\sqrt{2}$ $(Hh^{-1})\min \approx 1{,}23$ $\sigma_0^2(\sigma_0^2 - 1) = H^2h^{-2}$
Два магнитных шара		Независимо от $a_1a_2^{-1}$ устойчивости нет при любых R
Сплюснутый сфероид — диполь		Система неустойчива
Два идеально проводящих токовых кольца Ψ_1, $\Psi_2 = \mathrm{const}$		$\Psi_1\Psi_2^{-1} \neq 1$ $\frac{a_1}{R}, \frac{a_2}{R} \lessgtr \frac{1}{2}$

Таблица 10.2

Тип магнитной системы	Эскиз	Область устойчивых траекторий
Два длинных цилиндрических магнита		Независимо от $l_1l_2^{-1}$ устойчивости нет
Вытянутый сфероид — диполь		Система неустойчива

122

Продолжение табл. 10.2

Тип магнитной системы	Эскиз	Область устойчивых траекторий
Сплюснутый сфероид — диполь		$Ra^{-1} < \sqrt{3}$
Диск — диполь		$Ra^{-1} < \sqrt{3}$
Два магнитных шара		Система неустойчива
Два идеально проводящих токовых кольца		$\Psi_1\Psi_2^{-1} \neq 1$ $\frac{a_2}{R}, \frac{a_1}{R} \lessgtr \frac{1}{2}$

Следовательно, получен «недозволенный» потенциал $U = -\frac{1}{R^{\varepsilon}}$, $\varepsilon > 2$ с неустойчивыми траекториями. Этот вывод подчиняется общей закономерности табл. 10.1 (т. е. устойчивости систем с малым отношением площади магнитного полюса к квадрату расстояния между полюсами; для кольца это отношение, наоборот, большое).

Если кольца соосны и близко расположены, удобно пользоваться такой формулой взаимной индуктивности [28, с. 148]:

$$L_{12} = \mu_0 a\left[\left(1 + \frac{3}{4}\xi^2 - \frac{15}{64}\xi^4 + \frac{35}{256}\xi^6 + \cdots\right)\ln\frac{4}{\xi} - \right.$$
$$\left. - 2 - \frac{1}{4}\xi^2 + \frac{31}{128}\xi^4 - \frac{247}{1536}\xi^6 + \cdots\right] \quad (\xi = 2aR^{-1}). \quad (10.13)$$

Тогда в предположении нулевых моментов инерции условие устойчивой траектории $S = 3R^{-1}U_r + U_{rr} > 0$, записанное относительно ξ, имеет вид

$$\left(6 - \frac{45}{8}\xi^2 + \frac{105}{16}\xi^4 + \cdots\right)\ln\frac{4}{\xi} - \frac{13}{2} + \frac{261}{32}\xi^2 -$$
$$- \frac{1233}{128}\xi^4 - \frac{2}{\xi^2} + \cdots > 0. \quad (10.14)$$

В интервале $0 < \xi \leqslant 1$ оно удовлетворяется, если

$$\xi > 0{,}85. \quad (10.15)$$

123

... при некотором отношении между размером орбиты r_0 и размером магнита $2l$ происходит «переход» через закон $U \sim -\frac{1}{r^2}$, разделяющий взаимодействия с устойчивыми и неустойчивыми траекториями. Для одинаковых длинных цилиндрических магнитов это соответствует условию (5.4.7).

Если размеры одного магнита намного больше размеров другого, вместо (5.4.7) используем условие

$$l_2 r_0^{-1} > 0,5. \tag{5.4.8}$$

Если импульсы собственного вращения магнитов намного больше импульса орбитального вращения

$$M_i \gg M_0 > 0, \quad i = 1,\ 2,$$

что реализуемо выбором, например, скорости собственного вращения магнитов, то коэффициенты $U_{\theta_i\theta_i}$ и определитель (5.4.5) будут положительными при любых β и γ.

Таким образом, планетарная конфигурация двух цилиндрических магнитов, совершающих быстрое собственное вращение вокруг магнитной оси, перпендикулярной к плоскости орбиты, является устойчивой, если только размер невозмущенной орбиты не превосходит размера, сравнимого с размером магнита.

Если размеры орбиты большие по сравнению с размером магнита, устойчивость невозможна (нарушается условие (5.4.6)), и мы приходим к известному результату неустойчивости движения пары магнитных моментов [26].

Nonlinear stability of the orbitron relative equilibria

Theorem

Consider the relative equilibria introduced in Proposition 7. Then:

(i) *The regular relative equilibria of the standard orbitron in part* **(ii)** *of Proposition 7, that is, those for which* $\mathbf{x}_0 \neq \mathbf{0}$*, are* $\mathbb{T}^2$*–stable whenever the following three inequalities are satisfied:*

$$\frac{2}{3} < \frac{r^2}{h^2} < 4, \tag{20}$$

$$\operatorname{sign}(\xi_1^0) I_3 \xi_2 < - \mid \xi_1^0 \mid \left(I_1 - I_3 + \frac{2}{3} M \frac{(r^2 + h^2) h^2}{3r^2 - 2h^2} \right), \tag{21}$$

where $r^2 = \|\mathbf{x}_0\|^2$*,* $\xi_1^0 = \pm \left(- \frac{3h\mu q \mu_0}{2\pi M D(\mathbf{x}_0)^{5/2}} \right)^{1/2}$*, and* $\mu q < 0$*. The singular relative equilibria (*$\mathbf{x}_0 = \mathbf{0}$*) are always formally unstable, in the sense that the stability form exhibits a nontrivial signature.*

Theorem (Continued)

(ii) *The regular relative equilibria of the generalized orbitron in part* **(iii)** *of Proposition 7 are* $\mathbb{T}^2$*–stable whenever the following conditions hold:*

$$\mu f_1' < 0, \tag{22}$$

$$\mu\left(2f_1' + r^2 f_1''\right) < 0, \tag{23}$$

$$\mu f_2'' < 0, \tag{24}$$

$$\operatorname{sign}(\xi_1^0) I_3 \xi_2 < -|\xi_1^0| \left((I_1 - I_3) + \frac{1}{2} M \left(\frac{f_0}{f_1'} + 4r^2 \frac{f_1'}{f_2''} \right) \right), \tag{25}$$

where $r^2 = \|\mathbf{x}_0\|^2$*,* $f \in C^\infty(\mathbb{R}^2)$ *is the function such that* $B_z(x,y,z) = f(r^2,z)$*,* $f_0 = f(r^2,0)$*,* $f_1' = \left.\dfrac{\partial f(v,z)}{\partial v}\right|_{v=r^2,z=0}$*,* $f_1'' = \left.\dfrac{\partial^2 f(v,z)}{\partial v^2}\right|_{v=r^2,z=0}$*,* $f_2'' = \left.\dfrac{\partial^2 f(v,z)}{\partial z^2}\right|_{v=r^2,z=0}$*, and* $\xi_1^0 = \pm\left(-\dfrac{2}{M}\mu f_1'\right)^{1/2}$*.*

Theorem (Continued)

The singular branch ($\mathbf{x}_0 = \mathbf{0}$) is $\mathbb{T}^2$–stable if the following conditions are satisfied:

$$\mu f_1' < 0, \tag{26}$$

$$\mu f_2'' < 0, \tag{27}$$

$$\xi_1^2 < -\frac{2}{M}\mu f_1', \tag{28}$$

$$\mathrm{sign}(\xi_1)\Pi_0 > \frac{I_1\xi_1^2 - \mu f_0}{|\xi_1|}, \tag{29}$$

where $\Pi_0 = I_3(\xi_1 - \xi_2)$ and we use the same notation as above for f_0, f_1', and f_2'', replacing $v = r^2$ by $v = 0$. When $\mu f_0 < 0$ and $\dfrac{f_0}{f_1'} < \dfrac{2}{M} I_1$, the conditions (28) and (29) can be replaced by the following single ξ_1–independent optimal condition:

$$|\Pi_0| > 2\sqrt{-\mu f_0 I_1}. \tag{30}$$

This optimal condition is achieved by using the spatial velocities $\xi_1 = \pm(-\mu f_0/I_1)^{1/2}$; the positive (resp. negative) sign for the velocity corresponds to positive (resp. negative) values of Π_0.

- Conditions (26)–(29) can be used in the design of magnetic fields capable of confining magnetic rigid bodies that do not exhibit spatial rotation.
- This is the working principle of devices such as magnetic contactless flywheels or levitrons. In the case of flywheels, up until now only actively controlled versions have been developed.
- As to the levitron, the potentials that have been considered so far [Dullin 1999, 2004, Marsden, Krechetnikov 2006] do not allow to conclude nonlinear stability using the methods put at work in Theorem 9 and only the spectral stability of the corresponding linearized systems has been considered.

The use of the energy-momentum method provides sufficient but not necessary nonlinear stability conditions. The complementary spectral stability analysis of the linearized system is required.

Linear stability/instability analysis of relative equilibria

- Standard equilibria: examine the spectral stability of the linearization at the equilibrium of the vector field in question.
- Regular relative equilibria: examine the spectral stability of the linearization of the reduced Hamiltonian vector field at the equilibrium corresponding to the relative equilibrium in the symplectic Marsden–Weinstein reduced space.
- Singular case: there exist reduced spaces that generalize the Marsden–Weinstein reduced space, the equivalence between G_μ-stability of a relative equilibrium and standard nonlinear stability of the corresponding reduced equilibrium does not hold anymore, which makes necessary the formulation of a criterion that provides a linear stability analysis tool for relative equilibria whose formulation does not need reduction.

Proposition

Let G be a Lie group acting canonically and properly on the symplectic manifold (M, ω) and suppose that there exists a coadjoint equivariant m. m. $\mathbf{J} : M \to \mathfrak{g}^*$ that can be associated to it. Let $h \in C^\infty(M)^G$ be a G–invariant Hamiltonian and let $m \in M$ be a relative equilibrium of the corresponding G–equivariant Hamiltonian vector field X_h with velocity $\xi \in \mathfrak{g}$. Consider a G_m–invariant stability space W such that

$$\ker T_m\mathbf{J} = W \oplus T_m\left(G_\mu \cdot m\right),$$

with $\mu := \mathbf{J}(m)$ and $G_\mu \subset G$ the coadjoint isotropy of $\mu \in \mathfrak{g}^*$. Then:

(i) *(W, ω_W) with $\omega_W := \omega(m)|_W$ is a symplectic vector subspace of $(T_mM, \omega(m))$.*

(ii) *There exists a symplectic slice (S, ω_S) at $m \in M$ such that $(T_mS, \omega_S(m)) = (W, \omega_W)$.*

(iii) *The Hamiltonian vector field $X_{h_S^\xi} \in \mathfrak{X}(S)$ in S associated to the Hamiltonian function $h_S^\xi := \left(h - \mathbf{J}^\xi\right)\big|_S$ exhibits an equilibrium at the point $m \in S \subset M$.*

Proposition (Continued)

(iv) *The linearization* $X'_{h^\xi_S} \in \mathfrak{X}(T_m S) = \mathfrak{X}(W)$ *of* $X_{h^\xi_S}$ *at* $m \in S$ *coincides with the linear Hamiltonian vector field* X_Q *on* (W, ω_W) *that has as Hamiltonian vector field the stability form*

$$Q(w) := \mathbf{d}^2 \left(h - \mathbf{J}^\xi\right)(m)(w, w), \qquad w \in W.$$

(v) *Suppose that the two tangent spaces* $T_m\left(G_\mu \cdot m\right)$ *and* $T_m\left(G \cdot m\right)$ *coincide. Then*

$$T_m M = W \oplus W^\omega. \tag{31}$$

Additionally, let $h^\xi := h - \mathbf{J}^\xi \in C^\infty(M)$ *be the augmented Hamiltonian and let* $X'_{h^\xi} \in \mathfrak{X}(T_m M)$ *be the linearization of the Hamiltonian vector field* X_{h^ξ} *at* m. *Then*

$$X_Q = \mathbb{P}_W X'_{h^\xi} \mathbf{i}_W, \tag{32}$$

where $\mathbf{i}_W : W \hookrightarrow T_m M$ *is the inclusion,* $\mathbb{P}_W : T_m M \longrightarrow W$ *is the projection according to* (31), *and* X'_{h^ξ} *is the linearization of* X_{h^ξ} *at* m.

(vi) *If the linear vector field* X_Q *is spectrally unstable in the sense that it exhibits eigenvalues with a nontrivial real part, then the relative equilibrium* $m \in M$ *of* X_h *is nonlinearly* K*–unstable, for any subgroup* $K \subset G$.

Proposition

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let T^*G be its cotangent bundle endowed with the canonical symplectic form. Consider now the body coordinates expression $G \times \mathfrak{g}^*$ of T^*G and let $h \in C^\infty(G \times \mathfrak{g}^*)$ be a Hamiltonian function whose associated Hamiltonian vector field X_h exhibits an equilibrium at point $(g, \mu) \in G \times \mathfrak{g}^*$. Then the linearization $X_{Q^g} : \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g} \times \mathfrak{g}^*$ for any $(\xi, \tau) \in \mathfrak{g} \times \mathfrak{g}^*$ is given by:

$$X_{Q^g}(\xi, \tau) = \left(\pi_{\mathfrak{g}^*}\left(\mathrm{Hess}(\xi, \tau)\right), -\pi_{\mathfrak{g}}\left(\mathrm{Hess}(\xi, \tau)\right) + \mathrm{ad}^*_{\pi_{\mathfrak{g}^*}\mathrm{Hess}(\xi,\tau)}\mu\right), \tag{33}$$

where $\pi_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}$, $\pi_{\mathfrak{g}^*} : \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}^*$ are the canonical projections and $\mathrm{Hess} : \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g} \times \mathfrak{g}^*$ is the linear map associated to the Hessian of h^g at (e, μ) by the relation

$$\langle \mathrm{Hess}(\xi, \tau), (\eta, \rho) \rangle = \mathbf{d}^2 h^g(e, \mu)((\xi, \tau), (\eta, \rho)), \quad (\xi, \tau), (\eta, \rho) \in \mathfrak{g} \times \mathfrak{g}^*.$$

Let $h \in C^\infty(T^*(SE(3)))$ be a Hamiltonian function and let X_h be the corresponding Hamiltonian vector field that we assume has an equilibrium at the point $\mathbf{z}_0 = ((A_0, \mathbf{x}_0), (\mathbf{\Pi}_0, \mathbf{p}_0))$, that is, $\mathbf{d}h(\mathbf{z}_0) = 0$. Let $g = (A_0, \mathbf{x}_0) \in SE(3)$ and let $\mathbf{z} = ((I, \mathbf{0}), (\mathbf{\Pi}_0, \mathbf{p}_0))$; clear that $\mathbf{z}_0 = \varphi_g(\mathbf{z})$. Let $\mathrm{Hess}(\mathbf{z}) : \mathfrak{se}(3) \times \mathfrak{se}(3)^* \to \mathfrak{se}(3) \times \mathfrak{se}(3)^*$ be the linear map associated to the Hessian of $h \circ \varphi_g$ at $\mathbf{z}$, that is, for any $\mathbf{v}, \mathbf{w} \in T_{\mathbf{z}}(T^*SE(3)) \simeq \mathfrak{se}(3) \times \mathfrak{se}(3)^*$, $\langle \mathbf{v}, \mathrm{Hess}(\mathbf{z})\mathbf{w} \rangle = \mathbf{d}^2(h \circ \varphi_g)(\mathbf{z})(\mathbf{v}, \mathbf{w})$. Now, given $\mathbf{v} = (\delta A, \boldsymbol{\delta}\mathbf{x}, \boldsymbol{\delta}\mathbf{\Pi}, \boldsymbol{\delta}\mathbf{p}) \in \mathfrak{se}(3) \times \mathfrak{se}(3)^*$, define the projections:

$$\begin{array}{rcl} \boldsymbol{\pi}_{\delta A} : \mathfrak{se}(3) \times \mathfrak{se}(3)^* & \longrightarrow & \mathbb{R}^3 \\ (\delta A, \boldsymbol{\delta}\mathbf{x}, \boldsymbol{\delta}\mathbf{\Pi}, \boldsymbol{\delta}\mathbf{p}) & \longmapsto & \delta A \end{array}$$

By Proposition 14 the linearization X_h' of X_h at $\mathbf{z}_0$ is given by

$$X_h' = \Phi_g \circ X_{h^g}' \circ \Phi_{g^{-1}}, \tag{34}$$

where $X_{h^g}' : \mathfrak{se}(3) \times \mathfrak{se}(3)^* \simeq \mathbb{R}^{12} \to \mathfrak{se}(3) \times \mathfrak{se}(3)^* \simeq \mathbb{R}^{12}$ is the linear map

$$X_{h^g}' = \begin{pmatrix} \boldsymbol{\pi}_{\delta\mathbf{\Pi}} \mathrm{Hess}(\mathbf{z}_0) \\ \boldsymbol{\pi}_{\delta\mathbf{p}} \mathrm{Hess}(\mathbf{z}_0) \\ -\boldsymbol{\pi}_{\delta A} \mathrm{Hess}(\mathbf{z}_0) + \widehat{\mathbf{\Pi}}_0 \boldsymbol{\pi}_{\delta\mathbf{\Pi}} \mathrm{Hess}(\mathbf{z}_0) + \widehat{\mathbf{p}}_0 \boldsymbol{\pi}_{\delta\mathbf{p}} \mathrm{Hess}(\mathbf{z}_0) \\ -\boldsymbol{\pi}_{\delta\mathbf{x}} \mathrm{Hess}(\mathbf{z}_0) + \widehat{\mathbf{p}}_0 \boldsymbol{\pi}_{\delta\mathbf{\Pi}} \mathrm{Hess}(\mathbf{z}_0) \end{pmatrix}. \tag{35}$$

Theorem

Consider the relative equilibria introduced in Proposition 7. Then:

(i) *In the case of the standard orbitron in part* **(ii)***:*

  (a) *The regular relative equilibria that do not satisfy the Kozorez relation ($r^2/h^2 < 4$) are unstable; the stability condition is sharp. The conditions in (20) and (21) are not sharp, i.e. there are regions in parameter space that do not satisfy them and where the linearized system is spectrally stable.*

  (b) *The singular relative equilibria are nonlinearly unstable.*

(ii) *In the case of the generalized orbitron in part* **(iii)***:*

  (a) *The regular relative equilibria that do not satisfy the generalized Kozorez relation (23) ($\mu\left(2f_1' + r^2 f_2''\right) < 0$), are unstable; the stability condition is sharp. The conditions (22), (24), and (25) are not sharp.*

  (b) *The spectral stability of the singular relative equilibria is equivalent to:*

$$\mu f_1' < 0, \;\; \mu f_2'' < 0, \;\; \Pi_0^2 > -4\mu f_0 I_1, \tag{36}$$

  where $\Pi_0 = I_3(\xi_1 - \xi_2)$. The conditions (26) and (27) are sharp, the remaining conditions are not.

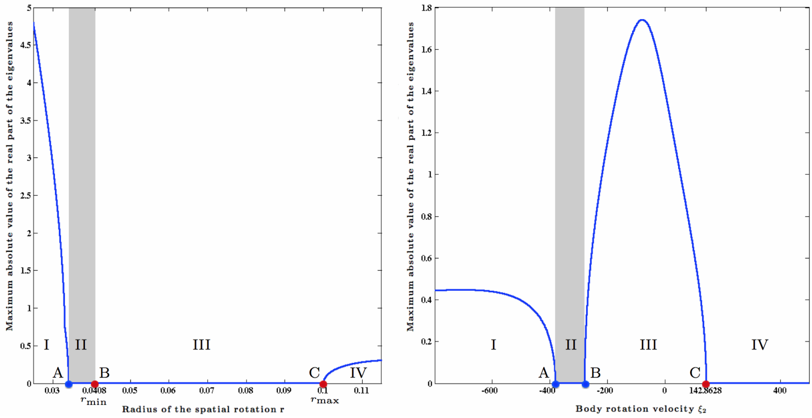
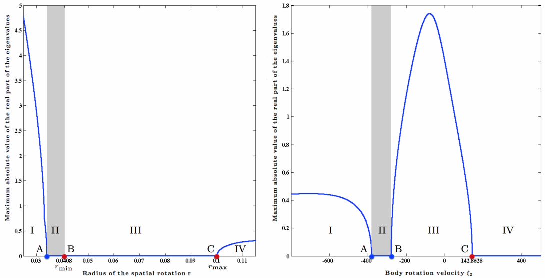


Figure : Standard orbitron with $h = 0.05$ m, $M = 0.0068$ kg, $\mu_0 = 4\pi \cdot 10^{-7}$ N·A^{-2}, $\mu = -0.18375$ A·m^2, $q = 17.58$ A·m, $I_1 = 0.17 \cdot 10^{-6}$ kg·m^2, $I_3 = 0.1 \cdot 10^{-6}$ kg·m^2. The red bullets indicate the critical values of r (m) and ξ_2 (rad·s^{-1}) determined by the stability conditions. The grey bands correspond to the stability gaps (the system is spectrally stable while the stability form exhibits a nontrivial signature).



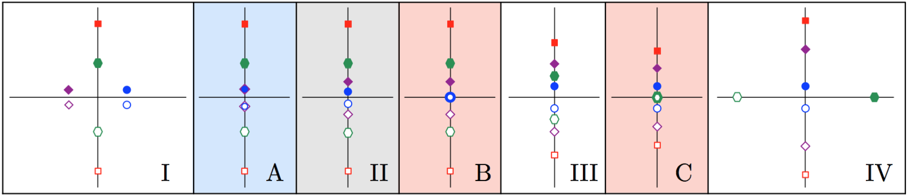


Figure : Evolution of the eigenvalues with r.

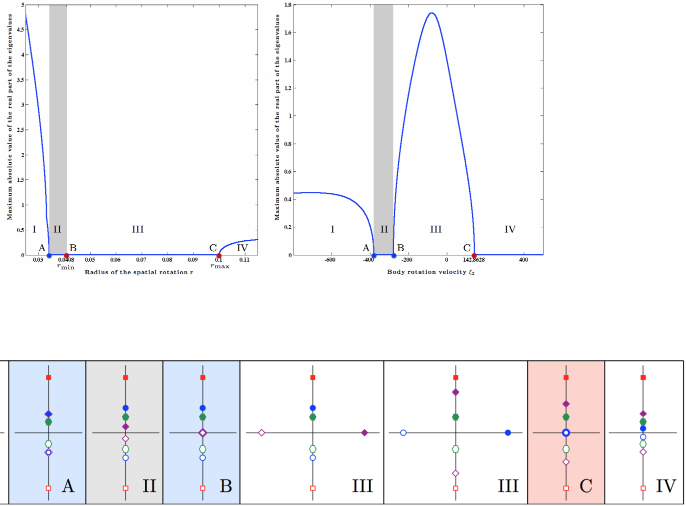


Figure : Evolution of the eigenvalues with ξ_2.

RPOs: symmetric energy-integrals method

Theorem (The symmetric energy–integrals method)

$(M, \{\cdot,\cdot\}, G, \mathbf{J} : M \to \mathfrak{g}^*, h : M \to \mathbb{R})$ Hamiltonian system with a symmetry given by the Lie group G acting properly on M. $h \in C^\infty(M)$ is G–invariant and $\mathbf{J}$ is equivariant. Let $m \in M$ be a RPO such that $\mathbf{J}(m) = \mu \in \mathfrak{g}^*$ and G_μ is compact. Then, if there is a set of G_μ–invariant conserved quantities $C_1, \ldots, C_n \in C^\infty(M)$, for which

$$\mathbf{d}(C_1 + \ldots + C_n)(m) = 0, \quad \textit{and} \quad \mathbf{d}^2(C_1 + \ldots + C_n)(m)|_{W \times W}$$

is definite for some (and hence for any) W such that

$$\ker \mathbf{d}C_1(m) \cap \ldots \cap \ker \mathbf{d}C_n(m) \cap \ker T_m\mathbf{J} = W \oplus (\mathrm{span}\{X_h(m)\} + T_m(G_m \cdot m)), \tag{37}$$

then m is a G_μ–stable RPo. If $\dim W = 0$, then m is always a G_m–stable RPP.

G-invariant Poincaré sections

Definition

Let $X \in \mathfrak{X}(M)^G$. A **G–invariant local transversal section** of X at $m \in M$ is a G–invariant submanifold S of codimension one with $m \in S$ such that for all $z \in S$, $X(z)$ is not contained in $T_z S$.

If $m \in M$ is a RPO with relative period $\tau > 0$, phase shift $g \in G$, and S is a G–invariant local transversal section at m then, a **G–equivariant Poincaré map** of the RPP m is a mapping $\Theta : W_0 \to W_1$ satisfying:

(RPM1) W_0, $W_1 \subset S$ are open G–invariant neighborhoods of m in S and Θ is a G–equivariant diffeomorphism;

(RPM2) there is a continuous G–invariant function, called the **period function**, such that for all $z \in W_0$, $(z, \tau - \delta(z)) \in \mathcal{D}_X$, and $\Theta(z) = F(z, \tau - \delta(z))$. The open set $\mathcal{D}_X \subset M \times \mathbb{R}$ is the domain of the flow $F : \mathcal{D}_X \subset M \times \mathbb{R} \to M$ of X;

(RPM3) if $t \in (0, \tau - \delta(z))$, then $F(z, t) \notin W_0$.

Theorem (Existence and uniqueness of G–equivariant Poincaré maps)

Due to Field [1980, 1991]. Let m, M, and X be as in Definition.

(i) *There exists a G–invariant local transversal section S and a G–equivariant Poincaré map $\Theta : W_0 \to W_1$ for $m \in M$.*

(ii) *If $\Theta : W_0 \to W_1$ is a G–equivariant Poincaré map for m in the G–invariant local transversal section S and similarly $\Theta' : W_0' \to W_1'$ for $m' := F_{t_0}(m)$ in S', then Θ and Θ' are locally G–equivariantly conjugate, that is, the diagram*

$$\begin{array}{ccc} \Theta^{-1}(W_2) \cap W_2 & \xrightarrow{\Theta} & W_2 \cap \Theta(W_2) \\ \mathcal{H}\downarrow & & \downarrow\mathcal{H} \\ W_2' & \xrightarrow{\Theta'} & S' \end{array}$$

commutes.

RPO example: the Manev potential

- Potential of the form $V(r) = \frac{k}{r} + \frac{B}{r^2}$.
- Manev [1924, 25, 30]. Diacu, Mioc, Stoica [1999], Delgado et al [1996].
- Proxy for relativistic correction.
- $SO(3)$ invariant. It reduces to Kepler with a momentum shift. Reduced space is 2 dimensional. Obvious orbital stability.